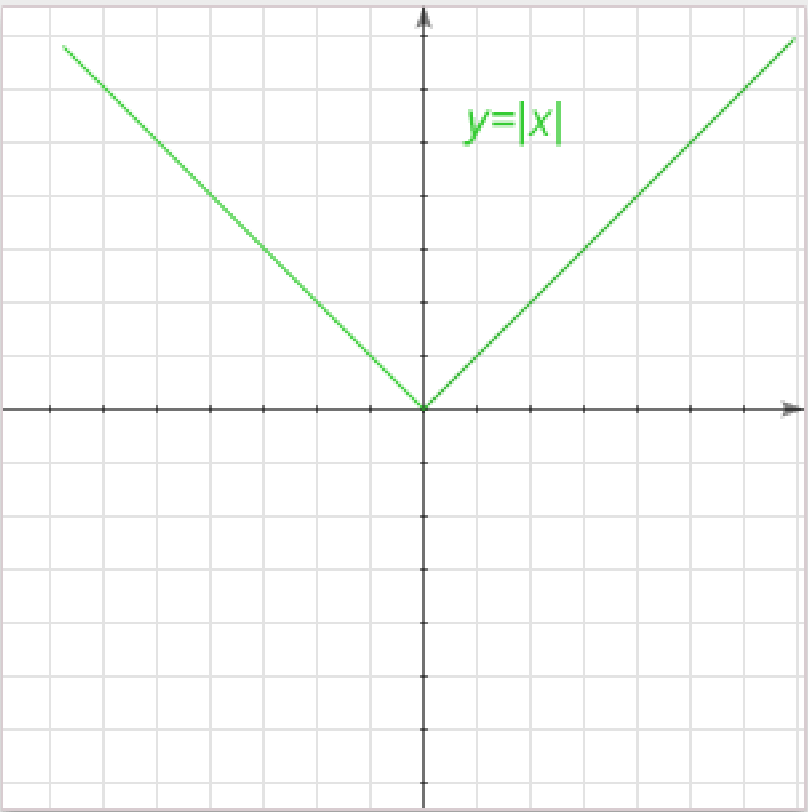
Definition

The absolute value (or modulus) $|x|$ of a real number x is x's numerical value without regard to its sign.

For example, $|3|=3$; $|-12|=12$; $|-1.3|=1.3$

Graph:



Important properties:

$|x|\geq 0$

$|x|=\sqrt{x^2}$

$|0|=0$

$|-x|=|x|$

$|x|+|y|\geq|x+y|$

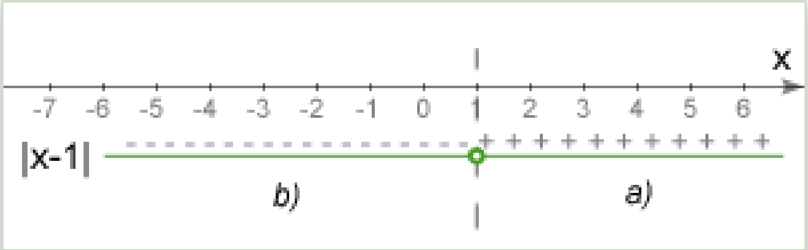
3-steps approach:

General approach to solving equalities and inequalities with absolute value:

1. Open modulus and set conditions.
To solve/open a modulus, you need to consider 2 situations to find all roots:

- Positive (or rather non-negative)
- Negative

For example, $|x-1|=4$
a) Positive: if $(x-1)\geq0$, we can rewrite the equation as:
$x-1=4$
b) Negative: if $(x-1)<0$, we can rewrite the equation as:
$-(x-1)=4$
We can also think about conditions like graphics. $x=1$ is a key point in which the expression under modulus equals zero. All points right are the first conditions $(x>1)$ and all points left are second conditions $(x<1)$.



2. Solve new equations:

a) $x-1=4$ --> x=5

b) $-x+1=4$ --> x=-3

3. Check conditions for each solution:

a) $x=5$ has to satisfy initial condition $x-1>=0$. $5-1=4>0$. It satisfies. Otherwise, we would have to reject x=5.

b) $x=-3$ has to satisfy initial condition $x-1<0$. $-3-1=-4<0$. It satisfies. Otherwise, we would have to reject x=-3.

3-steps approach for complex problems

Let's consider following examples,

Example #1

Q.: $|x+3|-|4-x|=|8+x|$. How many solutions does the equation have?

Solution: There are 3 key points here: -8, -3, 4. So we have 4 conditions:

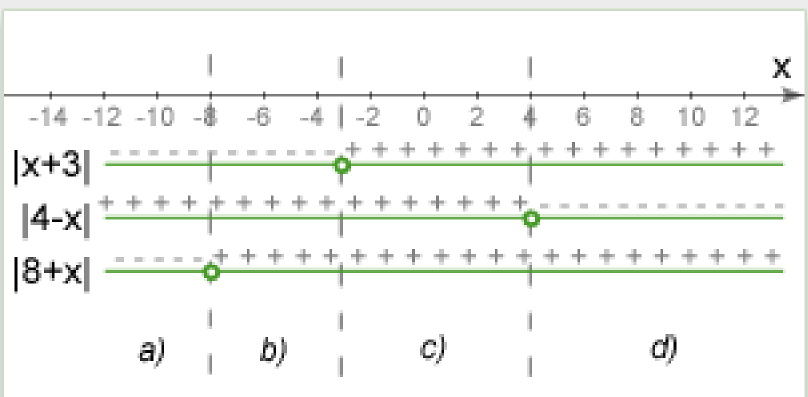
a) $x<-8$. $-(x+3)-(4-x)=-(8+x)$ --> $x=-1$. We reject the solution because our condition is not satisfied (-1 is not less than -8)

b) $-8 \leq x < -3$. $-(x+3)-(4-x)=(8+x)$ --> $x=-15$. We reject the solution because our condition is not satisfied (-15 is not within (-8,-3) interval.)

c) $-3 \leq x < 4$. $(x+3)-(4-x)=(8+x)$ --> $x=9$. We reject the solution because our condition is not satisfied (-15 is not within (-3,4) interval.)

d) $x \geq 4$. $(x+3)+(4-x)=(8+x)$ --> $x=-1$. We reject the solution because our condition is not satisfied (-1 is not more than 4)

(Optional) The following illustration may help you understand how to open modulus at different conditions.



Answer: 0

Example #2

Q.: $|x^2-4|=1$. What is x?

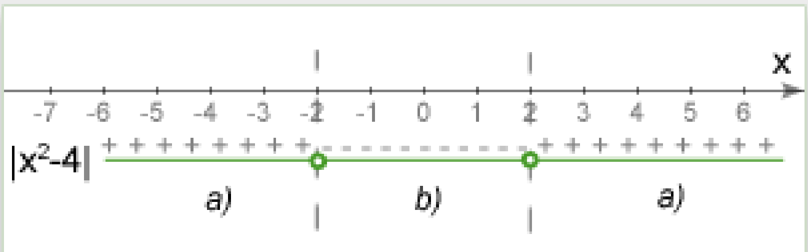
Solution: There are 2 conditions:

a) $(x^2-4) \geq 0$ --> $x \leq -2$ or $x \geq 2$. $x^2-4=1$ --> $x^2=5$. x e $\{-\sqrt{5},$

$\sqrt{5}\}$ and both solutions satisfy the condition.

b) $(x^2-4)<0$ --> $-2<x<2$. $-(x^2-4)=1$ --> $x^2=3$. x e $\{-\sqrt{3}, \sqrt{3}\}$ and both solutions satisfy the condition.

(Optional) The following illustration may help you understand how to open modulus at different conditions.



Answer: $-\sqrt{5}, -\sqrt{3}, \sqrt{3}, \sqrt{5}$

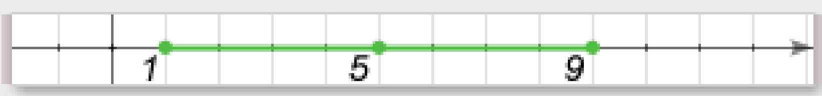
Tip & Tricks

The 3-steps method works in almost all cases. At the same time, often there are shortcuts and tricks that allow you to solve absolute value problems in 10-20 sec.

I. Thinking of inequality with modulus as a segment at the number line.

For example,

Problem: 1<x<9. What inequality represents this condition?



A. |x|<3
B. |x+5|<4
C. |x-1|<9

D. |-5+x|<4

E. |3+x|<5

Solution: 10sec. Traditional 3-steps method is too time-consume technique. First of all we find length (9-1)=8 and center (1+8/2=5) of the segment represented by 1<x<9. Now, let's look at our options. Only B and D has 8/2=4 on the right side and D had left site 0 at x=5. Therefore, answer is D.

II. Converting inequalities with modulus into range expression.

In many cases, especially in DS problems, it helps avoid silly mistakes.

For example,

|x|<5 is equal to x e (-5,5).

|x+3|>3 is equal to x e (-inf,-6)&(0,+inf)

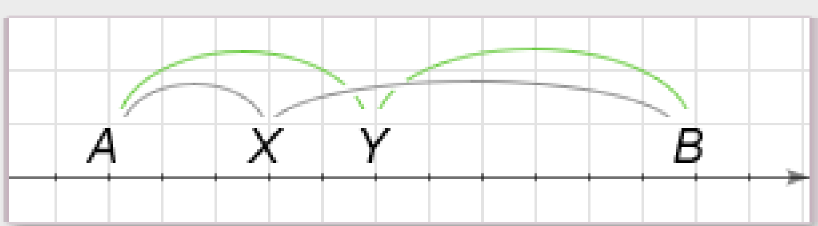
III. Thinking about absolute values as distance between points at the number line.

For example,

Problem: A<X<Y<B. Is |A-X| <|X-B|?

1) |Y-A|<|B-Y|

Solution:



We can think about absolute values here as distance between points. Statement 1 means than distance between Y and A is less than Y and B. Because X is between A and Y, distance between |X-A| < |Y-A| and at the same time distance between X and B will be larger than that between Y and B (|B-Y|<|B-X|). Therefore, statement 1 is sufficient.

Pitfalls

The most typical pitfall is ignoring third step in opening modulus - always check whether your solution satisfies conditions.