# INEQUATIONS (INEQUALITIES): Prologue

During GMAT exam, where time is precious, complicated inequalities often create a difficult situation. Purpose of this article is to narrate the paradigms of inequalities and to state the methods to solve the inequalities.

This Article is actually a digital version of my handwritten notes collected from various sources. Most of the input is from Indian School Text Books on Maths(CBSE Board) and remaining is from GMAT Math Strategy Guides of Elite Test Prep Companies and Guides for Indian CAT.

# Inequations: Definition and Characterictics

The term **Equation** is defined as a statement involving variable(s) and the sign of equality(=)

Similarly, we can define the term **Inequation** as a statement involving variable(s) and the sign of inequality viz, >, ≥, <, or ≤

An Inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

Some Examples of Inequations

$5x - 3 > 0$

$x^2 - 5x + 4 \leq 0$

$ax^2 + by^2 < 1$

$x^3 + 6x^2 + 11x + 6 \leq 0$

## *Linear Inequations with one variable*

Let *a* be a non-zero real number and *x* be a variable.

Then Inequations of the form $ax + b < 0$, $ax + b \leq 0$, $ax + b > 0$, $ax + b \geq 0$ are known as linear Inequations in one variable

Examples : $9x - 15 > 0$, $2x - 3 \leq 0$

## *Linear Inequations with two variables*

Let *a, b* be non-zero real numbers and *x, y* be a variables.

Then Inequations of the form $ax + by < c$, $ax + by \leq c$, $ax + by > c$, $ax + by \geq c$ are known as linear Inequations in two variables x and y

Examples : $2x + 3y \leq 6$, $2x + y \geq 6$

## *Quadratic Inequation*

Let *a* be a non-zero real number.

Then Inequations of the form $ax^2 + bx + c < 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$, $ax^2 + bx + c \geq 0$ are known as Quadratic Inequations with one variable.

Examples : $x^2 + x - 6 < 0$, $2x^2 + 3x + 1 > 0$

The equation $ax^2 + by^2 < 1$ is a sample quadric Inequation with two variables.

# Solution of an Inequation

**Definition :- A Solution of an Inequation is the value(s) or the range of values of the variable(s) that makes the inequation a true statement**

Consider the Inequation $\frac{3-2x}{5} < \frac{x}{3} - 4$

For x = 9, we have, LHS = $\frac{3-2*9}{5}$ = -3 and, RHS = $\frac{9}{3}$ - 4 = -1

Clearly -3 < -1 that means LHS < RHS, which is true

So x = 9 is a solution of the given inequation

For x = 6, LHS = $\frac{-9}{5}$ and, RHS = -2 ---------> LHS > RHS ---------> So x = 6 is not a solution of the given Inequation.

We can verify that any real number greater than 7 is a solution of the given Inequation.

Let us now consider the Inequation $x^2 + 1 < 0$

We know that

$x^2 \geq 0$ for all real values of x

$x^2 + 1 \geq 1$ for all real values of x--------------(Added 1 to both sides of ineqation)

So, there is no real value of x which makes the Inequation $x^2 + 1 < 0$ a true statement. Hence, it has no solution.

## *Solving an Inequation*

It is the process of obtaining all possible solutions of an Inequation. *Substitution by numbers is a lengthy process, and it is an impossible process for complicated quadratic equations*. Best way to solve Inequations is shown in the following section.

## *Solution Set*

The set of all possible solutions of an Inequation is known as its solution set.

For Example

The solution set of the Inequation $x^2 + 1 \geq 0$ is the set of all real numbers

The solution set of the Inequation $x^2 + 1 < 0$ is empty/null set

# Solving Linear Inequation with one variable

As mentioned earlier that solving an Inequation is the process of obtaining all possible solutions of the variable.

In the process of solving an Inequation, **we use mathematical simplifications which are governed by the following rules**

Rule 1 :- Same Number may be added to (or subtracted from) both sides of an Inequation without changing the sign of inequality. Example: $x > 5 \rightarrow x + 2 > 5 + 2$

Rule 2 :- Both sides of an Inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an Inequation are multiplied or divided by a negative number. Examples: $x > 5 \rightarrow 3x > 15$; $x > 1 \rightarrow -x < -1$

Rule 3 :- Any term of an Inequation may be taken to the other side with its sign changed without affecting the sign of inequality. Example: $\frac{x}{3} - 2 < \frac{x}{6} \rightarrow \frac{x}{3} - \frac{x}{6} < 2$

Rule 4 :- In inequalities never multiply for a term unless you know it does not equal 0 AND you know its sign.

**Type I :- Algorithm to solve the Inequations of the form $ax + b < 0$, $ax + b \le 0$, $ax + b > 0$, $ax + b \ge 0$**

Step I :- Obtain the linear Inequation.

Step II :- Collect all terms involving the variable on one side of the Inequation and the constant terms on other side

Step III :- Simplify both sides of inequality in their simplest forms to reduce the Inequation in the form $ax < b$, or $ax \le b$, or $ax > b$, or $ax \ge b$

Step IV :- Solve the Inequation obtained in step III by dividing both sides of the Inequation by the coefficient of the variable (of x)

Step V :- Write the solution set obtained in step IV in the form of an interval on the real line.

Example 1 :- Solve the Inequation $2x - 4 \leq 0$

Step 1) $2x - 4 \leq 0$

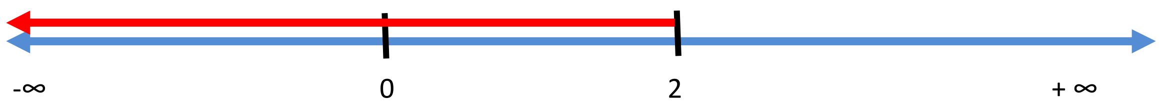
Step 2) $(2x - 4) + 4 \leq 0 + 4$ -------------------------[ Adding 4 on both sides ]

Step 3) $2x \leq 4$

Step 4) $\frac{2x}{2} \leq \frac{4}{2}$

Step 5) $x \leq 2$

Any real number less than or equal to 2 is a solution of the given Inequation.




Hence the solution set of the given Inequation is $(-\infty, 2]$

**NOTE :-** The Interval [x, y] stands for the values between x and y, both inclusive, where as The interval (x, y) stands for the values between x and y, both exclusive

Example 2 :- $-3x + 12 < 0$

Step 1) $-3x + 12 < 0$

Step 2) $-3x < -12$ -------------------------[ Transposing 12 on right side ]

Step 3) $\frac{-3x}{-3} > \frac{-12}{-3}$ -------------------------[ Dividing both sides by -3, and hence the sign of inequality changes ]

Step 4) $x > 4$

Any real number greater than 4 is a solution of the given Inequation.


Hence the solution set of the given Inequation is $(4,\infty)$

Example 3 :- $\frac{1}{x-2} < 0$

$\frac{1}{x-2} < 0$ ---------> x-2 < 0 -----------------[ If $\frac{a}{b} < 0$ and a > 0 then b < 0]

x < 2 ----------> Solution set is ( -∞, 2)

## Type II :- Algorithm to solve the Inequations of the form $\frac{ax+b}{cx+d} < k$, $\frac{ax+b}{cx+d} \leq k$, $\frac{ax+b}{cx+d} > k$, $\frac{ax+b}{cx+d} \geq k$

Step I :- Obtain the Inequation.

Step II :- Transpose all terms on LHS

Step III :- Simplify LHS of the equation obtained in step II to obtain an Inequation of the form

$\frac{px+q}{rx+s} < 0$, $\frac{px+q}{rx+s} \leq 0$, $\frac{px+q}{rx+s} > 0$, $\frac{px+q}{rx+s} \geq 0$

Step IV :- Make Coefficient of *x* positive in numerator and denominator if they are not.

Step V :- Equate Numerator and denominator separately to zero and obtain the values of *x*. These values of *x* are generally called critical points.

Step VI :- Plot the critical points obtained in step V on real line. These points will divide the real line In three regions.

Step VII:- In the rightmost region the expression on LHS of the Inequation obtained in step IV will be Positive and in other regions it will be alternatively negative and positive. So, mark positive Sign in the right most region and then mark alternatively negative and positive signs in Other regions.

Step VIII:- Select Appropriate region on the basis of the sign of the information obtained in step IV. For +ve inequality signs (>, ≥), the range will be the region with +ve signs, i.e., the rightmost segment and leftmost segment. For -ve inequality signs (<, ≤), the range will be the region with -ve signs, i.e., the range between critical points.

Write these regions in the form of intervals to obtain the desired solution sets of the given Inequation.

Example 1 :- $\frac{x-2}{x+5} > 2$

Step I :- $\frac{x-2}{x+5} > 2$

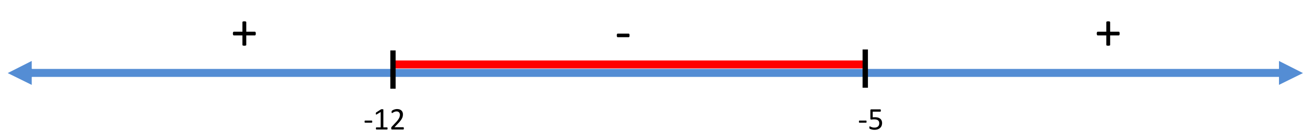
Step II :- $\frac{x-2}{x+5} - 2 > 0$

Step III :- $\frac{-x-12}{x+5} > 0$

Step IV :- $\frac{x+12}{x+5} < 0$ ---------------[Multiplying by -1 to make coefficient of *x* positive in the expression In numerator]

Step V :- x+12 < 0 -----> x < -12 and x+5 < 0 --------> x < -5 Therefore x = -12, -5 are the critical points.

Step VI and Step VII :-




Step VIII:- The Real line is divided in to three regions and the signs of LHS of Inequation (step IV) marked. Since the Inequation in (step IV) possesses less than sign which means that LHS of the Inequation is negative. So, the solution set of the given Inequation is the union of the regions Containing negative sign.

Hence the solution set of given Inequation is -12 < x < -5

Example 2 :- $\frac{x-3}{x-5} > 0$

$\frac{x-3}{x-5} > 0$

x-3 > 0 ------> x > 3 --------------------[ Equating x-3 to zero]

x-5 > 0 ------> x > 5 --------------------[ Equating x-5 to zero]

Upon equating x-3 and x-5 separately to zero, we obtain x = 3, 5 as critical points.

Plot these points on Real Line


The Real line is divided into three regions. Since the Inequation possesses greater than sign which means that LHS of the Inequation is positive. So, the solution set of the given Inequation is the union of the regions Containing positive sign.

Hence x > 5 or x < 3 -----------> (-∞, 3) U (5, ∞)

Example 3 :- $\frac{2x+4}{x-1} \geq 5$

1. $\frac{2x+4}{x-1} \geq 5$

2. $\frac{2x+4}{x-1} - 5 \geq 0$

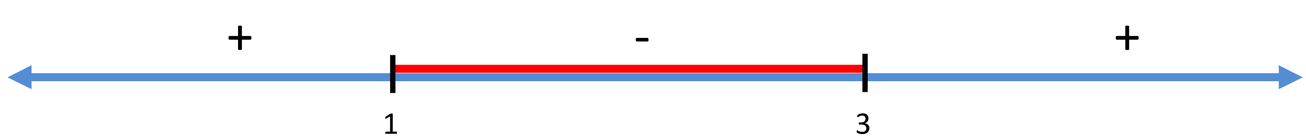
3. $\frac{(2x+4)-5(x-1)}{x-1} \geq 0$ → $\frac{-3x+9}{x-1} \geq 0$

4. $\frac{3x-9}{x-1} \leq 0$ ----------------[ Multiplying both sides by -1]

   $\frac{3(x-3)}{x-1} \leq 0$

5. $\frac{x-3}{x-1} \leq 0$ -----------------[ dividing both sides on Inequation by 3]

6. x = 3, 1



Sign of original Inequation obtained in Step 4 is negative so the solution set will be union of the regions

Containing negative sign.

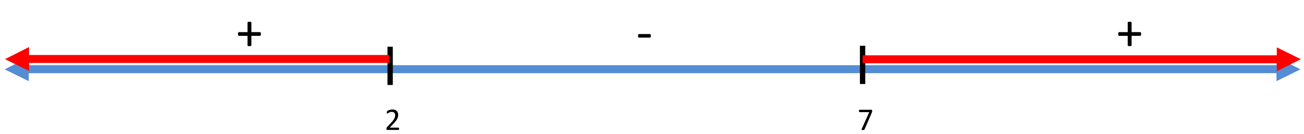
Therefore Solution Set is $1 < x \leq 3$

<u>NOTE :-</u> When x=1, the Inequation becomes $6/0 \geq 5$ and in the GMAT division by zero is not defined hence we need to exclude x=1 from solution set.

Example 4 :- $\frac{x+3}{x-2} \leq 2$

1. $\frac{x+3}{x-2} \leq 2$
2. $\frac{x+3}{x-2} - 2 \leq 0$
3. $\frac{x+3-2x+4}{x-2} \leq 0 \rightarrow \frac{-x+7}{x-2} \leq 0$
4. $\frac{x-7}{x-2} \geq 0$ -------------------[Multiplying both sides by -1]
5. $x-7 \geq 0$ ---------> $x \geq 7$
   $x-2 \geq 0$ ---------> $x \geq 2$

x = 7 and 2 are the critical points.




The Real line is divided into three regions. Since the Inequation (obtained in step 4) possesses greater than sign which means that LHS of the Inequation is positive. So, the solution set of the given Inequation is the union of the regions containing positive sign.

Hence $x \geq 7$ or $x \leq 2$ -----------> $(-\infty, 2] \cup [7, \infty)$

# Courtesy for the Information

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## Special Thanks to

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**Regards,**

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