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Reflections on Senior Mathematics
Permutations without formulas or tears

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In how many ways can 5 boys and 4 girls sit in a row if Katie and Christine want to sit together?

Do these kinds of problems intimidate and confuse you? Permutations and combinations was the one topic that caused me anxiety when I was studying for my HSC. It made little sense to me, and like many students with a probability phobia, I could never be sure if my final answer was correct! And judging by the number of calls to the HSC Advice Line on this very subject, many students today still experience a lack of confidence when using counting techniques to solve probability problems.

To overcome my limitations, I researched and studied this topic thoroughly the first time I taught a 3 Unit class. I knew there had to be a more straightforward, intuitive approach to teaching this topic, hopefully one free of jargon and formula. I'm happy to report that there is, and this became most self-evident some years later on a day when half of my Year 11 class were away on an excursion. I decided to give the remaining half a quick preview of their next 'difficult' topic, but surprised them and myself when I covered most of the permutations and combinations theory in that hour, and without mentioning ^nP_r or ^nC_r.

When I related this story to my head teacher afterwards, he agreed that yes, permutations is best taught using a 'lists-and-boxes' problem-solving approach, starting from basic principles and relying less upon fancy formulas. What follows are extracts from my teaching notes, which have been refined several times, with the hope that they may help you alleviate some of your students' fears (and perhaps your own) next time you teach this topic.

Introduction

There are five horses in a race - A, B, C, D and E. Suppose you have to bet on which two horses come first and second, **in the correct order**. From the five horses, how many different bets (1st - 2nd pairings) are possible?

AB BA CA DA EA
AC BC CB DB EB
AD BD CD DC EC
AE BE CE DE ED

Number of arrangements = **20**. These arrangements are called **permutations**.

Now suppose instead that you need to choose the two horses that come first or second, but you don't have to state which one comes first and which once comes second. In other words, **the order is not important**. Now how many selections are possible?

AB BC CD DE AC
BD CE AD BE AE

Number of selections = **10**. These selections are called **combinations**.

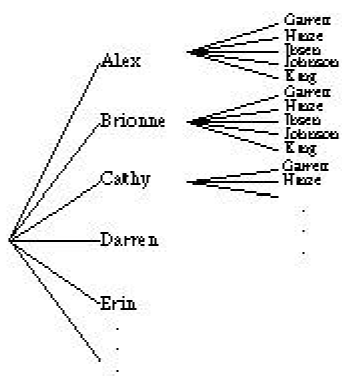
Notice how there are only half as many combinations as permutations because **order is not important** with combinations (e.g., AB is the same as BA).

The multiplication principle for counting arrangements

1.From this list of given names and surnames, how many different arrangements of given name - surname pairs are possible?

Given name	Surname
Alex	Garrett
Brionne	Hinze
Cathy	Ibsen
Darren	Johnson
Erin	King
Fiona	

Using a tree diagram, but not completing it ...



For each given name, there are 5 possible surnames.

Total possible arrangements is 6 x 5 = **30**.

2. From this menu, calculate how many different 3-course dinners are possible.

Entrée	Main course	Dessert
Pumpkin soup	Garlic prawns	Pavlova
Calamari rings	Steak Diane	Black Forest cake
Lasagne	Roast lamb	Chocolate mousse
	Chicken Dijon	Mangoes and

		ice cream
	Grilled perch	

Total possible dinner arrangements is 3 x 5 x 4 = **60** (and I've tried every one!!).

The **basic counting principle** is:

1.If A can be arranged in ***m*** different ways and B can be arranged in ***n*** different ways, then the number of possible arrangements of A and B together is ***m*** x ***n***.

2.More generally, if

A can be arranged in ***a*** different ways,

B can be arranged in ***b*** different ways,

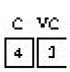
C can be arranged in ***c*** different ways, and so on;

then the total number of possible arrangements of ABC ... together is ***a*** x ***b*** x ***c*** ....

Permutations

1. A girls' school is electing a captain and vice-captain. There are four candidates: Amy, Betty, Caroline and Deane. How many possible arrangements of captain/vice-captain are there?

Instead of using a tree diagram to count the possibilities, we can draw boxes for the two positions.



There are 4 ways of choosing the captain, but once the captain is determined, there are only 3 ways of choosing the vice-captain.

Total possible arrangements = 4 x 3 = **12**.

These arrangements are called **permutations**. The number of permutations possible has the notation ${}^{n}P_{r}$ where *n* is the number of items or people available, and *r* is the number of places available. In the above example there were 4 candidates and 2 places.

${}^{n}P_{r} = {}^{4}P_{2} = 4 \times 3 = 12$.

Note ${}^{n}P_{r}$ is found by multiplying backward from *n, r* times.

2.8 people are to be photographed, but there are only 5 seats available. In how many ways can 8 people be seated in 5 seats?

8 people, 5 places.

$^8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6720$ arrangements.

Permutations with restrictions

In mathematics, '!' (read 'factorial') means a special product.

For example, 5! = 5 x 4 x 3 x 2 x 1 ('5 factorial')

x! means multiplying backward from *x* down to 1. There is also an x! key on your calculator.

This factorial notation is good for abbreviating numerical expressions in permutation problems.

1.In how many different ways can 5 boys and 4 girls sit together in a row?

2.What if boys and girls must alternate?

3.What if boys and girls sit in separate groups?

4.What if Katie and Christine want to sit together (they always do)?

We can solve this problem by drawing boxes to represent seats or positions. Boxes are like an advanced representation of the levels (branches) of a tree diagram.

1.9 seats, 9 places. $^9P_9 = 9! = 362880$ arrangements.

9	8	7	6	5	4	3	2	1

2.

B	G	B	G	B	G	B	G	B

5	4	4	3	3	2	2	1	1

Boys: 5 seats, 5 places, Girls: 4 seats, 4 places.

5! x 4! = 2880 arrangements.

3.

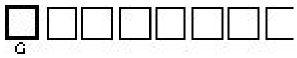
B	B	B	B	B	G	G	G	G

or

G	G	G	G	B	B	B	B	B

2 x 5! 4! = 2 x 2880 = 5760 arrangements.

4. Let both girls be considered as one person/seat.



Now there are only '8 people' to arrange on 8 seats. 8! = 40 320.

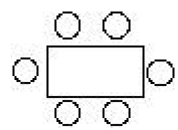
But for the '2-girls seat' GG, the two girls can sit in 2 different ways: G_1G_2 or G_2G_1 (Katie and Christine can swap seats). So for each of the 8! arrangements, there are 2 possibilities for GG.

2 x 8! = 80 640 arrangements.

Permutations with special conditions

(a) Arrangements around a circle

In how many ways can 6 people be arranged around a table, if the order around the table is all that matters?



This is different to 6 people in a line, because the 6 people in a circle can all move one (or more) places to the left, without affecting the **order** of seating. In other words, A B C D E F, B C D E F A, C D E F A B, etc., are ALL the same circular arrangement because they have the same order.

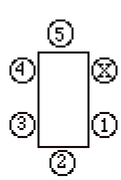
If the seats were in a line, there would be 6! = 720 possible arrangements. However, for each of these arrangements, you can move every person left one seat and the order would still be retained. In fact, you can continue doing this until you get back to where you started - that would be 6 shifts. The answer 6! is 6 times too high, so we must divide by 6.

Total possible arrangements =

$$\frac{6!}{6}=\frac{6.5.4.3.2.1}{6}=5.4.3.2.1=5!=120.$$

Generally, n items in a circle (e.g. seats, keys on a key ring) can be arranged in $(n - 1)!$ ways.

Alternative method: Another way of thinking of the above example is to say that it doesn't matter where the first person (X) sits, since only the order around him/her is important. Total possible arrangements = 5! = 120.



(b) Permutations of *n* items, some alike

1.How many 7-letter permutations are possible from the letters of the word LOLLIES?

Solution

The answer is less than 7!, because L is a repeated letter, so some of the arrangements will be the same, for example, E I L_1 L_2 L_3 O S, E I L_1 L_2 L_3 O S, E I L_1 L_2 L_3 O S, etc., are the same.

For each arrangement of 7 letters, the 3 L's
E I **L L L** O S

can be arranged in 3! ways, so the 7! needs to be divided by 3!.

No. of permutations = $\frac{7!}{3!} = \frac{5040}{6} = 840.$

From n items, of which p are alike of one kind, q are alike of another kind, and so on, the number of possible permutations is $\frac{n!}{p!q! \cdots}$

2. (1989 HSC) Let each different arrangement of all the letters of DELETED be called a word.

(a) How many words are possible?

(b) In how many of these words will the D's be separated?

Solution

(a) DELETED has 2 D's, 3 E's. Permutations $= \frac{7!}{2!3!} = \frac{5040}{12} = 420.$

(b) To count how many words have separated D's, we can count how many words have D's **together** and subtract from (a). Let DD be counted as one letter, giving 6 letters.

Permutations $= \frac{6!}{3!} = \frac{720}{6} = 120.$

No. of words with separated D's = 420 - 120 = 300.

Combinations

Three students need to be selected from a group of 8 students to represent the school at a conference.

How many combinations of 3 representatives are possible?

8 students, 3 places. There are ${}^8P_3 = 8 \times 7 \times 6 = 336$ **permutations**, but this isn't the answer to the problem. If A, B, C, D, ... represent the 8 students, then the permutations ABC, ACB, BAC, BCA, CAB, CBA are the same selection. These 3 students can change places within the group but it's still the same group. Because of this, we must divide our permutations answer by 3! = 6.

No. of combinations $= \frac{{}^8P_3}{3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = \frac{336}{6} = 56.$

Combinations are a special type of permutation. The number of combinations is smaller because the order of arrangements is not important, e.g. ABC = ACB = BAC = ... , so we divide by $r!$ where r is the number of places available.

Combinations have the formula and notation $^nC_r = \frac{^nP_r}{r!}$ where n is the number of items available and r is the number of places available.

So for selecting 3 students from 8 students to go to a conference:

$$^8C_3 = \frac{^8P_3}{3!} = \frac{8\times7\times6}{3\times2\times1} = \frac{336}{6} = 56.$$

Permutations vs combinations

Permutations are **arrangements** of items, while combinations are **selections** of items. When you **arrange** something, you organize them in a particular order or place, but when you **select** something, you simply pick out or choose them, without considering their order or place. When selecting from a group of items, the order of the items selected is not important. ^nC_r is sometimes called '*n* choose *r*', for example, $^8C_3$ is sometimes read '8 choose 3'. Sometimes, ^nC_r is also written as $\binom{n}{r}$, for example, $^8C_3$ can also be written $\binom{8}{3}$.

Permutations are **ordered arrangements:** there are more of them because ABC, ACB, BAC, etc. are considered **different**. For example, a 'trifecta' bet on a horse race (first 3 horses in correct order) is a permutation. Combinations are **unordered selections:** there are fewer of them because ABC, ACB, BAC, etc. are considered **the same**. For example, a Lotto bet (6 numbers to come up in any order) is a combination.

Formulas for ^nP_r and ^nC_r

Formulas for ^nP_r and ^nC_r can be written using $x!$ notation.

For example: $^7P_4 = 7\times6\times5\times4 = \frac{7\times6\times5\times4\times(3\times2\times1)}{(3\times2\times1)} = \frac{7!}{3!}$.

Generally, $\boxed{^nP_r = \frac{n!}{(n-r)!}}$

for example: $^7C_4 = \frac{7\times6\times5\times4}{4\times3\times2\times1} = \frac{7\times6\times5\times4}{4\times3\times2\times1}\left(\frac{3\times2\times1}{3\times2\times1}\right) = \frac{7!}{4!3!}$.

Generally: $\boxed{^nC_r = \binom{n}{r} = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!r!}}$

Most calculators also have ^nP_r and ^nC_r keys, so permutations and combinations can be calculated in three different ways: by multiplying backward, by using the $n!$ formulas, or by using the calculator keys.

Investigations

1. Why is $^{n}P_{n} = n!$?

2. Why is $^{n}C_{n} = 1$?

3. Why is $^{n}C_{r} = {}^{n}C_{n-r}$? (symmetrical property)

4. Why is $^{n}C_{1} = n$?

5. Why is $^{n}P_{1} = n$?

Combinations with restrictions

1. (1996 HSC) A committee of 3 men and 4 women is to be formed from a group of 8 men and 6 women. In how many ways can this be done?

Solution

No. of ways 3 men can be chosen from a group of 8 men = $^{8}C_{3}$.

No. of ways 4 women can be chosen from a group of 6 women = $^{6}C_{4}$.

No. of possible committees $= {}^{8}C_{3} \times {}^{6}C_{4} = 840$.

2. Three letters are randomly chosen from the word TUESDAY. How many possible selections are there? How many of these selections have:

(a) exactly one vowel?

(b) exactly 2 vowels?

(c) at least 2 vowels?

Solution

No. of selections $= {}^{7}C_{3} = 35$.

(a) 'Exactly one vowel 'means 1 vowel and 2 consonants. Drawing dashes to represent these:

$\underline{V}\ \underline{\overline{V}}\ \underline{\overline{V}}$ where V = vowel, and $\overline{V}$ = **not** vowel.

TUESDAY has 3 vowels and 4 consonants.

No. of selections $= {}^{3}C_{1} \times {}^{4}C_{2} = 3 \times 6 = 18$.

Note: For convenience, we'll use **boxes** for permutations and **dashes** for combinations.

(b) 'Exactly 2 vowels': $\underline{V}\ \overline{\underline{V}}\ \overline{\underline{V}}$

No. of selection ${}^{3}C_{2} \times {}^{4}C_{1} = 3 \times 4 = 12.$

(c) 'At least 2 vowels' means 2 or 3 vowels.

No. of selections of 3 vowels $(\underline{V}\ \underline{V}\ \underline{V}) = {}^{3}C_{3} = 1.$ Total selections = 12 + 1 = 13.

Binomial probability

With binomial probability, we are concerned with repeated probability trials in which there are only two possible outcomes: a **success** with probability p, and a **failure** with probability q, where $p + q = 1$.

These are also called **Bernoulli trials** after the Swiss mathematician James Bernoulli (1654 -1705). Some examples of outcomes in Bernoulli trials are HEAD or TAIL, WIN or LOSE, TRUE or FALSE, BOY or GIRL, DEFECTIVE or NON-DEFECTIVE.

The probability of hitting a target is $\frac{1}{4}$. Garrett takes 5 shots at the target.

Calculate the probability that he scores:

(a) no hits

(b) only 1 hit

(c) 2 hits

(d) 3 hits

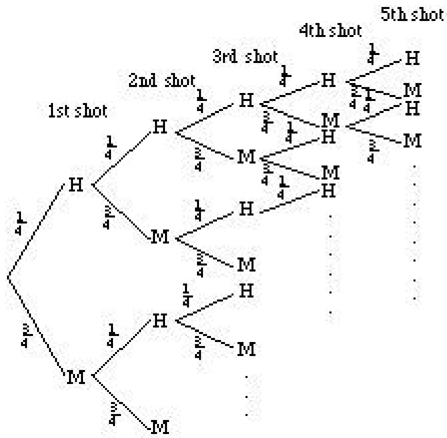
(e) 4 hits

(f) 5 hits.

Solution

$p = \frac{1}{4},\ q = \frac{3}{4}.$

Drawing a tree diagram but not completing it ...



Let X represent the number of hits out of 5.

(a) $P(X=0)=\left(\frac{3}{4}\right)^5=\frac{243}{1024}$. x x x x x

(b) $P(X=1)=\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4\times 5$ ↖ No. of different ways of Getting 1 correct from 5. ✓ x x x x

$=\frac{405}{1024}$.

(c) $P(X=2)=\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3\times {}^5C_2$ ↖ No. of different ways of Getting 2 correct from 5. ✓✓ x x x

$=\frac{135}{512}$.

(d) $P(X=3)=\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2\times {}^5C_3=\frac{45}{512}$. ✓✓✓ x x

(e) $P(X=4)=\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)\times {}^5C_4=\frac{15}{1024}$. ✓✓✓✓ x

(f) $P(X=5)=\left(\frac{1}{4}\right)^5=\frac{1}{1024}$. ✓✓✓✓✓

The general pattern is $P(x=r)=p^r q^{n-r}\,{}^nC_r$ where *n* is the total number of trials and *r* is the number of successes. Note that the sum of the probabilities equals 1.

$$\frac{243}{1024}+\frac{405}{1024}+\frac{135}{512}+\frac{45}{512}+\frac{15}{1024}+\frac{1}{1024}=1.$$

If *p* = *P*(success) and *q* = *P*(failure) = 1 - *p*, and the Bernoulli trial is repeated *n* times, then the probability of *r* successes is

$$P(X=r)={}^nC_r\,p^r q^{n-r}$$