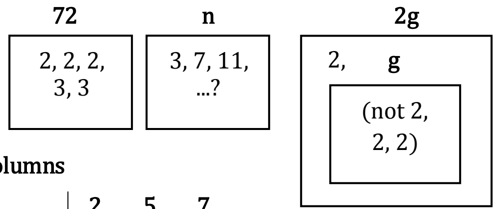
ALGEBRA

Basic concepts

- If you add or subtract multiples of N, the result is a multiple of N.
- If you add or subtract a multiple of N to a non-multiple of N, the result is a non-multiple of N.
- If you add or subtract two non-multiples of N, the result could be either a multiple of N or a non-multiple of N.
- Factor foundation rule: if A is a factor of B and B is a factor of C, then A is a factor of C.
- Every N^{th} integer is divisible by N.
- GCF of M and N cannot be larger than the difference between M and N. $GCF(M,N) * LCM(M,N) = M * N$
- Consecutives multiples of N have a GCF of N.
- All prime numbers except 2 and 5 end in 1, 3, 7 or 9.
- Prime numbers above 3 are of the form $6n \pm 1$.
- The power of the factor p of a given number n is $\frac{n}{p^1} + \frac{n}{p^2} + \cdots + \frac{n}{p^k}$ for all $p^k < n$.

Factors



Prime columns

	2	5	7	
100	2^2	5^2	-	
140	2^2	5^1	7^1	
250	2^1	5^3	-	
GCF	2^1	5^1		$= 2^1 \cdot 5^1 = 10$
LCM	2^2	5^3	7^1	$= 2^1 \cdot 5^3 \cdot 7^1 = 3{,}500$

Remainders

1. Can be added/subtracted as long as you correct excess/negative remainders
2. Can be multiplied as long as you correct excess at the end.

Remainders and decimals

$$y = divisor \cdot quotient + remainder$$

$$y = xq + r$$

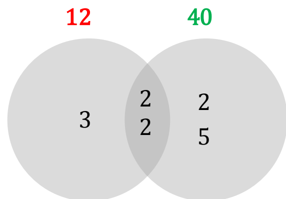
$$\frac{y}{x} = q + \frac{r}{x}$$

$$4.35 \quad \rightarrow \quad 0.35 = \frac{r}{x} = \frac{35}{100} = \frac{7}{20} \quad \rightarrow \quad 20r = 7x$$

The remainder (r) must have 7 as a factor.

In order for the decimal version of a fraction to terminate, the fraction's denominator in fully reduced form must have a prime factorization that consists of only 2's and/or 5's.

GCF / LCM



$GCF = 2 \cdot 2 = 4$
$LCM = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 120$

$4 \cdot 120 = 12 \cdot 40$

1) How many different prime factors?
$1400 = 2^3 5^2 7^1 = \mathbf{3}$
2) How many total prime factors (length)
$1400 = 2^3 5^2 7^1 = \mathbf{6}$
3) How many total factors:
$1400 = 2^3 5^2 7^1 = (3+1)(2+1)(1+1) = \mathbf{32}$

If total factors = ODD → Perfect Square
If total factors ≠ ODD → Non Square

Perfect Square

$36 = 2^2 3^2 \rightarrow (2+1)(2+1)$

9 factors

Small	Large
1	36
2	18
3	12
4	9
6	6

Non Square

$32 = 2^5 \rightarrow (5+1)$

6 factors

Small	Large
1	32
2	16
4	6

Any number whose prime factorization contains only even powers must be a perfect square:

$8{,}100 = 2^2 3^4 5^2 = (2^1 3^2 5^1)(2^1 3^2 5^1) = 90^2$

N! is a multiple of all the integers from 1 to N.

Positive integer n leaves a remainder of 4 after division by 6 and a remainder of 3 after division by 5. If n is greater than 30, what is the remainder that n leaves after division by 30?

A. 3 B. 12 C. 18 D. 22 E. 28

$n = 6q + 4 \quad \rightarrow \quad n = 4, 10, 16, 22, \mathbf{28}, 34, \ldots$
$n = 5q + 3 \quad \rightarrow \quad n = 3, 8, 13, 18, 23, \mathbf{28}, 33, \ldots$
$n = 30q + 28$

Digits

Most of the time, it is necessary to break down by units, tens, cents, etc… then simplify. Some interesting property may appear.

Two-digit number A: $\boxed{X}\boxed{Y} = 10X + Y$
Two-digit number B: $\boxed{Y}\boxed{X} = 10Y + X$

$A - B = 10X + Y - 10Y - X = 9X - 9Y = 9(X - Y)$.

In this case, reversing the digits of a two-digit number and subtracting one from another leads to the result being a multiple of 9.

However, sometimes it is necessary an out-of-the-box approach:

AB + CD = AAA, where AB and CD are two-digit numbers and AAA is a three digit number; A, B, C, and D are distinct positive integers. In the addition problem above, what is the value of C?
(A) 1
(B) 3
(C) 7
(D) 9
(E) Cannot be determined

AAA could range from 111 to 999, as A can be any positive integer from 1 to 9.
However, for two two-digit numbers to form a three-digit number, and considering that A, B, C and D are distinct, the maximum three-digit number that can be achieved is given by $97 + 86 = 183 < 222$, so AAA must be 111.
Knowing that 1B + CD = 111: CD = 111 – 12 = 99 and CD = 111 – 19 = 92. No matter what value we assign to B, CD bust be in the 90’s range to sum 111, and thus C must be 9. Answer is D.

ARITHMETIC PROGRESSIONS – EVENLY SPACED SETS

Sequence $a_1,\ a_2,\ \dots\ a_n$, so that $a_n = a_{n-1} + d$:

$$a_n = a_1 + d(n-1)$$

$$Mean = \frac{a_1 + a_n}{2}$$

$$\sum_{i=1}^{n} a_i = n \cdot Mean = n \cdot \frac{a_1 + a_n}{2}$$

$$\sum_{i=1}^{n} a_i = n \cdot \frac{2a_1 + d(n-1)}{2}$$

Particular cases:

$$1 + 2 + 3 + \cdots + n = n\frac{(1+n)}{2}$$

$$2 + 4 + 6 + \cdots + n = \sum_{i=1}^{2n} a_i = n(n+1)$$

The first 2000 even numbers range from $\mathbf{2}$ to $\mathbf{2n = 4,000}$. Its sum is $\mathbf{n(n+1) = 2,000 \cdot 2,001 = 4,002,000}$.

$$1 + 3 + 5 + \cdots + n = \sum_{i=1}^{2n-1} a_i = n^2$$

The first 2000 odd numbers range from $\mathbf{1}$ to $\mathbf{2n - 1 = 3,999}$. Its sum is $\mathbf{n^2 = 4,000,000}$.

Counting consecutive integers: $(Last\ -\ First\ +\ 1)$.

From 5 to 56, there are $56 - 5 + 1 = \mathbf{52}$ numbers.

Counting evenly spaced sets: $\frac{(Last - First)}{Increment} +\ 1.$

From 35 to 1999, there are $\frac{1999-35}{2} + 1 = 983$ odd integers.

From 35 to 1999, there are $\frac{1999-35}{5}+1=393.8 \rightarrow$ 393 multiples of 5.

For any set of consecutive integers with an ODD number of items, the sum of all the integers is ALWAYS a multiple of the number of items.

For any set of consecutive integers with an EVEN number of items, the sum of all the items is NEVER a multiple of the number of items.

MEAN & SETS

$$\boldsymbol{Mean_{old}} = \frac{\sum Items}{\#Items}$$

$$\boldsymbol{Mean_{new}} = \frac{\sum Items + New\ Item(s)}{\#\ Items + \#\ New\ Item(s)}$$

$$\boldsymbol{Change\ in\ the\ Mean} = \boldsymbol{Mean_{new}} - \boldsymbol{Mean_{old}} = \frac{New\ Item(s) - Mean_{old}}{\#\ New\ Item(s)}$$

A PhD student in English finished six books during the past week, increasing his average number of books read per week by 1. Assuming that the PhD's new average number of books read per week is 4, what is the total number of books the PhD student has read (including this past week)?

$$A_{old} = 3 = \frac{n}{w} \qquad n = 3w \rightarrow 4w + 4 = 3w + 6 \rightarrow w = 2 \rightarrow n = 2$$

$$A_{new} = 4 = \frac{n+6}{w+1} \qquad The\ total\ number\ of\ books = n + 6 = 12\ (6\ old + 6\ new)$$

Matt gets a $1,000 commission on a big sale. This commission alone raises his average commission by $150. If Matt's new average commission is $400, how many sales has Matt made?

$$A_{old} = 250 = \frac{n}{s} \qquad n = 250s \rightarrow 400s + 400 = 250s + 1000 \rightarrow s = \frac{600}{150} = 4 \rightarrow n = 1000$$

$$A_{new} = 400 = \frac{n+1000}{s+1} \qquad Total\ sales\ = 1000(old) + 1000(new) = 2000$$

STANDARD DEVIATION

$Set\ with\ n\ elements \equiv \{x_i\}$
$Median = \bar{x}$

$$Standard\ Deviation\ of\ n\ terms\ \equiv \sigma = \sqrt{variance} = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}}$$

- $\sigma \geq 0$ with $\sigma = 0$ only if all the elements in the set are equal, otherwise $\sigma > 0$.
- Multiplying or dividing all elements of the set by the same number (increasing or decreasing all of them by the same percentage) will result in the standard deviation changing in the same way.

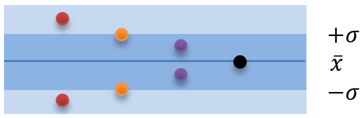
$$\frac{\{x_i\}}{a} \Rightarrow \sigma' = \frac{\sigma}{a}$$

- Adding or subtracting a number to all elements of the set will not cause the standard deviation to change.

$$\{x_i + a\} \Rightarrow \sigma' = \sigma$$

- If a new element is added to the set $\{x_i\}$ and the standard deviation of the new set $\{\{x_i\}, y\}$ is σ', then:

$\sigma' > \sigma$ if $|y - \bar{x}| > \sigma$
$\sigma' \approx \sigma$ if $|y - \bar{x}| = \sigma$
$\sigma' < \sigma$ if $|y - \bar{x}| < \sigma$
σ' is the lowest if $y = \bar{x}$



DIRECTLY / INVERSELY PROPORTIONAL

Y directly proportional to X means that Y increases as X does:

$$Y = \boldsymbol{k}\, X \qquad 6 = \mathbf{2} \cdot 3 \qquad 8 \uparrow = \mathbf{2} \cdot 4 \uparrow$$

Y inversely proportional to X means that Y increases as X decreases:

$$Y = \frac{\boldsymbol{k}}{X} \qquad 6 = \frac{\mathbf{12}}{2} \qquad 3 \downarrow = \frac{\mathbf{12}}{4\uparrow}$$

Y is directly proportional to A and indirectly proportional to B:

$$Y = \frac{A}{B}$$

SPECIAL PRODUCT & DIVISIBILITY

$$x^2 - y^2 = (x+y)(x-y)$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x-y)^2 = (x-y)(x-y) = x^2 - 2xy + y^2$$

$x^n - y^n$ *is always divisible by* $(x-y)$

$x^n - y^n$ *is always divisible by* $(x+y)$ *if n is even*

$x^n + y^n$ *is always divisible by* $(x+y)$ *if n is odd, and not divisible by* $(x+y)$ *if n is even*

EXPONENTS

$$x^a x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad x^{-a} = (x^a)^{-1} = \frac{1}{x^a}$$

$$a^x b^x = (ab)^x \qquad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x \qquad x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

$$\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}} \qquad \sqrt[n]{x}\sqrt[n]{y} = \sqrt[n]{xy} \qquad \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a = x^{\frac{a}{b}}$$

DST PROBLEMS

$Distance = Speed * Time$

Same direction: subtract speeds.
Opposite direction: sum speeds.

POPULATION PROBLEMS

$Final\ Population\ =\ S * P^{t/I}$

$S = Starting\ Population$
$P = Progression\ (doubles = 2, triples = 3, etc.)$
$t/I = total\ amount\ of\ iterations$
$t = Time$
$I = Intervals$

INTERVAL ARITHMETICS

$[a,b]+[c,d]=[a+c,b+d]$
$[a,b]-[c,d]=[a-d,b-c]$
$[a,b]\times[c,d]=[\min(ac,ad,bc,bd)\,,\max(ac,ad,bc,bd)]$
$[a,b]/[c,d]=[min(a/c,a/d,b/c,b/d), max\,(a/c,a/d,b/c,b/d)]$

INEQUALITIES

Inequalities can only be added if the signs are in the same direction, and can only be subtracted if the signs are in the opposite direction (the sign to keep is the one in the inequality you subtract from).

$$\begin{array}{c} 2<3 \\ \underline{3<4} \\ 2+3<3+4 \end{array} \qquad \begin{array}{c} 3>2 \\ \underline{2<3} \\ 2-3<3-2 \\ -1<1 \end{array}$$

Absolute values can't be moved in inequalities unless sign is defined (by trying both positive and negative inequality).

ABSOLUTE VALUES

For any real number x, the absolute value (or modulus) of x is denoted by$|x|$ and defined as:

$$|x|=\begin{cases} x & if\ x\geq 0 \\ -x & if\ x<0 \end{cases}$$

$|x|=0$

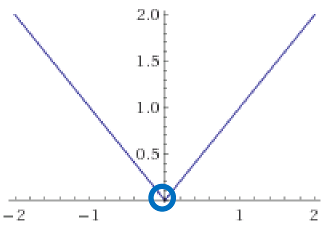
Break-point: $x=0$

Interval: $(-\infty,0),[0,\infty)$

$$|x|=\begin{cases} x & if\ x\geq 0 \\ -x & if\ x<0 \end{cases}$$

Interval $(-\infty,0)$:
$x=0\notin(-\infty,0)$ **NOK!**

Interval $[0,\infty)$:
$x=0\in[0,\infty)$ **OK!**

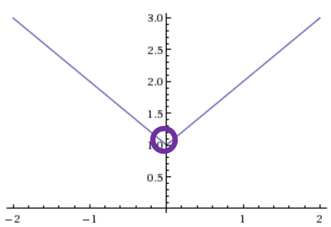


There is one solution.

$|x|+1=0$

then $|x|=-1$

It can clearly be seen that there is no solution, as no absolute value can be negative. However:



Break-point: $x=0$

$$|x|=\begin{cases} x\ if\ x\geq 0 \\ -x\ if\ x<0 \end{cases}$$

Interval $(-\infty,0)$:
$-x=-1$
$x=1\notin(-\infty,0)$ **NOK!**

Interval $[0,\infty)$:
$x=-1\notin[0,\infty)$ **NOK!**

No conditions are met, so there are no solutions.

$\mathbf{|x+3| = 1}$

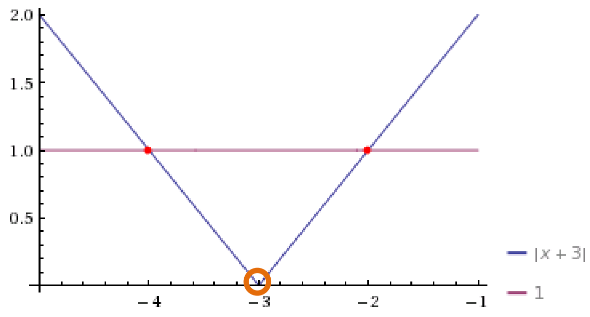
Break-point:
$x + 3 = 0 \rightarrow$ $\mathbf{x = -3}$

Interval: $(-\infty, -3), [-3, \infty)$

$$|x+3| = \begin{cases} x+3 \text{ } if \text{ } x \geq -3 \\ -(x+3) \text{ } if \text{ } x < -3 \end{cases}$$

<u>Interval $(-\infty, -3)$:</u>
$-x - 3 = 1$
$x = -4 \in (-\infty, -3)$ **OK!**

<u>Interval $[-3, \infty)$:</u>
$x + 3 = 1$
$x = -2 \in [-3, \infty)$ **OK!**



Both conditions are met, so both solutions are valid.

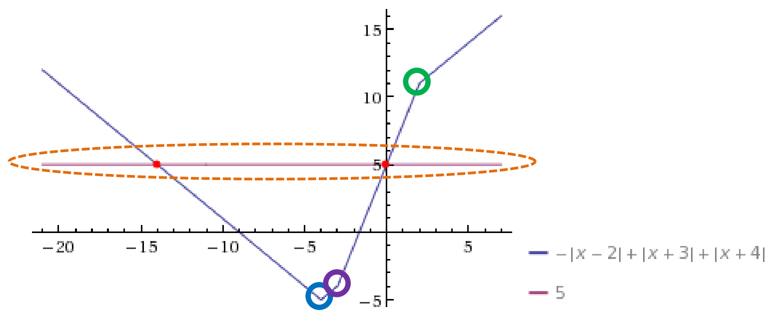
$|x+4| + |x+3| - |x-2| = \mathbf{5}$

Break-points:

$x + 4 = 0 \rightarrow$ $\mathbf{x = -4}$
$x + 3 = 0 \rightarrow$ $\mathbf{x = -3}$
$x - 2 = 0 \rightarrow$ $\mathbf{x = 2}$

Intervals:

$(-\infty, -4), (-4, -3), (-3, 2), (2, \infty)$



$$|x+4| = \begin{cases} \mathbf{x+4 \text{ } if \text{ } x \geq -4} \\ \mathbf{-(x+4) \text{ } if \text{ } x < -4} \end{cases}$$

$$|x+3| = \begin{cases} \mathbf{x+3 \text{ } if \text{ } x \geq -3} \\ \mathbf{-(x+3) \text{ } if \text{ } x < -3} \end{cases}$$

$$|x-2| = \begin{cases} \mathbf{x-2 \text{ } if \text{ } x \geq 2} \\ \mathbf{-(x-2) \text{ } if \text{ } x < 2} \end{cases}$$

<u>Interval $(-\infty, -4)$:</u>

$-(x+4) - (x+3) + (x-2) = 5$
$-x - 4 - x - 3 + x - 2 = 5$
$x = -14 \in (-\infty, -4)$ **OK!**

<u>Interval $(-4, -3)$:</u>

$(x+4) - (x+3) + (x-2) = 5$
$x + 4 - x - 3 + x - 2 = 5$
$x = 6 \notin (-4, -3)$ **NOK!**

<u>Interval $(-3, 2)$</u>

$(x+4) + (x+3) + (x-2) = 5$
$x + 4 + x + 3 + x - 2 = 5$
$x = 0 \in (-3, 2)$ **OK!**

<u>Interval $(2, \infty)$:</u>

$(x+4) + (x+3) - (x-2) = 5$
$x + 4 + x + 3 - x + 2 = 5$
$x = -4 \notin (2, \infty)$ **NOK!**

Only two solutions are valid.

COMBINATORICS

<u>By definition:</u>
0! = 1
n = total amount of objects
p = amount of objects taken at a time to permute

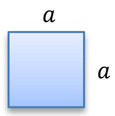
	ORDER MATTERS		ORDER DOESN'T MATTER
	ARRANGE **PERMUTATIONS**	ARRANGE & PICK (NOT USING ALL) **VARIATIONS**	PICK (NOT USING ALL) **COMBINATIONS**
WITHOUT REPETITION	$\boldsymbol{P_n = P_n^n = n!}$ How many ways are there to arrange 3 letters A, B, C? $P_3 = 3! = 6$ $\begin{bmatrix} ABC & BCA & CAB \\ ACB & BAC & CBA \end{bmatrix}$	$\boldsymbol{V_p^n = P_p^n = \frac{n!}{(n-p)!}}$ How many words of 2 different letters can you make with 4 letters A, B, C, D? $V_2^4 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$ $\begin{bmatrix} - & AB & AC & AD \\ BA & - & BC & BD \\ CA & CB & - & CD \\ DA & DB & DC & - \end{bmatrix}$	$\boldsymbol{C_p^n = \frac{n!}{p!\,(n-p)!} = \binom{n}{p}}$ How many ways are there to pick 2 different letters out of 4 letters A, B, C, D? $C_2^4 = \frac{4!}{2!\,(4-2)!} = \frac{4!}{2!\,2!} = 6$ $\begin{bmatrix} - & AB & AC & AD \\ - & - & BC & BD \\ - & - & - & CD \\ - & - & - & - \end{bmatrix}$
WITH REPETITION	$\boldsymbol{P_{n_1,\dots,n_k} = \frac{(\Sigma\, n_i)!}{\Pi(n_i!)}}$ How many ways are there to arrange 2 letters A and 2 letters B? $P_{2,2} = \frac{(2+2)!}{2! \cdot 2!} = 6$ $\begin{bmatrix} AABB & ABAB & ABBA \\ BAAB & BABA & BBAA \end{bmatrix}$	$\boldsymbol{\bar{V}_p^n = n^p}$ How many words of 2 letters can you make with 4 letters A, B, C, D? $\bar{V}_2^4 = 2^4 = 16$ $\begin{bmatrix} AA & AB & AC & AD \\ BA & BB & BC & BD \\ CA & CB & CC & CD \\ DA & DB & DC & DD \end{bmatrix}$	$\boldsymbol{\bar{C}_p^n = C_p^{n+p-1} = \frac{(n+p-1)!}{p!\,(n-1)!}}$ How many ways are there to pick 2 letters out of 4 letters A, B, C, D? $\bar{C}_2^4 = C_3^5 = \frac{(4+2-1)!}{2!\,(4-1)!} = \frac{5!}{2!\,3!} = 10$ $\begin{bmatrix} AA & AB & AC & AD \\ - & BB & BC & BD \\ - & - & CC & CD \\ - & - & - & DD \end{bmatrix}$

GEOMETRY

Areas

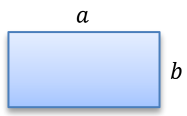
Square

$A = a^2$



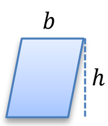
Rectangle

$A = ab$



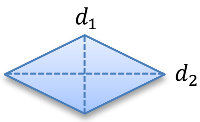
Parallelogram

$A = bh$



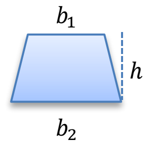
Rhombus

$A = bh$



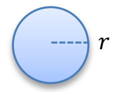
Trapezoid

$A = \frac{h}{2}(b_1 + b_2)$



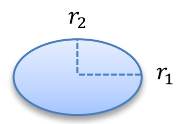
Circle

$A = \pi r^2$



Ellipse

$A = \pi r_1 r_2$

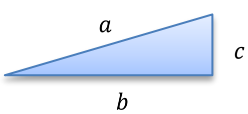


Max Area / Perimeter: of all the quadrilaterals, the square has the largest area and the minimum perimeter. For triangles, place two sides perpendicular to maximize the area.

Triangles & Polygons

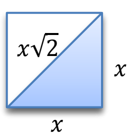
- Sum of the three angles of a triangle = 180.
- Sum of interior angles of a polygon $= (n-2)180$ ... OR ... divide the polygon into triangles.
- Angles correspond to their opposite sides. If twos sides are equal, their angles are also equal.
- The sum of any two sides is greater than the third side.
- Similar triangles with side lengths in ratio $a{:}\,b$ will have their areas in ratio $a^2{:}\,b^2$.

General rule: $a^2 = b^2 + c^2$



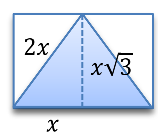
Isosceles triangle $(90-45-45)$

$a = x\sqrt{2}$
$b = x$
$c = x$



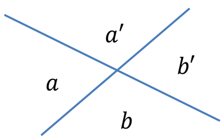
Equilateral triangle $(60-60-60)$

$a = 2x$
$b = x\sqrt{3}$
$c = x$

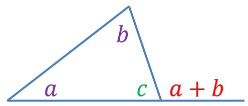


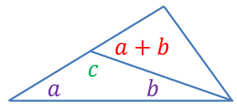
The above formulas apply only to rectangle triangles. If a triangle is not rectangle, divide it in rectangle triangles to solve.

Lines and angles



(a, b) and (a', b') are supplementary angles because they add up to 180°





Given any two angles of a triangle, the third angle's supplementary angle is equal to the sum of the first two angles.

Coordinate plane – Lines and points

$Line\ formula$

$$General\ form: \boldsymbol{Ax + By + C = 0}$$

$$\boldsymbol{By = -Ax - C}$$

$$\boldsymbol{y = -\frac{A}{B}x - \frac{C}{B}}$$

$$Slope - intercept\ form: \boldsymbol{y = mx + b}$$

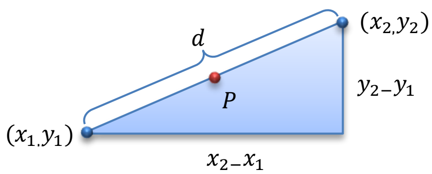
m: slope
b: y-intercept

Parallel lines (||) have the same slope: $-\frac{A_1}{B_1} = -\frac{A_2}{B_2}$

Perpendicular lines (⊥) have negative reciprocal slope $-\frac{A_1}{B_1} = \frac{B_2}{A_2} \rightarrow A_1A_2 - B_1B_2 = 0$

$Distance\ between\ two\ points\ (x_1, y_1)\ \&\ (x_2, y_2)$: $\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$

$Midpoint\ between\ two\ points\ (x_1, y_1)\ \&\ (x_2, y_2)$: $\boldsymbol{P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)}$



$Distance\ between\ a\ point\ (x_0, y_0)\ and\ a\ line$: $\boldsymbol{D = \frac{|Ax_0 + Bx_0 + C|}{\sqrt{A^2+B^2}}}$

$Slope\ of\ a\ line = \frac{\Delta y}{\Delta x} = \frac{\boldsymbol{y_1 - y_2}}{\boldsymbol{x_1 - x_2}}$

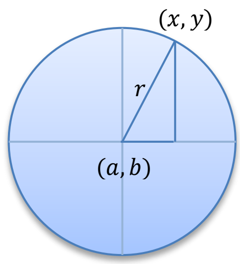
Find the equation of a line using two points of that line:

1. Find slope = m
2. Plug one of the points (x, y) in the equation to find b.

Find two points of a line using the equation of that line [$\boldsymbol{slope \neq 0\ (perpendicular),\ undefined\ (vertical)}$]:

1. Plug $x = 0$ to find the x-intercept point, $(0, y)$
2. Plug $y = 0$ to find the y-intercept point, $(x, 0)$

Coordinate Plane – Circles



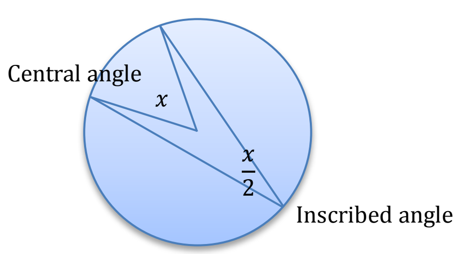
Equation of a circle (applies to any point in the perimeter):

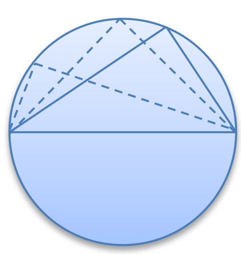
$(x - a)^2 + (y - b)^2 = r^2$

If the circle is centered at the origin $(a, b) = (0,0)$:

$x^2 + y^2 = r^2$

Other considerations





If one of the sides of an inscribed triangle is a Diameter of the circle, then the triangle must be rectangle.

Cylinder's area: top circle + bottom circle + rectangle.

Coordinate Plane – Parabolas

$y = ax^2 + bx + c$

y–intercept: c $(x = 0)$

x–intercept: solve the equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\begin{aligned} \sqrt{b^2 - 4ac} > 0 &\rightarrow \text{2 intercepts on x} \\ = 0 &\rightarrow \text{1 intercept on x} \\ = 0 &\rightarrow \text{no intercepts on x} \end{aligned}$$

Vertex: $\left(-\frac{b}{2a}, y\right)$ → find x then substitute and find y

$a \uparrow \Rightarrow width \downarrow$ 

$a > 0$ 

$a > 0$