

Definition

Number Theory is concerned with the properties of numbers in general, and in particular integers.

As this is a huge issue we decided to divide it into smaller topics. Below is the list of Number Theory topics.

GMAT Number Types

GMAT is dealing only with ***Real Numbers***: Integers, Fractions and Irrational Numbers.

INTEGERS

Definition

Integers are defined as: all negative natural numbers $\{..., -4, -3, -2, -1\}$, $\{0\}$, and positive natural numbers $\{1, 2, 3, 4, ...\}$.

Note that integers do not include decimals or fractions - just whole numbers.

Even and Odd Numbers

An even number is an *integer* that is "evenly divisible" by 2, i.

e., divisible by 2 without a remainder.

An even number is an integer of the form $n = 2k$, where k is an integer.

An odd number is an *integer* that is not evenly divisible by 2.

An odd number is an integer of the form $n = 2k + 1$, where k is an integer.

Zero is an even number.

even $\pm$ even = even;

even $\pm$ odd = odd;

odd $\pm$ odd = even;

even $\times$ even = even;

even $\times$ odd = even;

odd $\times$ odd = odd.

Division of two integers can result into an even/odd integer or a fraction.

IRRATIONAL NUMBERS

Fractions (also known as rational numbers) can be written as *terminating* (ending) or *repeating* decimals (such as 0.5, 0.76, or 0.333333....). On the other hand, all those numbers that can be written as non-repeating, non-terminating decimals are non-rational, so they are called the "irrationals". Examples

would be $\sqrt{2}$ ("the square root of two") or the number pi ($\pi \approx 3.14159\dots$, from geometry). The rationals and the irrationals are two totally separate number types; there is no overlap.

Putting these two major classifications, the rationals and the irrationals, together in one set gives you the "real" numbers.

POSITIVE AND NEGATIVE NUMBERS

A positive number is a real number that is greater than zero. A negative number is a real number that is smaller than zero.

Zero is not positive, nor negative.

Prime Numbers

A Prime number is a natural number with exactly two distinct natural number divisors: 1 and itself. Otherwise a number is called a *composite* number. Therefore, **1 is not a prime**, since it only has one divisor, namely 1. A number $n > 1$ is prime if it cannot be written as a product of two factors a and b , both of which are greater than 1: $n = ab$.

The first twenty-six prime numbers are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101

- **Note: only positive numbers can be primes.**
- There are infinitely many prime numbers.
- The **only even prime** number is 2, since any larger even number is divisible by 2. Also **2 is the smallest prime**.
- **All prime numbers except 2 and 5 end in 1, 3, 7 or 9**, since numbers ending in 0, 2, 4, 6 or 8 are multiples of 2 and numbers ending in 0 or 5 are multiples of 5. Similarly, **all prime numbers above 3 are of the form $6n-1$ or $6n+1$** , because all other numbers are divisible by 2 or 3.
- Any nonzero natural number n can be factored into primes, written as a product of primes or powers of primes. Moreover, this **factorization is unique** except for a possible reordering of the factors.
- **Prime factorization:** every positive integer greater than 1 can be written as a product of one or more prime integers in a way which is unique. For instance integer n with three unique prime factors a , b , and c can be expressed as $n = a^p * b^q * c^r$, where p , q , and r are powers of a , b , and c , respectively and are ≥ 1 .
Example: $4200 = 2^3 * 3 * 5^2 * 7$.
- **Verifying the primality** of a given number n can be done by trial division, that is to say dividing n by all integer numbers

smaller than $\sqrt{n}$, thereby checking whether n is a multiple of $m < \sqrt{n}$.

Example: Verifying the primality of 161: $\sqrt{161}$ is little less than 13, from integers from 2 to 13, 161 is divisible by 7, hence 161 is not prime.

- If n is a positive integer greater than 1, then there is always a prime number p with $n < p < 2n$

Factors

A divisor of an *integer* n , also called a factor of n , is an *integer* which evenly divides n without leaving a remainder. In general, it is said m is a factor of n , for non-zero integers m and n , if there exists an integer k such that $n = km$.

- 1 (and -1) are divisors of every integer.
- Every integer is a divisor of itself.
- Every integer is a divisor of 0, except by convention 0 itself.
- Numbers divisible by 2 are called even and numbers not divisible by 2 are called odd.
- A positive divisor of n which is different from n is called a *proper divisor*.

- An integer $n > 1$ whose only proper divisor is 1 is called a prime number. Equivalently, one would say that a prime number is one which has exactly two factors: 1 and itself.
- Any positive divisor of n is a product of prime divisors of n raised to some power.
- If a number equals the sum of its proper divisors, it is said to be a *perfect number*.

Example: The proper divisors of 6 are 1, 2, and 3: $1+2+3=6$, hence 6 is a perfect number.

There are some elementary rules:

- If a is a factor of b and a is a factor of c , then a is a factor of $(b+c)$. In fact, a is a factor of $(mb+nc)$ for all integers m and n .
- If a is a factor of b and b is a factor of c , then a is a factor of c .
- If a is a factor of b and b is a factor of a , then $a=b$ or $a=-b$.
- If a is a factor of bc , and $\gcd(a,b)=1$, then a is a factor of c .
- If p is a prime number and p is a factor of ab then p is a factor of a or p is a factor of b .

Finding the Number of Factors of an Integer

First make prime factorization of an integer $n = a^p * b^q * c^r$, where a , b , and c are prime factors of n and p , q , and r are their powers.

The number of factors of n will be expressed by the formula $(p+1)(q+1)(r+1)$. **NOTE:** this will include 1 and n itself.

Example: Finding the number of all factors of 450:

$$450 = 2^1 * 3^2 * 5^2$$

Total number of factors of 450 including 1 and 450 itself is $(1+1)*(2+1)*(2+1) = 2*3*3 = 18$ factors.

Finding the Sum of the Factors of an Integer

First make prime factorization of an integer $n = a^p * b^q * c^r$, where a , b , and c are prime factors of n and p , q , and r are their powers.

The sum of factors of n will be expressed by the formula:

$$\frac{(a^{p+1}-1)*(b^{q+1}-1)*(c^{r+1}-1)}{(a-1)*(b-1)*(c-1)}$$

Example: Finding the sum of all factors of 450: $450 = 2^1 * 3^2 * 5^2$

The sum of all factors of 450 is

$$\frac{(2^{1+1}-1)*(3^{2+1}-1)*(5^{2+1}-1)}{(2-1)*(3-1)*(5-1)} = \frac{3*26*124}{1*2*4} = 1209$$

Greatest Common Factor (Divisor) - GCF (GCD)

The greatest common divisor (gcd), also known as the greatest common factor (gcf), or highest common factor (hcf), of two or more non-zero integers, is the largest positive integer that divides the numbers without a remainder.

To find the GCF, you will need to do prime-factorization. Then, multiply the common factors (pick the lowest power of the common factors).

- Every common divisor of a and b is a divisor of $\gcd(a, b)$.
- $a \cdot b = \gcd(a, b) \cdot \text{lcm}(a, b)$

Lowest Common Multiple - LCM

The lowest common multiple or lowest common multiple (lcm) or smallest common multiple of two integers a and b is the smallest positive integer that is a multiple both of a and of b . Since it is a multiple, it can be divided by a and b without a remainder. If either a or b is 0, so that there is no such positive integer, then $\text{lcm}(a, b)$ is defined to be zero.

To find the LCM, you will need to do prime-factorization. Then multiply all the factors (pick the highest power of the common

factors).

Perfect Square

A perfect square, is an integer that can be written as the square of some other integer. For example $16=4^2$, is an perfect square.

There are some tips about the perfect square:

- The number of distinct factors of a perfect square is ALWAYS ODD.
- The sum of distinct factors of a perfect square is ALWAYS ODD.
- A perfect square ALWAYS has an ODD number of Odd-factors, and EVEN number of Even-factors.
- Perfect square always has even number of powers of prime factors.

Divisibility Rules

2 - If the last digit is even, the number is divisible by 2.

3 - If the sum of the digits is divisible by 3, the number is also.

4 - If the last two digits form a number divisible by 4, the number is also.

5 - If the last digit is a 5 or a 0, the number is divisible by 5.

6 - If the number is divisible by both 3 and 2, it is also divisible by 6.

7 - Take the last digit, double it, and subtract it from the rest of the number, if the answer is divisible by 7 (including 0), then the number is divisible by 7.

8 - If the last three digits of a number are divisible by 8, then so is the whole number.

9 - If the sum of the digits is divisible by 9, so is the number.

10 - If the number ends in 0, it is divisible by 10.

11 - If you sum every second digit and then subtract all other digits and the answer is: 0, or is divisible by 11, then the number is divisible by 11.

Example: to see whether 9,488,699 is divisible by 11, sum every second digit: $4+8+9=21$, then subtract the sum of other digits: $21-(9+8+6+9)=-11$, -11 is divisible by 11, hence 9,488,699 is divisible by 11.

12 - If the number is divisible by both 3 and 4, it is also divisible by 12.

25 - Numbers ending with 00, 25, 50, or 75 represent numbers divisible by 25.

Factorials

Factorial of a positive integer n , denoted by $n!$, is the product of all positive integers less than or equal to n . For instance $5! = 1 * 2 * 3 * 4 * 5$.

Trailing zeros:

Trailing zeros are a sequence of 0's in the decimal representation (or more generally, in any positional representation) of a number, after which no other digits follow.

125000 has 3 trailing zeros;

The number of trailing zeros in the decimal representation of $n!$, the factorial of a non-negative integer n , can be determined with this formula:

$$\frac{n}{5} + \frac{n}{5^2} + \frac{n}{5^3} + \dots + \frac{n}{5^k}, \text{ where } k \text{ must be chosen such that } 5^k < n.$$

It's easier if you look at an example:

How many zeros are in the end (after which no other digits follow) of $32!$?

$$\frac{32}{5} + \frac{32}{5^2} = 6 + 1 = 7 \text{ (denominator must be less than } 32, 5^2 = 25 \text{ is less)}$$

Hence, there are 7 zeros in the end of $32!$

The formula actually counts the number of factors 5 in $n!$, but since there are at least as many factors 2, this is equivalent to the number of factors 10, each of which gives one more trailing zero.

Finding the number of powers of a prime number p , in the $n!$.

The formula is:

$$\frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} \dots \text{till } p^x < n$$

What is the power of 2 in 25!?

$$\frac{25}{2} + \frac{25}{4} + \frac{25}{8} + \frac{25}{16} = 12 + 6 + 3 + 1 = 22$$

Finding the power of non-prime in $n!$:

How many powers of 900 are in 50!

Make the prime factorization of the number: $900 = 2^2 * 3^2 * 5^2$, then find the powers of these prime numbers in the $n!$.

Find the power of 2:

$$\frac{50}{2} + \frac{50}{4} + \frac{50}{8} + \frac{50}{16} + \frac{50}{32} = 25 + 12 + 6 + 3 + 1 = 47$$

$$= 2^{47}$$

Find the power of 3:

$$\frac{50}{3} + \frac{50}{9} + \frac{50}{27} = 16 + 5 + 1 = 22$$

$$=3^{22}$$

Find the power of 5:

$$\frac{50}{5} + \frac{50}{25} = 10 + 2 = 12$$

$$=5^{12}$$

We need all the prime {2,3,5} to be represented twice in 900, 5 can provide us with only 6 pairs, thus there is 900 in the power of 6 in 50!.

Consecutive Integers

FRACTIONS

Definition

Fractional numbers are ratios (divisions) of integers. In other words, a fraction is formed by dividing one integer by another integer. Set of Fraction is a subset of the set of Rational Numbers.

Fraction can be expressed in two forms *fractional representation* $\left(\frac{m}{n}\right)$ and *decimal representation* $(a.bcd)$.

Fractional representation

Fractional representation is a way to express numbers that fall in between integers (note that integers can also be expressed in fractional form). A fraction expresses a part-to-whole relationship in terms of a numerator (the part) and a denominator (the whole).

- The number on top of the fraction is called *numerator* or *nominator*. The number on bottom of the fraction is called *denominator*. In the fraction, $\frac{9}{7}$, 9 is the numerator and 7 is denominator.
- Fractions that have a value between 0 and 1 are called *proper fraction*. The numerator is always smaller than the denominator. $\frac{1}{3}$ is a proper fraction.
- Fractions that are greater than 1 are called *improper fraction*. Improper fraction can also be written as a mixed number. $\frac{5}{2}$ is improper fraction.
- An integer combined with a proper fraction is called *mixed number*. $4\frac{3}{5}$ is a mixed number. This can also be written as an improper fraction: $\frac{23}{5}$

Converting Improper Fractions

- Converting Improper Fractions to Mixed Fractions:
 1. Divide the numerator by the denominator

2. Write down the whole number answer

3. Then write down any remainder above the denominator

Example #1: Convert $\frac{11}{4}$ to a mixed fraction.

Solution: Divide $\frac{11}{4} = 2$ with a remainder of 3. Write down the 2 and then write down the remainder 3 above the denominator 4, like this: $2\frac{3}{4}$

- Converting Mixed Fractions to Improper Fractions:

1. Multiply the whole number part by the fraction's denominator

2. Add that to the numerator

3. Then write the result on top of the denominator

Example #2: Convert $3\frac{2}{5}$ to an improper fraction.

Solution: Multiply the whole number by the denominator:

$3 \times 5 = 15$. Add the numerator to that: $15 + 2 = 17$. Then write that down above the denominator, like this: $\frac{17}{5}$

Reciprocal

Reciprocal for a number x , denoted by $\frac{1}{x}$ or x^{-1} , is a number which when multiplied by x yields 1. The reciprocal of a fraction $\frac{a}{b}$ is $\frac{b}{a}$. To get the reciprocal of a number, divide 1 by the number. For example reciprocal of 3 is $\frac{1}{3}$, reciprocal of $\frac{5}{6}$ is $\frac{6}{5}$.

Operation on Fractions

- **Adding/Subtracting fractions:**

To add/subtract fractions with the same denominator, add the numerators and place that sum over the common denominator.

To add/subtract fractions with the different denominator, find the Least Common Denominator (LCD) of the fractions, rename the fractions to have the LCD and add/subtract the numerators of the fractions

- **Multiplying fractions:** To multiply fractions just place the product of the numerators over the product of the denominators.

- **Dividing fractions:** Change the divisor into its reciprocal and then multiply.

Example #1: $\frac{3}{7} + \frac{2}{3} = \frac{9}{21} + \frac{14}{21} = \frac{23}{21}$

Example #2: Given $\frac{3}{5}$, take the reciprocal of 2. The reciprocal is $\frac{1}{2}$. Now multiply: $\frac{3}{5} * \frac{1}{2} = \frac{3}{10}$.

Decimal Representation

The decimals has ten as its base. Decimals can be *terminating* (ending) (such as 0.78, 0.2) or *repeating* (recurring) decimals (such as 0.333333....).

Reduced fraction $\frac{a}{b}$ (meaning that fraction is already reduced to its lowest term) can be expressed as terminating decimal *if and only* b (denominator) is of the form $2^n 5^m$, where m and n are non-negative integers. For example: $\frac{7}{250}$ is a terminating decimal 0.028, as 250 (denominator) equals to $2 \cdot 5^3$. Fraction $\frac{3}{30}$ is also a terminating decimal, as $\frac{3}{30} = \frac{1}{10}$ and denominator $10 = 2 \cdot 5$.

Converting Decimals to Fractions

• To convert a terminating decimal to fraction:

1. Calculate the total numbers after decimal point
2. Remove the decimal point from the number
3. Put 1 under the denominator and annex it with "0" as many as the total in step 1
4. Reduce the fraction to its lowest terms

Example: Convert 0.56 to a fraction.

1: Total number after decimal point is 2.

2 and 3: $\frac{56}{100}$.

4: Reducing it to lowest terms: $\frac{56}{100} = \frac{14}{25}$

• To convert a recurring decimal to fraction:

1. Separate the recurring number from the decimal fraction
2. Annex denominator with "9" as many times as the length of the recurring number

3. Reduce the fraction to its lowest terms

Example #1: Convert 0.393939... to a fraction.

1: The recurring number is 39.

2: $\frac{39}{99}$, the number 39 is of length 2 so we have added two nines.

3: Reducing it to lowest terms: $\frac{39}{99} = \frac{13}{33}$.

• **To convert a mixed-recurring decimal to fraction:**

1. Write down the number consisting with non-repeating digits and repeating digits.

2. Subtract non-repeating number from above.

3. Divide 1-2 by the number with 9's and 0's: for every repeating digit write down a 9, and for every non-repeating digit write down a zero after 9's.

Example #2: Convert 0.2512(12) to a fraction.

1. The number consisting with non-repeating digits and repeating digits is 2512;

2. Subtract 25 (non-repeating number) from above: 2512-25=2487;

3. Divide 2487 by 9900 (two 9's as there are two digits in 12 and 2 zeros as there are two digits in 25): $2487/9900=829/3300$.

Rounding

Rounding is simplifying a number to a certain place value. To round the decimal drop the extra decimal places, and if the

first dropped digit is 5 or greater, round up the last digit that you keep. If the first dropped digit is 4 or smaller, round down (keep the same) the last digit that you keep.

Example:

5.3485 rounded to the nearest tenth = 5.3, since the dropped 4 is less than 5.

5.3485 rounded to the nearest hundredth = 5.35, since the dropped 8 is greater than 5.

5.3485 rounded to the nearest thousandth = 5.349, since the dropped 5 is equal to 5.

Ratios and Proportions

Given that $\frac{a}{b} = \frac{c}{d}$, where a, b, c and d are non-zero real numbers, we can deduce other proportions by simple Algebra. These results are often referred to by the names mentioned along each of the properties obtained.

$$\frac{b}{a} = \frac{d}{c} \text{ - } invertendo$$

$$\frac{a}{c} = \frac{b}{d} \text{ - } alternendo$$

$$\frac{a+b}{b} = \frac{c+d}{d} \text{ - } componendo$$

$$\frac{a-b}{b} = \frac{c-d}{d} \text{ - } dividendo$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \text{ - } componendo \& \text{ dividendo}$$

EXPONENTS

Exponents are a "shortcut" method of showing a number that was multiplied by itself several times. For instance, number a multiplied n times can be written as a^n , where a represents the base, the number that is multiplied by itself n times and n represents the exponent. The exponent indicates how many times to multiple the base, a , by itself.

Exponents one and zero:

$a^0 = 1$ Any nonzero number to the power of 0 is 1.

For example: $5^0 = 1$ and $(-3)^0 = 1$

$a^1 = a$ Any number to the power 1 is itself.

Powers of zero:

If the exponent is positive, the power of zero is zero: $0^n = 0$, where $n > 0$.

If the exponent is negative, the power of zero (0^n , where $n < 0$) is undefined, because division by zero is implied.

Powers of one:

$1^n = 1$ The integer powers of one are one.

Negative powers:

$$a^{-n} = \frac{1}{a^n}$$

Powers of minus one:

If n is an even integer, then $(-1)^n = 1$.

If n is an odd integer, then $(-1)^n = -1$.

Operations involving the same exponents:

Keep the exponent, multiply or divide the bases

$$a^n * b^n = (ab)^n$$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$(a^m)^n = a^{mn}$$

$$a^{mn} = a^{(m^n)} \text{ and not } (a^m)^n$$

Operations involving the same bases:

Keep the base, add or subtract the exponent (add for multiplication, subtract for division)

$$a^n * a^m = a^{n+m}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

Fraction as power:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Exponential Equations:

When solving equations with *even exponents*, we must consider both positive and negative possibilities for the solutions.

For instance $a^2 = 25$, the two possible solutions are 5 and -5 .

When solving equations with *odd exponents*, we'll have only one solution.

For instance for $a^3 = 8$, solution is $a = 2$ and for $a^3 = -8$, solution is $a = -2$.

Exponents and divisibility:

$a^n - b^n$ is ALWAYS divisible by $a - b$.

$a^n - b^n$ is divisible by $a + b$ if n is even.

$a^n + b^n$ is divisible by $a + b$ if n is odd, and not divisible by $a + b$ if n is even.

LAST DIGIT OF A PRODUCT

Last n digits of a product of integers are last n digits of the product of last n digits of these integers.

For instance last 2 digits of $845 * 9512 * 408 * 613$ would be the last 2 digits of $45 * 12 * 8 * 13 = 540 * 104 = 40 * 4 = 160 = 60$

Example: The last digit of $85945 * 89 * 58307 = 5 * 9 * 7 = 45 * 7 = 35 = 5$?

LAST DIGIT OF A POWER

Determining the last digit of $(xyz)^n$:

1. Last digit of $(xyz)^n$ is the same as that of z^n ;
2. Determine the cyclicity number c of z ;
3. Find the remainder r when n divided by the cyclicity;
4. When $r > 0$, then last digit of $(xyz)^n$ is the same as that of z^r and when $r = 0$, then last digit of $(xyz)^n$ is the same as that of z^c , where c is the cyclicity number.

- Integer ending with 0, 1, 5 or 6, in the integer power $k > 0$, has the same last digit as the base.

- Integers ending with 2, 3, 7 and 8 have a cyclicity of 4.

- Integers ending with 4 (eg. $(xy4)^n$) have a cyclicity of 2.

When n is odd $(xy4)^n$ will end with 4 and when n is even $(xy4)^n$ will end with 6.

- Integers ending with 9 (eg. $(xy9)^n$) have a cyclicity of 2.

When n is odd $(xy9)^n$ will end with 9 and when n is even $(xy9)^n$ will end with 1.

Example: What is the last digit of 127^{39} ?

Solution: Last digit of 127^{39} is the same as that of 7^{39} . Now we should determine the cyclicity of 7:

1. $7^1=7$ (last digit is 7)
2. $7^2=9$ (last digit is 9)
3. $7^3=3$ (last digit is 3)
4. $7^4=1$ (last digit is 1)
5. $7^5=7$ (last digit is 7 again!)

...

So, the cyclisity of 7 is 4.

Now divide 39 (power) by 4 (cyclisity), remainder is 3. So, the last digit of 127^{39} is the same as that of the last digit of 7^{39} , is the same as that of the last digit of 7^3 , which is 3.

ROOTS

Roots (or radicals) are the "opposite" operation of applying exponents. For instance $x^2=16$ and square root of $16=4$.

$$\sqrt{x}\sqrt{y} = \sqrt{xy}$$

Even Root

Odd Root

Square Root

General rule: $\sqrt{x^2} = |x|$, when $x \leq 0$ $\sqrt{x^2} = -x$ and when $x \geq 0$ $\sqrt{x^2} = x$

When the GMAT provides the square root sign for an even root, such as a square root, then the only accepted answer is the positive root.

That is, $\sqrt{16} = 4$, NOT +4 or -4. In contrast, the equation $x^2 = 16$ has TWO solutions, +4 and -4.

Even roots have only a positive value on the GMAT.

Odd roots will have the same sign as the base of the root. For example, $\sqrt[3]{125} = 5$ and $\sqrt[3]{-64} = -4$.