Modulus or Absolute Values |x|

Many times students scared seeing the modules sign in an inequality question. Indeed, dealing with inequality question that involves expressions in modules can be tricky. However once you studied the theory and mastered the results pertaining to modules, any such question will be appeared like an ordinary Inequality question.

Some Important Results

In all there are 6 results out of which 4 are basic and remaining 2 are derived from the first two basic results.

4 basic inequality with absolute values to remember:

(A) $|x| \leq a \Rightarrow -a \leq x \leq a$
(B) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$
(C) $|x| < 0 \Rightarrow$ No solution (Remember :- $|x| \leq 0$ can have a solution for $x=0$)
(D) $|x| \geq 0 \Rightarrow -\infty < x < \infty$ (*all real numbers*)

Result 1 :- If *a* is a positive real number, then $|x| < a \rightarrow -a < x < a$

Let's Prove this result

We have $|x| < a$

We know that $|x| = (x \text{ if } x \geq 0)$ and $(-x \text{ if } x < 0)$

So, we consider the following cases :

CASE I :- When $x \geq 0$

In this case $|x| = x$ -----------> Therefore $x < a$

Thus in this case the solution set of the given Inequation is given by

$X \geq 0$ and $x < a$ -----------> $0 \leq x < a$

CASE I :- When $x < 0$

In this case $|x| = -x$ -----------> Therefore $-x < a$ --------> $x > -a$

Thus in this case the solution set of the given Inequation is given by

$X < 0$ and $x > -a$ -----------> $-a < x < 0$

Combining two cases together, we get $(-a < x < 0)$ or $(0 \leq x < a)$

Once we take extreme points, we get $-a < x < a$

Result 2 :- If *a* is a positive real number, then $|x| > a \rightarrow x > a$ or $x < -a$

We have $|x| > 0$

CASE I :-   When $x \geq 0$

In this case $|x| = x$ -----------> Therefore $x > a$

Thus in this case the solution set of the given Inequation is given by

$x \geq 0$ and $x > a$ -----------> $x > a$

CASE I :-   When $x < 0$

In this case $|x| = -x$ --------------> Therefore $-x > a$ -----------> $x < -a$

Thus in this case the solution set of the given Inequation is given by

$x < 0$ and $x < -a$ ---------------> $x < -a$

Combining two cases, we get $x > a$ or $x < -a$

Result 3 :- For $|x| < 0 \rightarrow$ No Solution,

Result 4 :- For $|x| \geq 0 \rightarrow -\infty < x < \infty$ (all real numbers)

Here are another two results which are derived from above basic results

Result 5 :- Let *r* be positive real and *a* be a fixed number, then $|x - a| < r \rightarrow (a - r) < x < (a + r)$

We know that (From Result 1) $|x| < a = -a < x < a$

We have $|x - a| < r$ ------------> $-r < x-a < r$ ----------> $a - r < x < a + r$

Result 6 :- Let *r* be positive real and *a* be a fixed number, then $|x - a| > r \rightarrow x < a - r$ or $x > a + r$

We know that (From Result 2) $|x| > a = x > a$ or $x < -a$

We have $|x - a| > r$ ------------> $x - a > r$ -------> $x > a + r$ ----------or ---------> $x - a < -r$ -------> $x < a - r$

Example 1 :- $|3x-2| \le \frac{1}{2}$

We know that $|x-a| \le r = a - r \le x \le a + r$

$|3x-2| \le \frac{1}{2}$ → $2 - \frac{1}{2} \le 3x \le 2 + \frac{1}{2}$ → $\frac{3}{2} \le 3x \le \frac{5}{2}$ → $\frac{1}{2} \le x \le \frac{5}{6}$ → $[\frac{1}{2}, \frac{5}{6}]$

Example 2 :- $|x - 2| \ge 5$

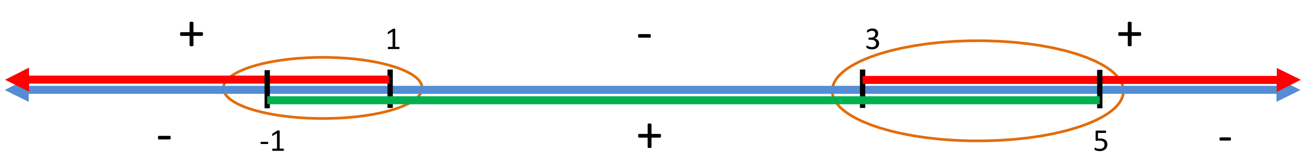
$|x - a| \ge r = x \ge a + r$ or $x \le a - r$

$|x - 2| \ge 5$ → $x \ge 2 + 5$ or $x \le 2 - 5$ → $x \ge 7$ or $x \le -3$ → $(-\infty, -3]$ or $[7, \infty)$

Example 3 :- $1 \le |x - 2| \le 3$

$1 \le |x - 2|$ → $x \ge 3$ or $x \le 1$

$|x - 2| \le 3$ → $-1 \le x \le 5$



Solution of inequality $-1 \le x \le 1$ and $3 \le x \le 5$ → $[-1, 1] \cup [3, 5]$

Example 4 :- $|\frac{2}{x-4}| > 1$ and x ≠ 4

→ $\frac{2}{|x-4|} > 1$ --[$|\frac{a}{b}| = \frac{|a|}{|b|}$]

→ 2 > |x-4|----------------- (We can multiply both sides of inequality by |x-4|, since we know |x-4|>0 always for all x ≠ 4)

→ -2 + 4 < x < 4 + 2 → 2 < x < 6

We know x ≠ 4. Hence the solution set of inequality will be (2<x<4) U (4<x<6)

Example 5 :- If $\frac{|x|}{|3|} > 1$, which of the following must be true?

A. x > 3
B. x < 3
C. x = 3
D. x ≠ 3
E. x < -3

→ |x| > |3| ----------------------------[Since |3| will always positive, we can multiply both sides by |3|]

→ |x| - |3| > 0 → |x|- 3 > 0 → |x| > 3

→ x > 3 or x < -3

A) x > 3 Incorrect. This could be true. (x can be greater than 3 or less than -3)
B) X < 3 Incorrect. This could be true. (x can be greater than 3 or less than -3)
C) X = 3 Incorrect. This can never be true. (x can be greater than 3 or less than -3)
D) X ≠ 3 Correct. This must be true
E) X <-3 Incorrect. This could be true. (x can be greater than 3 or less than -3)

EXAMPLE 6 :- $|1 – x| > 3$

$|x – 1| > 3$ ----------------------------[$|1 – x| = |x – 1|$]

$x > 4$ or $x < -2$

Example 7 :- http://gmatclub.com/forum/x-x-x-which-of-the-following-must-be-true-about-x-13943.html#p772618

if $\frac{x}{|x|} < x$, Which of the following must be true about *x*

→ $\frac{x}{|x|} < x$ → Since $|x|$ has to be positive, we can multiply both sides of an inequality by $|x|$

→ $x < x|x|$ → $x|x| - x > 0$ → $x(|x| - 1) > 0$

<u>Case I</u> :- x and ($|x|$-1) are positive

In this case $x > 0$ and $|x| - 1 > 0$ → $x > 1$ or $x < -1$ -----------{ Applying $|x| - a > r$ → $x > a + r$ or $x < a – r$}

Therefore Range of x is $x > 1$

Case II :- x and ($|x|$-1) are negative

In this case $x < 0$ and $|x| - 1 < 0$ → $-1 < x < 1$ ------------{ Applying $|x| - a < r$ → $a – r < x < a + r$}

Therefore range of x is $-1 < x < 0$

So we have that $-1 < x < 0$ or $x > 1$

A) $x > 1$ Incorrect
B) $x > -1$ Correct
C) $|x| < 1$ Incorrect
D) $|x| = 1$ Incorrect
E) $|x|^2 > 1$ Incorrect

Solving Quadratic Inequation

Here is the way to tackle Quadratic Inequalities questions.

Example 1 :- $3x^2 – 7x + 4 \le 0$

$3x^2 – 7x + 4 \le 0$ → 3x2 – 3x – 4x + 4 ≤ 0 → 3x(x-1) - 4(x-1) ≤ 0 → (3x-4)(x-1) ≤ 0

we get 1 and 4/3 as critical points. We will place them on number line.

\+ 1 - 4/3 +

Since the number line is divided into three regions, now we can get 3 ranges of x i) x < 1 ii) 1 ≤ x ≤ 4/3 iii) x > 4/3

At this point we should understand that for the inequality (3x-4)(x-1) ≤ 0 to hold true, exactly one of (3x-4)and (x-1) should be negative and other one be positive.

Let's examine 3 possible ranges one by one.

i) If x > 4/3, obviously both the factors i.e. (3x-4)and (x-1) will be positive and in that case inequality would not hold true. So this cannot be the range of x
ii) If x is between 1 and 4/3 both inclusive, (3x-4)will be negative or equal to zero and (x-1) will be positive or equal to zero. Hence with this range inequality holds true. Correct.
iii) If x < 1, both (3x-4)and (x-1) will be negative hence inequality will not hold true.

So the range of x that satisfies the inequality $3x^2 – 7x + 4 \le 0$ is (1 ≤ x ≤ 4/3)

Let's look at another question

Example 2 :- $x^2 – 14x – 15 > 0$

$x^2 – 14x – 15 > 0$ → $x^2 – 15x + x – 15 > 0$ → $x^2 + x - 15x – 15 > 0$ → (x+1)(x – 15) > 0

We have -1 and 15 as critical points. For the inequality above be true we would need both the factors either positive or negative

\+ -1 - 15 +

We can see here that whenever x takes the value greater than 15 or less than -1 both the factors becomes positive or negative respectively thereby satisfying the inequality. However whenever x takes the value between -1 and 15 one factor becomes positive and other one becomes negative. In that case inequality does not hold true. Hence the solution of inequality is x > 15 or x < -1

Algorithm to solve Quadratic Inequations $ax^2 + bx + c$ > or < 0

1) Obtain the Inequation
2) Obtain the factors of Inequation
3) Place them on number line. The number line will get divided into the three regions
4) Mark the leftmost region with + sign and rest two regions with – and + signs respectively.
5) If the Inequation is of the form $ax^2 + bx + c < 0$, the region having minus sign will be the solution of inequality
6) If the Inequation is of the form $ax^2 + bx + c > 0$, the region having plus sign will be the solutions of inequality

Most useful methods for solving quadratic inequalities –Proposed by legendary Members

BUNUEL :- http://gmatclub.com/forum/if-x-is-an-integer-what-is-the-value-of-x-1-x-2-4x-94661.html#p731476

VeritasPrepKARISHMA :- http://gmatclub.com/forum/inequalities-trick-91482.html#p804990

Zarrolou :- http://gmatclub.com/forum/tips-and-tricks-inequalities-150873.html#p1211920

Example 3 :- http://gmatclub.com/forum/if-9-x-2-0-which-of-the-following-describes-all-153160.html#p1227496

If $9-x^2 \geq 0$, which of the following describes all possible values of x ?

A. 3 ≥ x ≥ -3
B. x ≥ 3 or x ≤ -3
C. 3 ≥ x ≥ 0
D. -3 ≥ x
E. 3 ≤ x

$x^2-9 \leq 0$ → $(x-3)(x+3) \leq 0$ → $-3 \leq x \leq 3$ → Option A

Example 4 :- http://gmatclub.com/forum/if-x-is-an-integer-what-is-the-value-of-x-1-x-2-4x-94661.html#p728428

If x is an integer, what is the value of x?

(1) $x^2 - 4x + 3 < 0$
(2) $x^2 + 4x +3 > 0$

S1) $x^2 - 4x + 3 < 0$ → $x^2 - x - 3x + 3 < 0$ → $x(x-1)-3(x-1)<0$ → $(x-1)(x-3)<0$ → $1<x<3$
→ since x is an integer, x has to be 2. Sufficient.

S2) $x^2 + 4x + 3 > 0$ → $x^2 + x + 3x + 3 > 0$ → $x(x+1)+3(x+1)>0$ → $(x+1)(x+3)>0$ → $x < -3$ or $x > -1$ → x can take multiple values such as 0, 5, -4 etc… Insufficient.

Choice A.

Example 5 :- http://gmatclub.com/forum/is-k-2-k-147133.html#p1181488

Is $k^2 + k - 2 > 0$?

(1) $k < 1$
(2) $k < -2$

We have $k^2 + k - 2 > 0$ → $k^2 + k - 2 > 0$ → $k^2 - k + 2k - 2 > 0$ → $k(k-1)+2(k-1) > 0$ → $(k-1)(k+2)>0$ → $K>1$ or $k<-2$

So the question can be rephrased as 'Is K greater than 1 or less than -2'?

S1) $k < 1$ k may be less than -2 or may not, Insufficient

S2) $k < -2$ k is less than -2, sufficient.

Choice B

Example 6 :- http://gmatclub.com/forum/if-y-x-1-x-and-x-0-is-xy-152900.html#p1225963

If $y=\frac{|x+1|}{x}$ **and** $x\neq0$**, is** $xy>0$**?**

A) $x^2+2x+1>0$

B) $y\neq0$

First let's simplify the inequality

$\frac{|x+1|}{x} - y = 0$ ----------> $\frac{|x+1|-xy}{x} = 0$ -----> we know $x\neq0$, then |x+1|-xy must be zero. Hence |x+1| - xy = 0 --------> |x+1| = xy

We are asked whether xy>0 --------> Whether |x+1| > 0 ? --------> We know the expression within modules can either be zero or greater than zero. For xy to be greater than zero |x+1| has to be greater than zero.

|x+1| will be zero only when x=-1 and for any other value of x, |x+1| will always be greater than zero

So the question can be rephrased as whether $x \neq -1$ true?

Statement 1) x^2 + 2x + 1 > 0

This is an quadratic inequality.

Rule :- For any quadratic inequation ax^2 + bx + c > 0, if b^2 – 4ac = 0 and a > 0 then the inequality holds true outside the interval of roots

In our case b^2 – 4ac = 4 – 4 = 0 and a > 0 so x^2 + 2X + 1 > 0 will hold true for all values beyond the Root(s) of equation (Towards any direction - Positive or Negative)

x^2 + 2x + 1 = 0 --------> x(x+1) +1(x + 1) = 0 ---------> (x+1)(x+1) = 0 ----------> x=Root = -1

So x^2 + 2x + 1 > 0 will hold true for any of x except for -1

This reveals that $x \neq -1$ and xy>0 ----------------> **Sufficient**

Statement 2) $y \neq 0$

From the question stem we know $x \neq 0$

As per Statement 2, $y \neq 0$-------------->That means both X and Y are nonzero.

|x+1| = xy

xy can be either Positive or Negative

|x+1| can be Zero or Positive

Combining both these inferences we can conclude that XY must be Positive. **Sufficient**

Answer = D

Example 7 :- http://gmatclub.com/forum/which-of-the-following-values-of-x-satisfy-both-inequalities-143295.html#p1148838

(x —1)(x + 3) > 0
(x +5)(x—4) < 0

Which of the following values of x satisfy both inequalities shown?

I. -6
II. -4
III. 2
IV. 5

A. I and II only
B. I and III only
C. II and III only
D. II and IV only
E. I, II, III, and IV

1st Inequality :- (x —1)(x + 3) > 0 → x > 1 or x < -3

2nd Inequality :- (x +5)(x—4) < 0 → -5 < x < 4

Solution = (-5 < x < -3) U (1 < x < 4)

Choice C

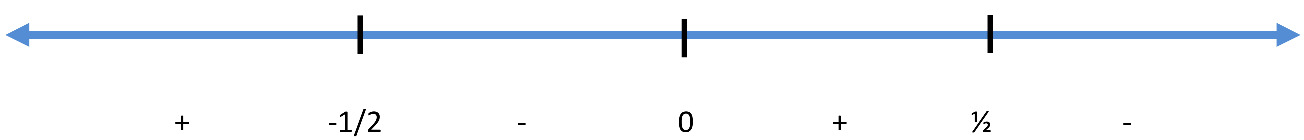
Example 8 :- http://gmatclub.com/forum/which-of-the-following-represents-the-complete-range-of-x-108884.html

Which of the following represents the complete range of x over which $x^3 - 4x^5 < 0$?

A. $0 < |x| < ½$
B. $|x| > ½$
C. $–½ < x < 0$ or $½ < x$
D. $x < –½$ or $0 < x < ½$
E. $x < –½$ or $x > 0$

First Factories the expression

$X^3(1-4x^2) < 0$ → $x^3(1-2x)(1+2x)<0$ → Critical points are -1/2, 0, ½



Recall the steps mentioned earlier

4) Mark the leftmost region with + sign and rest two regions with – and + signs respectively.
5) If the Inequation is of the form $ax^2 + bx + c < 0$, the region having minus sign will be the solution of inequality
6) If the Inequation is of the form $ax^2 + bx + c > 0$, the region having plus sign will be the solutions of inequality

Hence, Range of x is $(-1/2 < x < 0)$ and $(x > ½)$

Choice C

Courtesy for the Information

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Author of CBSE Math Books

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Aluminus – IIM Bangalore

Special Thanks to

BUNUEL, doe007, Zarrolou

Regards,

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