

Experiment, Sample Space and Event

DEFINITIONS

Experiment

An experiment is a set of processes/terms, which are carried out under stipulated conditions to study the phenomena associated with it. Broadly, there can be two types of experiments:

- (i) Experiments with definite outcome: These type of experiments are certain in nature. Outcomes of experiments are known in advance.
- (ii) Experiments with indefinite outcome: These type of experiment are uncertain in nature, we cannot predict the outcome with certainty.

Random Experiment

An experiment whose all possible outcomes (results) are known in advance, but the result of any specific performance cannot be predicted before completion of the experiment.

Illustration:

Consider the following experiments.

- (i) Tossing a coin
- (ii) Rolling a die
- (iii) Drawing a card from a pack of well-shuffled pack of 52 cards.

Solution:

All the experiments have more than one possible outcome. All outcomes are known in advance. Hence all these experiments are random experiments.

Sample Space

The set of all possible outcomes of an experiment (or trial) is called the sample space. It is usually denoted by S .

A die is thrown. The upper face can show any of the six numbers (1, 2, 3, 4, 5, 6). Due to this, we get a sample space consisting of 6 elements.

A card is drawn from a well-shuffled pack of cards. We get a sample space consisting of 52 elements.

For example, for the experiment of tossing a coin, the sample space is a set $S = \{\text{head, tail}\}$. For the experiment of tossing two coins, the sample space is $S = \{\text{head - head, head - tail, tail - head, tail -$

tail}.

Consider the experiment of shooting a tiger until there is a hit. The sample space is the countably infinite set $S = \{h, mh, mmh, \dots\}$ where h denotes a hit, m denotes a miss and mh denotes a miss followed by a hit and so on.

Event

The appearance of a particular outcome, which we may find in the sample space, is an event. Any subset of the sample space is called an event.

Choosing an ace from a pack of well-shuffled cards is an event. This event occurs 4 times in a pack of card i.e. in sample space. Observing an even number when a dice is thrown is an event. This event occurs three times in the sample space of 6 elements. So, event is a subset of sample space.

Complement of an Event:

Let S be the sample space and A be some events, then the set of all out comes which are in S but not in A is called the complete of event A. It is denoted by A^c or A^c .

Illustration:

In rolling a die event A is described as numbers

$$A = \{2, 4, 6\}$$

$$A^c = \{1, 3, 5\}$$

Mutually Exclusive Events:

Let the two events be A and B. If occurrence of event A excludes the possibility of occurrence of B and vice-versa, then we say that A and N are mutually exclusive event i.e. they cannot occur simultaneously. It (i.e. mutual exclusion) can happen only when the sets of two events have no common point.

Illustration:

In rolling a die, event A is described as appearance of an even number and event B is described as appearance of an odd number.

$$A = \{2, 4, 6\}$$

$$B = \{1, 3, 5\}$$

Clearly if one says that A has occurred we will instantly conclude that B cannot occur (in that trial of course). So,

$$A \cap B = \Phi$$

Equally likely Events:

Two or more events are said to be equally likely when there is no reason to prefer one event over other i.e. they have equal (not same) number of points in their sets.

Illustration:

In rolling a die event A is described as number showing less than 4 and event B is described as number appearing greater than 3.

$$A = \{1, 2, 3\}$$

$$B = \{4, 5, 6\}$$

Clearly A and B have equal number of points in the sample space and hence are equally likely.

Exhaustive Events:

If n events $A_1, A_2, \dots, A_n$ related to any particular sample space are such that if we take union of the sets of all the n events, sample space is formed. i.e.

$$A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n = S$$

Illustration:

In rolling a die three events are described as follows

A_1 : even number appears

A_2 : less than 4 appears

A_3 : greater than 4 appears

Hence $A_1 : \{2, 4, 6\}$

$$A_2 : \{1, 2, 3\}$$

$$A_3 : \{5, 6\}$$

$A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6\}$ which is the sample space for the experiment of rolling a die.

Mutually Exclusive and Exhaustive Events.

Events $A_1, A_2, \dots, A_n$ are said to be mutually exclusive and exhaustive if they satisfy the condition for mutual exclusion and exhaustiveness both.

$$\text{i.e. } A_1 \cup A_2 \cup A_3 \dots \cup A_n = S$$

and $A_i \cap A_j = \Phi$, where $i = 1, 2, \dots, n$

$$j = 1, 2, \dots, n \text{ and } i \neq j.$$

i.e. mutually exclusive and exhaustive events are those events of which union is equal to sample space and occurrence of any one of them excludes the possibility of occurrence of all others.

Illustration:

In rolling a die, events A_1 , A_2 , A_3 and A_4 are described as follows

A_1 : Number less than 2 occurs.

A_2 : 2 occurs

A_3 : odd number greater than 1 occur.

A_4 : even number greater than 2 occur.

A_1 : {1}

A_2 : {2}

A_3 : {3, 5}

A_4 : {4, 6}

Clearly, $A_1 \cup A_2 \cup A_3 \cup A_4 = \{1, 2, 3, 4, 5, 6\} = S$

and $A_1 \cap A_2 = A_1 \cap A_3 = A_1 \cap A_4 = A_2 \cap A_3 = A_2 \cap A_4 = A_3 \cap A_4 = \Phi$

i.e. $A_i \cap A_j = \Phi$, $i \neq j$, $i, j = 1, 2, 3, \dots, n$.

Independent Events

If the occurrence or non occurrence of any event does not affect the sample space of occurrence of the other, then two events are said to be independent.

Equally likely events/outcomes

In an experiment, two or more event/outcomes are said to be equally likely, if they have the same chances associated with them. i.e. no one of them has more chance of occurrence than others.

Introduction To Probability

PROBABILITY

If n represents the total number of equally mutually exclusive and exhaustive possible outcomes of an experiment and m of them are favourable to the event A , then the probability of the event A is defined as

$$P(A) = \frac{n(E)}{n(S)} = \frac{m}{n}$$

This is known as classical definition of probability

Note:

(i) for any event $A \subseteq S$

(ii) $0 \leq P(A) \leq 1, P(A) \in \mathbb{R}$

(iii) $P(A) + P(\bar{A}) = 1$

(iv) If a cases are favourable to an event A and b cases are favourable to an event $\bar{A}$ (i.e. unfavourable to A) then $P(A) = a/(a+b)$ and $P(\bar{A}) = b/(b+a)$.

We say that odds in favour of A are a : b

and odds against of A are b : a

Rule 1: In the case of mutually exclusive events the chance of happening of one or other of them is the sum of the chances of the separate events. e.g. if A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$. If $E_1, E_2, \dots, E_n$ are mutually exclusive events, then $P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^n P(E_i)$.

Therefore, if there are n possible outcomes of an experiment which are mutually exclusive, exhaustive and equally likely then, since the probability associated with each outcome is the same (say x) and since they are mutually exclusive, the probability of occurrence of one of them is the sum nx which must be equal to 1 (because they are also exhaustive).

That is, $nx = 1 \Rightarrow x = 1/n$.

Corollary: Since A and A' are mutually exclusive events, $P(A \cup A') = P(A') = 1$.

If out of m + n equally likely, mutually exclusive and exhaustive cases, m cases are favourable to an event A and n are not favourable to an event A. m : n is called odds in favour of A, n : m is called odds against A, probability of occurrence of event A is $m/(m+n)$ and that of non-occurrence of it is $n/(m+n)$.

Notation: Let A and B be two events, then

(i) A' or $\bar{A}$ or A^c stands for the non-occurrence or negation of A.

(ii) $A \cup B$ stands for the occurrence of at least one of A or B.

(iii) $A \cap B$ stands for the simultaneous occurrence of A and B.

(iv) $A' \cap B'$ stands for the non-occurrence of both A and B.

(v) $A \subseteq B$ stands for "the occurrence of A implies occurrence of B".

Illustration:

A die is rolled. What is the probability that outcome will be an even number?

Solution:

When the die is rolled the possible outcomes are

$S = \{1, 2, 3, 4, 5, 6\}$ and $A =$ outcome will be even number i.e. $\{2, 4, 6\}$

So, $P(A) = n(E)/n(S) = 3/6 = 1/2$

Illustration:

Two dice are thrown simultaneously. What is the probability of obtaining a total score of seven?

Solution:

There are six possible ways as to the number of points on the first die; and to each of these ways, there corresponding 6 possible number of points on second die. Hence total number of ways

$$S = 6 \times 6 = 36$$

We now find out how many ways are favourable to the total of 7 points. This may happen only in following ways : (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3).

Hence, required Probability = $6/36 = 1/6$.

Illustration:

Suppose a die is rolled, can we find the probability that either odd number or a number divisible by 4 comes?

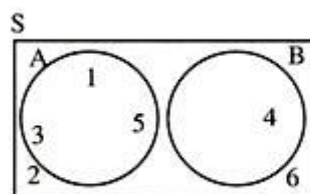
Solution:

When a die is rolled 1, 2, 3, 4, 5 and 6 are possible outcomes. So,
 $S = \{1, 2, 3, 4, 5, 6\}$.

Let event A : odd number i.e. $A = \{1, 3, 5\}$

event B : number divisible by 4 i.e. $B = \{4\}$

Here, we have represented the sample space and event in Venn Diagram.



We have to find probability of either A or B , i.e. $P(A \cup B)$

Clearly, $A \cup B = \{1, 3, 4, 5\}$ i.e. $A + B$

Hence, $P(A \cup B) = P(A) + P(B)$

$$= 3/6 + 1/6 = 2/3.$$

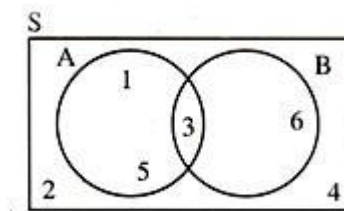
Now, think about a different situation: A die is rolled, can we calculate the probability that the number, which comes is either odd or divisible by 3?

Sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Define, event A : odd number i.e. $A = \{1, 3, 5\}$

event B : divisible by 3 i.e. $B = \{3, 6\}$

we can represent sample space and even in Venn diagram.



Now, we have to find probability that either A or B occurs i.e. $P(A \cup B)$

Clearly, $A \cup B = P\{1, 3, 5, 6\}$

Can we write $A \cup B = A + B$?

Let's do it, $A \cup B = 1, 3, 5, 3, 6$; this is incorrect for obvious reasons since we counted 3 twice.

From above illustration, it can be concluded that when we write

$A \cup B = A + B$ we count the common element twice, once while counting the elements of A and once while counting the elements of B. Therefore in order to get correct result we should subtract it (common) element once i.e.

$$A \cup B = A + B - A \cap B \quad \{A \cap B \text{ element common in both A and B}\}$$

$$= 1, 3, 5, 4, 6, - 3$$

$$= 1, 3, 5, 6$$

Hence the probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = (3/6) + (2/6) - (1/6) = 2/3$$

Note:

1. In previous case, $P(A \cup B) = P(A) + P(B)$ was correct because $A \cap B = \emptyset$. So, we get a very important result.

$P(A \cup B) = P(A) + P(B)$ when A and B are mutually exclusive events.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, when A and B are not mutually exclusive events.

2. This particular case of two events can easily be generalized for the three events A, B and C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Illustration:

The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c respectively. Out of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Prove that $m + p + c = 27/20$.

Solution:

$$1 - (1 - m)(1 - p)(1 - c) = 0.75 \quad \dots\dots\dots (i)$$

$$(1 - m)pc + m(1 - p)c + mp(1 - c) + mpc = 0.5 \quad \dots\dots\dots (ii)$$

$$(1 - m)pc + m(1 - p)c + mp(1 - c) = 0.4 \quad \dots\dots\dots (iii)$$

(ii) - (iii) gives : $mpc = 0.1 = 1/10$ From equation (i), we get

$$p + c + m - (mp + mc + pc) + mpc = 0.75 \quad \dots\dots\dots (iv)$$

From (ii),

$$pc + mc + mp - 3mpc + mpc = 0.5$$

$$\Rightarrow pc + mc + mp = 0.7$$

Substituting in (iv), we get

$$m + p + c = 0.75 - 0.1 + 0.7 = 1.35 = 135/100 = 27/20$$

INTERSECTION AND UNION OF SETS OF EVENTS

Let $E_1, E_2, \dots, E_n$ be n events, then $P(E_1 \cup E_2 \cup \dots \cup E_n)$, represents the probability of occurrence of at least one of the events from $E_1, E_2, \dots, E_n$ and $P(E_1 \cap E_2 \cap \dots \cap E_n)$ represents the occurrence of all the events together.

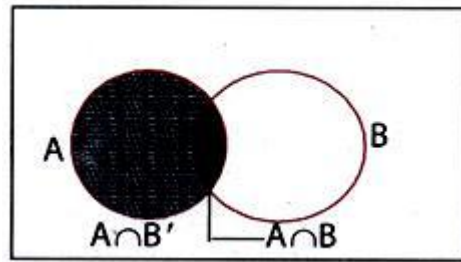
In general, we have

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^n P(E_i) - \sum_{1 \leq i < j \leq n} P(E_i \cap E_j) + \dots + (-1)^{n-1} P(E_1 \cap E_2 \cap \dots \cap E_n).$$

Corollary: If A and B are any two events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Note that $P(A \cap B^c) = P(A) - P(A \cap B)$.

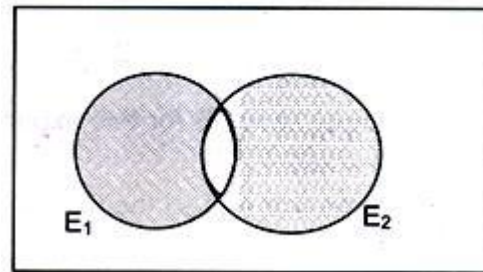


Also for any two events A and B,

$P(\text{exactly one of them occurs})$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= P(E_1 \cup E_2) - P(E_1 \cap E_2).$$



If E_1 , E_2 and E_3 be three events, then

(i) $P(\text{at least two of } E_1, E_2, E_3 \text{ occur})$

$$= P(E_2 \cap E_3) + P(E_3 \cap E_1) + P(E_1 \cap E_2) - 2P(E_1 \cap E_2 \cap E_3),$$

(ii) $P(\text{exactly two of } E_1, E_2, E_3 \text{ occur})$

$$= P(E_2 \cap E_3) + P(E_3 \cap E_1) + P(E_1 \cap E_2) - 3P(E_1 \cap E_2 \cap E_3),$$

(ii) $P(\text{exactly one of } E_1, E_2, E_3 \text{ occur})$

$$= P(E_1) + P(E_2) + P(E_3) - 2P(E_3 \cap E_1) + 2P(E_1 \cap E_2) + 3P(E_1 \cap E_3).$$

If events are mutually exclusive, $P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^n P(E_i)$.

Illustration:

In a single cast with fair dice, what is the chance of throwing

- (i) two 4's, (ii) a doublet,
- (iii) five - six, (iv) a sum of 7.

Solution:

(i) There are 6×6 equally likely cases

(as any face of any die may turn up)

=> 36 possible outcomes. For this event, only one outcome (4, 4) is favourable

=> probability = $1/36$

(ii) A doublet can occur in six ways {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

Therefore probability of a doublet = $6/36 = 1/6$.

(iii) Two favourable outcomes {(5, 6), (6, 5)}:

This probability = $2/36 = 1/18$.

(iv) A sum of 7 can occur in the following cases: {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)} which are 6 in number. Therefore probability = $6/36 = 1/6$.

Illustration:

Seven accidents occur in a week. What is the probability that they happen on the same day.

Solution:

Total number of cases = Total no. of ways in which 7 accidents can happen in a week = 7^7 .

Favourable number of cases out of these = number of those in which all 7 happen on one day (any day of the week) = 7. Hence the required probability = $7/7^7 = 1/7^6$.

Illustration:

From a bag containing 5 white, 7 red and 4 black balls a man draws 3 balls at random. Find the probability that all are white.

Solution:

Total number of balls in the bag = $5 + 7 + 4 = 16$

=> The total number of ways in which 3 balls can be drawn

$$= {}^{16}C_3 = (16 \cdot 15 \cdot 14) / (3 \cdot 2 \cdot 1) = 560.$$

Thus sample space S for this experiment has 560 outcomes

i.e. $n(S) = 560$.

Let E be the event of all the three balls being white. Total number of white balls is 5.

So the number of ways in which 3 white balls can be drawn = 5C_3
= $(5*4*3)/(3*2*1) = 10$.

Thus E has 10 elements of S, i.e. $n(E) = 10$

=> Probability of E, $P(E) = n(E)/n(S) = 10/560 = 1/56$.

Illustration:

From a pack of 52 cards two cards are drawn at random. Find the probability of the following events:

- (i) Both the cards are spades.
- (ii) One card is of spade and the other is a diamond.

Solution:

The total number of ways in which 2 cards can be drawn

$$= {}^{52}C_2 = (52*51)/(1*2) = 26 \times 51 = 1326.$$

=> Number of elements in the sample space S are $n(S) = 1326$.

- (i) Let the event that both cards are of spade, be denoted by E_1 . Then $n(E_1)$ = Number of elements in E_1 = Number of ways in which 2 cards can be selected out of 13 cards of spade = ${}^{13}C_2 = (13*12)/(1*2) = 78$.

Hence the probability of $E_1 = P(E_1) = n(E_1)/n(S) = 78/1326 = 1/17$.

- (ii) Let E_2 be the event that one card is of spade and one is of diamond. Then $n(E_2)$ = number of elements in E_2 = number of ways in which one card of spade can be selected out of 13 spades and one card of diamond can be selected out of 13 diamond cards

$$= {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 = 169$$

=> $P(E_2) = n(E_2)/n(S) = 169/1326 = 13/102$.

Illustration:

A has 3 shares in a lottery containing 3 prizes and 6 blanks: B has one share in a lottery containing one prize and 2 blanks. Compare their chances of success.

Solution:

Total number of tickets in the first lottery = $3 + 6 = 9$

A may select any three tickets out of these 9 tickets. Therefore, the number of element in the sample space S is given by

$$= n(S) = {}^9C_3 = (9*8*7)/(3*2*1) = 84.$$

Let E_1 be the event of winning the prize in the first lottery by A. So, $\overline{E_1}$ is the event of not winning the prize by A. Then number of elements

in $\overline{E_1}$ is given $n(\overline{E_1}) =$ number of ways of selecting 3 tickets out of six blank tickets $= {}^6C_3 = (6*5*4)/(1*2*3) = 20$.

=> The probability of not winning the prize by A $= P(\overline{E_1}) = n(\overline{E_1})/n(S) = 20/84 = 5/21$.

Since $P(E_1) + P(\overline{E_1}) = 1$, $P(E_1) = 1 - P(\overline{E_1})$ or $P(E_1) = 1 - 5/21 = 16/21$.

For second lottery

$n(S) =$ number of ways of selecting 1 ticket out of 3 tickets.

Or $n(S) = {}^3C_1 = 3$.

Let E_2 be the event of winning the prize by B.

$n(E_2) =$ number of ways of selecting one ticket out of 1 prize ticket
 $= {}^1C_1 = 1$

=> The probability of winning the prize by B

$= P(E_2) = n(E_2)/n(S) = 1/3$ so that the ratio of the probabilities of winning the prizes by A and B =
 $P(E_2)/P(E_1) = (1/3)/(16/21) = 7/16$.

Illustration:

From a pack of cards, four are drawn at random. What is the chance that there is one card of each suit?

Solution:

A pack of cards has 52 cards. Let S be the sample space. Then

$$n(S) = {}^{52}C_4 = (52*51*50*49)/(4*3*2*1) = 13 \times 17 \times 25 \times 49.$$

Let E be the event of drawing one card from each suit.

There are four different suits and each suit has 13 cards.

$n(E) =$ number of total ways of drawing one card from each suit.

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 \times 13 \times 13$$

$$\Rightarrow \text{Required probability} = P(E) = n(E)/n(S) = (13*13*13*13)/(13*17*25*49) = 2197/20825.$$

Illustration:

In a given race the odds in favour of four horses A, B, C, D are 1:3, 1:4, 1:5, 1:6 respectively. Assuming that, a dead heat is impossible, find the chance that one of them wins the race.

Solution:

Let $P(A)$, $P(B)$, $P(C)$ and $P(D)$ be the responsibilities of winning of the horses A, B, C and D respectively. Then

$$P(A) = 1/4, P(B) = 1/5, P(C) = 1/6, P(D) = 1/7.$$

Since the above events are mutually exclusive, the chance that one of them wins

$$= P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$$

$$= (1/4) + (1/5) + (1/6) + (1/7) .$$

Illustration:

Ten passengers get into an elevator on the ground floor of a 20 floor building. What is the probability that they will all get out at different floors?

Solution:

$$\text{Total no. of ways in which the 10 passengers can get out} = 20^{10}.$$

Favourable no. of ways = number of those cases in which not more than one passenger gets out at each floor = ${}^{20}P_{10} = 20!/10!$.

$$\text{Required probability} = 20!/(10!(20)^{10}).$$

Illustration:

There are three events A, B and C out of which one and only one can happen. The odds are 8 to 3 against A, 5 to 2 against B. Find odds against C.

Solution:

Let the total no. of cases = $m + n + p$:

m are in favour of A, n in favour of B & p in favour of C.

We have $(n+p)/m = 8/3$, $(m+p)/n = 5/2$, so that

$$P(A') = (n+p)/(m+n+p) = 8/11, P(B') = (m+p)/(m+n+p) = 5/7$$

$$\Rightarrow P(A) = 3/11, P(B) = 2/7$$

$$\text{Also } P(A) + P(B) + P(C) = 1 \Rightarrow P(C) = 34/77$$

Conditional Probability

$P(A/B)$ denotes the probability of event A happening, given that event B has happened. It is the conditional probability of A, given B. While calculating $P(A/B)$, we assume that event B has occurred. It implies that the outcomes favourable to B become the total outcomes and hence outcomes favourable to $P(A/B)$ are outcomes common to A and B.

$$P(A/B) = P(A \text{ given } B)$$

$$= (\text{Total no. of favourable cases})/(\text{Total no. of cases}) = n(A \cap B)/n(B) = P(A \cap B)/P(B).$$

$$\text{Thus } P(A/B) = P(A \cap B)/P(B).$$

$$\Rightarrow (P(A \cap B) = P(B) \cdot (P(A/B) = P(A) \cdot (P(B/A)).$$

Note that if A and B are independent events then

$$P(A \cap B) = P(A) P(B)$$

$$\Rightarrow P(A/B) = P(A \cap B)/P(B) = P(A)P(B)/P(B) = P(A) \text{ which should be the case as occurrence of A does not depend upon B. Similarly } P(B/A) = P(B).$$

More often than not the probability of one event is affected by occurrence of other events. Let a bag contains 3 red, 2 green and 4 yellow balls. A ball is drawn and found to be red. Consider the following cases:

- (i) If the red ball is replaced then find the probability that the next ball to be drawn is yellow.
- (ii) What happens when red ball will not be replaced?

Let, R : red ball is first drawn

Y : yellow ball in next drawn.

In first case when the red ball is replaced.

$$P(Y) = (\text{Total number of yellow balls})/(\text{total number of balls}) = 4/9$$

In second case the red ball is not replaced, so,

$$P(Y) = (\text{Total number of yellow balls})/(\text{Total number of balls left}) = 4/8$$

The probability in these two cases is different. In the first case when the first ball was red and it replaced it does not affect the probability of yellow ball in second draw but in second case when the ball was not replaced the probability of yellow ball is changed. It happened because in second case the sample space is changed. So from here we can draw a conclusion that when occurrence of any event changes the sample space, the probability of other event change. Hence the probability also changes.

It means that, occurrence of any event change the sample space. Then Probability of other events change, in other words probability of some event depend or the occurrence of other events. This is known as "Conditional Probability".

Let A and B be the two events. If A has already happened (i.e. given) then probability of B can be found by using the formula

$$P(B/A) = P(A \cap B)/P(A)$$

If A and B are independent events, then

$$P(B/A) = P(B) \text{ i.e. } P(A \cap B) = P(A) P(B)$$

or, $P(A/B) = P(A)$

(i) n events $A_1, A_2, A_3, \dots, A_n$ are said to be pair-wise independent tiff P

$$(A_i \cap A_j) = P(A_i) P(A_j), \text{ where } 1, i = 1, 2, 3, \dots, n$$

(ii) These n events are said to be mutually independent iff

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) P(A_2) \dots P(A_n).$$

Note : If the set of n events related to a sample space are pair-wise independents, they must be mutually independent, but vice versa is not always true.

To apply total probability formula we must check whether A_1 and A_2 fulfill the following three conditions or not.

$$A_1 \cap A_2 = \Phi$$

$$\text{and } A_1 \cup A_2 = S.$$

$$A_1 \subset S \text{ and } A_2 \subset S.$$

Probability of an event which depends on more than one event

Suppose we have n events $A_1, A_2, \dots, A_n$ related to a sample space such that:

(i) They are mutually exclusive i.e.

$$A_i \cap A_j = \Phi, i, j = 1, 2, \dots, n, i \neq j.$$

(ii) They are exhaustive i.e.

$$A_1 \cup A_2 \cup A_3 \dots \cup A_n = S.$$

(iii) They are proper subsets of sample space S i.e.

$$A_i \subset S, i = 1, 2, \dots, n.$$

Then probability of an event A which depends on the events $A_1, A_2, \dots, A_n$ can be calculated as

$$P(A) = \sum_{i=1}^n P(A/A_i) P(A_i)$$

$$= P(A/A_1) P(A_1) + P(A/A_2) P(A_2) + \dots + P(A/A_n) P(A_n)$$

This is a formula for total probability.

Illustration:

Take the case of previous illustration in which a bag contains 3 Red, 2 Green and 4 Yellow balls. A ball is drawn and if it is Red or Yellow it is replaced otherwise not. Find the probability that second ball drawn is a yellow ball.

Solution:

First of all, we should be very clear about the event of which probability is to be found and the events on which it depends.

Let's A_1 : First ball is either red or yellow

A_2 : First ball is green

A : Second ball drawn is yellow

We have to calculate $P(A)$ which definitely depends on A_1 and A_2 .

Therefore, $P(A) = P(A/A_1) P(A_1) + P(A/A_2) P(A_2)$

$P(A/A_1)$ = Probability of A when A_1 is already happen.

= 4/9 (i.e. probability of A given A_1)

$P(A_1) = 7/9$

$P(A/A_2)$ = Probability of A when A_2 is already happened (Probability of A given A_2). = 4/8

$P(A_2) = 2/9$

Hence, $P(A) = 4/9 \times 7/9 + 4/8 \times 2/9 = 37/81$

Illustration:

If a dice is thrown, what is the probability of occurrence of a number greater than 1, if it is known that only odd numbers can come up.

Solution:

S =the sample space = {1, 2, 3, 4, 5, 6}

A =the event of occurrence of an odd number = {1, 3, 5}

B =the event of occurrence of a number greater than 1={2, 3, 4, 5, 6}

Here $A \cap B = \{3, 5\}$

so that $P(B/A) = P(A \cap B)/P(A) = (n(A \cap B))/(n(A)) = 2/3$.

Illustration:

In a college, 25% students failed in Mathematics, 15% students failed in Physics, and 10% failed in Mathematics and Physics. A student is selected at random:

- (i) If he failed in Physics, then find the chance of his failure in Mathematics,
- (ii) If he failed in Mathematics, then find the chance of his failure in Physics,
- (iii) Find the chance of his failure in Mathematics or Physics.

Solution:

Let E_1 and E_2 be the events of failure in mathematics and physics respectively. Let the total number of students appearing in the examination be 100.

Since 25% students failed in Mathematics, $n(E_1) = 25$.

$$\Rightarrow P(E_1) = n(E_1)/n(S) = 25/100 = 1/4.$$

Since 15% students failed in Physics, $n(E_2) = 15$

$$\Rightarrow P(E_2) = n(E_2)/n(S) = 15/100 = 3/20.$$

Again 10% students failed in Physics and Mathematics both, so $n(E_1 \cap E_2) = 10$

$$\Rightarrow P(E_1 \cap E_2) = n(E_1 \cap E_2)/n(S) = 10/100 = 1/10.$$

- (i) The chance of failure in Mathematics while he has failed in Physics is given by

$$P(E_1/E_2) = (P(E_1 \cap E_2))/P(E_2) = (1/10)/(3/20) = 2/3.$$

- (ii) The chance of failure in Physics while he has failed in Mathematics is given

$$\text{by } P(E_2/E_1) = (P(E_1 \cap E_2))/P(E_1) = (1/10)/(1/4) = 2/5.$$

- (iii) The chance of failure in Mathematics or Physics is:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

[since both the events are not mutually exclusive]

Independent Events

Independent Events

Events are said to be independent when the happening of any one of them does not affect the happening of any of the others/

(i) A and B are independent if

$$P(B/A) = P(B) \text{ and } P(A/B) = P(A).$$

(ii) The probability of the occurrence of several independent events is the product of their individual probabilities.

$$\begin{aligned} \text{i.e. If } E_1, E_2, \dots, E_n \text{ are } n \text{ independent events then } P(E_1 \cap E_2 \cap \dots \cap E_n) \\ = P(E_1)P(E_2)P(E_3)\dots P(E_n). \end{aligned}$$

Remark: The converse of this is also true i.e. if n given events satisfy the above condition then they will be independent.

Pairwise Independent Events

Three events E_1, E_2 and E_3 are said to pairwise independent if

$$P(E_1 \cap E_2) = P(E_1)P(E_2), P(E_2 \cap E_3) = P(E_2)P(E_3) \text{ and } P(E_3 \cap E_1) = P(E_3)P(E_1)$$

i.e. Events $E_1, E_2, E_3, \dots, E_n$ will be pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j) \quad i \neq j.$$

Three events are said to be mutually independent if

$$P(E_1 \cap E_2) = P(E_1)P(E_2), P(E_2 \cap E_3) = P(E_2)P(E_3) \text{ and } P(E_3 \cap E_1) = P(E_3)P(E_1)$$

$$\text{and } P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3).$$

If two events A and B are mutually exclusive,

$$P(A \cap B) = 0 \text{ but } P(A)P(B) \neq 0 \text{ (In general).}$$

$$\Rightarrow P(A \cap B) \neq P(A)P(B)$$

$\Rightarrow$ mutually exclusive events will not be independent.

So if two events are independent, they have to have some common element between them i.e. they cannot be mutually exclusive. Mutually exclusiveness is used when the events are taken from the same experiment and independence is used when the events are taken from different experiments.

For example

(i) Two fair dice are thrown. Let two events be 'first die shows an even number' and second die shows an odd number'. These two events are independent events since the result of the first die does not depend upon the result of 2nd die. But these events are not mutually exclusive since both the events may simultaneously occur.

(ii) From a pack of cards, (a) Two cards are drawn in succession without replacement. Let an event be 'card drawn is Ace'. Then the two events (i.e. two cards being aces) are dependent since drawing of 2nd card very much depends upon which is the first drawn card. (b) If the cards are drawn with replacement, the drawing of second card will not depend upon first card. So, two events will be independent.

Illustration:

An event A_1 can happen with probability p_1 and event A_2 can happen with probability p_2 . What is the probability that

- (i) exactly one of them happens.
- (ii) at least one of them happens (Given A_1 and A_2 are independent events).

Solution:

- (i) The probability that A_1 happens is p_1

=> The probability that event A_1 fails $1-p_1$. Also the probability that A_2 happens is p_2 .

Now, the chance that A_1 happens and A_2 fails is $p_1(1 - p_2)$ and the chance that A_1 fails and A_2 happens is $p_2(1 - p_1)$.

=> The probability that one and only one of them happens is

$$p_1(1 - p_2) + p_2(1 - p_1) = p_1 + p_2 - 2p_1p_2.$$

- (ii) The probability that both of them fail = $(1 - p_1)(1 - p_2)$

=> Probability that at least one happens = $1 - (1 - p_1)(1 - p_2)$
 $= p_1 + p_2 - p_1 p_2.$

Illustration:

A person draws a card from a pack of 52, replaces it and shuffles it. He continues doing it until he draws a spade. What is the chance that he has to make?

- (i) At least 3 trials
- (ii) Exactly 3 trials

Solution:

(i) For at least 3 trials, he has to fail at the first 2 attempts and then after that it doesn't make a difference if he fails or wins at the 3rd or the subsequent attempts.

Chance of success at any attempt = $1/4$

=> chance of failure = $3/4$

=> chance of failing in first 2 attempt = $(3/4)^2$ = Chance of success at any attempt = $1/4$

=> chance of failure = $3/4$

=> chance of failing in first 2 attempt = $(3/4)^2 = 9/16$.

(ii) For exactly 3 attempts, he has to fail in the first two attempts and succeed in the 3rd attempt.

The required probability = $(3/4)^2 \cdot 1/4 = 9/64$.

Illustration:

A, B and C in order, toss a coin. The one who gets a head first wins. Find their respective probabilities of winning.

Solution:

$P(\text{Head}) = 1/2 = P(\text{Tail})$ in any toss of the coin. A wins if he gets a head in the first trial or if he fails, then B & C fail and then he gets a head and so on.

Therefore, $P(A) = 1/2 + (1/2)^3 \cdot 1/2 + (1/2)^6 \cdot 1/2 + \dots$;

$= 1/2 [1 + 1/8 + 1/64 + \dots] = 1/2 \cdot 1/(1-1/8) = 4/7$.

Similarly $P(B) = 1/2 \cdot 1/2 + (1/2)^4 \cdot 1/2 + (1/2)^7 \cdot 1/2 + \dots = 2/7$

and $P(C) = (1/2)^2 \cdot 1/2 + (1/2)^5 \cdot 1/2 + \dots = 1/7$.

Total Probability Theorem

Let $A_1, A_2, \dots, A_n$ be a the set of mutually exclusive and exhaustive events and E be some event which is associated with $A_1, A_2, \dots, A_n$. Then probability that E occurs is given by

$$P(E) = \sum_{i=1}^n P(A_i)P(E/A_i).$$

Illustration:

A bag contains 3 white balls and 2 black balls another contains 5 white and 3 black balls. If a bag is chosen at random and a ball is drawn from it, what is the probability that is white?

Solution:

The probability that the first bag is chosen is $1/2$ and the chance of drawing a white ball from it is $3/5$.

=> Chance of choosing the first bag and drawing a white ball is $1/2 \cdot 3/5$.

Similarly, the chance of choosing the second bag and drawing a white ball is $1/2 \cdot 5/8$.

Hence the chance of randomly choosing a bag and drawing a white ball is $= 1/2 \cdot 3/5 + 1/2 \cdot 5/8$ (Mutually exclusive cases)

$$= 49/80.$$

Illustration:

Find the probability that a year chosen at random has 53 Sundays.

Solution:

Let $P(L)$ be the probability that a year chosen at random is leap

$$P(L) = 1/4 \\ \Rightarrow P(L^c) = 3/4.$$

Let $P(S)$ be the probability that a year chosen at random has 53 Sundays

Now $P(S/L)$ = Probability that a leap year has 53 Sundays.

A leap year has 366 days, 52 weeks + 2 days; the remaining 2 days may be Sunday-Monday, M-T, T-W.

W-Th, Th-F, F-Sat or Sat-S.

Out of the 7 possibilities, 2 are favourable.

$$\Rightarrow P(S/L) = 2/7. \text{ Similarly } P(S/L^c) = 1/7.$$

$$\Rightarrow P(S) = 1/4 \cdot 2/7 + 3/4 \cdot 1/7 = 5/28.$$

Baye's Theorem

Baye's theorem revises (reassigns) the probabilities of the events $A_1, A_2, \dots, A_n$, related to a sample space, when there is an information about the outcome beforehand. The earlier probabilities of the events $P(A_i)$, $i = 1, 2, \dots, n$ are called a priori probabilities and the probabilities of events calculated after the information A is received i.e. $P(A_i/A)$ is called posterior probabilities.

Conditions for the application of Baye's formula is that priori events i.e. $A_1, A_2, \dots, A_n$ of the sample space are exhaustive and mutually exclusive i.e.

$$A_1 \cup A_2 \cup \dots \cup A_n = S$$

$$\text{and } A_i \cap A_j = \Phi, i = 1, 2, \dots, n \text{ and } i \neq j$$

It's derivation is as follows

Let $B_1, B_2, \dots, B_n$ be n mutually exclusive and exhaustive events and A be any event in sample space, then

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n)$$

$$\Rightarrow P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \dots + P(A \cap B_n)$$

$$= P(B_1).P(A/B_1) + P(B_2).P(A/B_2) + \dots + P(B_n).P(A/B_n)$$

$$\text{Hence, } P(B_j/A) = (P(B_j).P(A/B_j)) / (\sum_{i=1}^n P(B_i).P(A/B_i))$$

If in a problem some event has already happened and then the probability of another event is to be found, it is an application of Baye's Theorem



Note: To recognize the question in which Baye's theorem is to be used, the key word is "is found to be".

Illustration:

A die is rolled and it is found that number turned up is an even number. Find the probability that it is 2.

Solution:

All possible events when we roll a die are

$$A_1 : 1 \text{ appears } P(A_1) = 1/6$$

$$A_2 : 2 \text{ appears } P(A_2) = 1/6$$

$$A_3 : 3 \text{ appears } P(A_3) = 1/6$$

$$A_4 : 4 \text{ appears } P(A_4) = 1/6$$

$$A_5 : 5 \text{ appears } P(A_5) = 1/6$$

A_6 : 6 appears $P(A_6) = 1/6$

An even number is turned up this is an information to us. You should revise the probability of priori events A_i in the light of information received. (It will be totally foolish if we don't revise the probabilities. Since the information of even numbers is available, it makes the probabilities of 1, 3, and 5 equal to zero).

Let, A : Even number has turned up.

We have to calculate the probability of A_2 when A is given. Since $A_1, A_2, \dots, A_6$ are exhaustive and mutually exclusive we can apply Baye's formula

$$P(A_2/A) = (P(A)P(A/A_2))/(P(A_1)P(A/A_1)+P(A_2)P(A/A_2)+\dots+P(A_6)P(A/A_6))$$

$P(A/A_1)$ means probability of A when A_1 is given i.e. probability of coming of an event number when 1 has appeared. Obviously it is zero.

Hence $P(A/A_1) = 0$

Similarly, $P(A/A_2) = 1, P(A/A_3) = 0, P(A/A_4) = 1, P(A/A_5) = 0, P(A/A_6) = 1$

Therefore, $P(A_2/A) = (1/6 \times 1)/(1/6 \times 0 + 1/6 \times 1 + 1/6 \times 0 + 1/6 \times 1 + 1/6 \times 0 + 1/6 \times 1) = 1/3$

Note: This problem is very simple and illustrated only to make the application of Baye's formula clear.

Illustration:

In a factor, machines A, B and C manufacture 15%, 25% and 60% of the total production of bolts respectively. Of the bolts manufactured by the machine A, B and C 4%, 2% and 3% are defective A bolt is drawn at random and is found to be defective. What is the probability that it was produced by B?

Solution:

In the formula $P(A_k/A) = (P(A_k)P(A/A_k))/(\sum_{i=1}^n P(A_i) \cdot P(A/A_i))$

A_i means all the possibilities, which can happen w.r.t. to the given event A, while A_k means the particular event whose probability w.r.t. B we are required to find

Let us take A = the event of bolt being defective.

A_1 = the bolt is produced by B.

A_2 = the bolt is produced by A.

A_3 = the bolt is produced C.

Required probability = $P(A_1/A) = (P(A_1)P(A/A_1))/(\sum_{i=1}^3 P(A_i) \cdot P(A/A_i))$

$$\begin{aligned}
& \Rightarrow P(A_1/A) \\
& = (25/100 \times 2/100) / ((15/100 \times 4/100) + (25/100 \times 2/100) + (60/100 \times 3/100)) \\
& = 50 / (60 + 50 + 180) = 5/29.
\end{aligned}$$

Illustration:

A bag contains 5 balls of unknown colours two balls are drawn at random and are found to be red. Find the probability that the bag contains exactly 4 red balls.

Solution:

First of all let's try to find out all the possibilities, which can happen w.r.t. to given events. Since two balls drawn are found to be red, so there are four possibilities. The bag contains only two red balls (say event A_2), bag contains 3 red balls (A_3), bag contains 4 red balls (A_4), or that all the balls are red balls (A_5). Let B : two balls are drawn and found to be red. So, it is obvious that we have to find the probability of A_4 given B i.e. required probability = $P(A_4/B)$

$$P(A_4/B) = (P(A_4) \cdot P(B/A_4)) / (P(A_2) \cdot P(B/A_2) + P(A_3) \cdot P(B/A_3) + P(A_4) \cdot P(B/A_4) + P(A_5) \cdot P(B/A_5))$$

Now as all these four possible cases are equally likely i.e. we cannot say that bag is more likely to contain 5 red balls, than 3 red balls, so priori probabilities of all these events will be equal i.e.

$$P(A_2) = P(A_3) = P(A_4) = P(A_5)$$

Since A_2, A_3, A_4 and A_5 are exhaustive

$$\therefore P(A_2) + P(A_3) + P(A_4) + P(A_5) = 1$$

$$\therefore P(A_4) = 1/4 \text{ (since } A_2, A_3, A_4 \text{ and } A_5 \text{ are equally likely also)}$$

Now $P(B/A_4)$ means that bag contains 4 red balls and we have to find the probability, that two balls drawn at random are red which is obviously ${}^4C_2/{}^5C_2$.

$$\begin{aligned}
\text{So } P(A_4/B) &= (1/4 \times ({}^4C_2/({}^5C_2))) / (1/4 \times ({}^2C_2/({}^5C_2)) + 1/4 \times ({}^3C_2/({}^5C_2)) + 1/4 \times ({}^4C_2/({}^5C_2)) + 1/4 \times ({}^5C_2/({}^5C_2))) \\
&= 12/39
\end{aligned}$$

Proof: This is so because

$$P(A_i/B) = P(A_i \cap B) / P(B) = P(A_i \cap B) / P((B \cap A_1) \cup P(B \cap A_2) \cup \dots (B \cap A_n)) = P(A_i \cap B) / P((B \cap A_1) \cup P(B \cap A_2) \cup \dots (B \cap A_n))$$

$$= P(A_i) \cdot P(B/A_i) / \sum_{i=1}^n P(A_i) P(B/A_i)$$



Note : In conditional probability the sample space is reduced to the set of samples/outcomes in the event which is given to have happened.

Illustration:

Each of three bags A, B, C contains white balls and black balls. A has a_1 white & b_1 black, B has a_2 white & b_2 black and C has a_3 white & b_3 black. A ball is drawn at random and is found to be white. Find the respective probability that it is from A, B & C.

Solution:

Here A_1, A_2, A_3 are the events that the bag picked are A, B, C respectively

E is the event that a white ball is drawn.

We are supposed to find $P(A_1/E), P(A_2/E), P(A_3/E)$.

$P(A_1/E) = P(A_1 \cup B)/P(E) = ((\text{Prob. that bag A is chosen and white is drawn})/(\text{Prob. that a bag is chosen at random and white is drawn}))$

$$= (P(A_1) \cdot P(E/A_1)) / (P(A_1) \cdot P(E/A_1) + P(A_2) \cdot P(E/A_2) + P(A_3) \cdot P(E/A_3))$$

$$= (1/3 \cdot a_1 / (a_1 + b_1)) / (1/3 \cdot [a_2 / (a_2 + b_2) + a_2 / (a_2 + b_2) + a_3 / (a_3 + b_3)]) = p_1 / (p_1 + p_2 + p_3)$$

Similarly, $P(A_2/E) = P_2 / P_1 + P_2 + P_3$, $P(A_3/E) = P_3 / P_1 + P_2 + P_3$

where $p_1 = a_1 / a_1 + b_1$, $p_2 = a_2 / a_2 + b_3$, $p_3 = a_3 / a_3 + b_3$.

Illustration:

A bag contains 5 balls and of these it is equally likely that 0, 1, 2, 3, 4, 5 are white. A ball is drawn and is found to be white. What is the chance that is the only white ball?

Solution:

Here, the again, is a problem of conditional probability.

The condition B that is given is that one ball is drawn and is white.

$$\text{Hence } P(1w/B) = (P(1w/B) \cdot P(1w)) / (P(0w)P(B/0w) + P(1w) \cdot p(B/1w) + \dots + P(5w) \cdot P(B/5w))$$

Where $P(B/1_w) \rightarrow$ Probability that B occurs when exactly 1 w ball is there = ${}^1C_1 / {}^5C_1$ and so on and $P(1w) = P(2w) = \dots = P(5w) = P(0w) = 1/6$

$$\Rightarrow \text{the required probability} = (1/6 ({}^5C_1 / ({}^5C_1))) / (1/6 [0 + ({}^1C_1 / ({}^5C_1) + ({}^2C_1 / ({}^5C_1) + \dots + ({}^5C_1 / ({}^5C_1))])$$

$$= 1/(0+1+2+...+5)=1/15.$$

Binomial Distribution

Binomial Distribution for Successive Events

As experiment may be repeated n times and each of these n trials are independent of one another. If each trial only gives one of the two possible outcomes, say 'success' (when required event take place) and failure (when required event does not take place) then it is called a Binomial Experiment. Let probability of success in any trial be p and that of failure be q , then $p + q = 1$

$$\text{Then } (p + q)^n = C_0 p^n + C_1 p^{n-1} q + \dots + C_r p^{n-r} q^r + \dots + C_n q^n$$

Then the probability of exactly k successes in n trials is given by

$$P_k = {}^n C_k q^{n-k} p^k$$

Some important facts related with binomial distribution

(i) The probability of getting at least k successes is

$$P(x \geq k) = \sum_{x=k}^n {}^n C_x p^x q^{n-x}$$

$$(ii) \sum_{x=k}^n {}^n C_x q^{n-x} p^x = (q + p)^n = 1$$

Illustration:

In a rainy season, there is 60% chance that it will rain on a particular day. What is the probability that there will exactly 4 rainy days in a week?

Solution:

Let probability of raining on a particular day $P(\text{success}) = 3/5$

And probability that no rain on a particular day $q(\text{failure}) = 2/5$

Probability of exactly 4 rainy days i.e. 4 success out of 7 trials

$$= {}^7C_4 (p)^4 (q)^3 = \left(\frac{7!}{4!3!} \right) * \left(\frac{3}{5} \right)^4 * \left(\frac{2}{5} \right)^3$$