Order of Operations

1. Parentheses: including absolute value signs
2. Exponents
3. Multiplication/Division: performed left to right
4. Addition/Subtraction: performed left to right

Expressions vs. Equations

An expression represents a value and can be simplified:

Combine like terms: $6z + 5z \rightarrow 11z$

Find common denominator:

$$\frac{1}{12} + \frac{3x^3}{4} \times \left(\frac{3}{3}\right) \rightarrow \frac{1}{12} + \frac{9x^3}{12} = \frac{9x^3+1}{12}$$

Pull out common factor: $2ab + 4b \rightarrow 2b(a + 2)$

Cancel common factors: $\frac{5y^3}{25y} \rightarrow \frac{y^2}{5}$

The value of the expression stays the same

NOTE: Equations contain an equals sign, expressions do not. Any change must be made to both sides of an equation.

Solving Simultaneous Equations

Substitution:

1. Solve one equation for an unknown.
2. Substitute this expression in second equation.
3. Solve second equation for other unknown variable(s).

Combination:

1. Line up terms of equations.
2. Multiply one equation by some number, if necessary, so one variable is eliminated when equations are added together.

NOTE: In linear equations, all variables have exponent of 1.

Solving Multiple Simultaneous Equations

Look for ways to simplify through shortcuts or symmetries to reduce number of steps to solve:

What is the sum of x,y,z?

$$x + y = 8 \quad x + z = 11 \quad y + z = 7$$

Notice the symmetry and add together:

$$\begin{array}{rcccccccc} & x & + & y & & & = & 8 \\ & x & & & + & z & = & 11 \\ + & & & y & + & z & = & 7 \\ \hline & 2x & + & 2y & + & 2z & = & 26 \end{array}$$

Therefore, $x + y + z$ is half of 26, or 13.

Absolute Value Equations

1. Isolate expression within absolute value brackets.
$$12 + |w - 4| = 30$$
$$|w - 4| = 18$$
2. Once equation is in form $|\, x \,| = a$ with $a > 0$, $x = \pm\, a$. Remove absolute value brackets, solve for 2 different cases.

Case 1: $x = a$ (x is positive)
$$w - 4 = 18$$
$$w = 22$$

Case 2: $x = -a$ (x is negative)
$$w - 4 = -18$$
$$w = -14$$

3. Check solutions by putting them into original equation! Possibility of failed solutions a peculiarity of absolute value equations.

Exponential Bases

0 raised to any power equals 0: $0^3 = 0 \times 0 \times 0 = 0$

1 raised to any power equals 1: $1^3 = 1 \times 1 \times 1 = 1$

Thus, if $x = x^2$, x must be either 0 or 1

If the base is a decimal or positive proper fraction or decimal between 0 and 1, the value of expression decreases as the exponent increases:

$$\left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16} \qquad (0.6)^2 = 0.36 \qquad (0.5)^4 = 0.0625$$

Combining Exponential Terms

To **multiply** terms with same base, **add** exponents:
$$x^2 \times x^3 = x^5 \qquad 5 \times 5^2 = 5^3$$
To **divide** terms with same base, **subtract** exponents:
$$\frac{4^6}{4^2} = 4^4$$
Divide something by itself, quotient is 1. Therefore, $a\hat{\ }0 = 1$.
$$\frac{a^3}{a^3} = a^{3-3} = a^0$$
A base raised to 0 power equals 1. Exception, base of 0. 0^0 is undefined because $\frac{0}{0}$ is undefined.

Zero Exponents

If a number is divided by itself, the quotient is 1:
$$\frac{x^3}{x^3} = 1 \qquad \frac{x^3}{x^3} = x^{3-3} = x^0$$
Therefore, $x^0 = 1$.

A base raised to the 0 power equals 1. The exception is a base of 0.

0^0 is undefined because $\frac{0}{0}$ is undefined.

Negative Exponents

An expression with a negative exponent is the reciprocal of the same expression with positive exponent:

$$\frac{y^2}{y^5} = y^{2-5} = y^{-3} = \frac{1}{y^3}$$

Factoring Out Common Terms

Normally, exponential terms being added or subtracted cannot be combined. However, a common term can be factored out if terms have same base:

$$11^3 + 11^4 \to 11^3(1 + 11) \to 11^3(12)$$

If $x = 4^{20} + 4^{21} + 4^{22}$, what is the largest prime factor of x?

$$x = 4^{20}(1 + 4^1 + 4^2)$$
$$x = 4^{20}(1 + 4 + 16)$$
$$x = 4^{20}(21)$$
$$x = 4^{20}(3 \times 7)$$

Exponents and the Sign of the Base

Even exponents hide the sign of the base. Any number raised to an even exponent becomes positive.

For any x, $\sqrt{x^2} = |x|$.

$a^2 = 17$ has two solutions: $\sqrt{17}$ and $-\sqrt{17}$.

Odd exponents keep the sign of the base.

If an equation has both variables with odd exponents and variables with even, it is likely to have two solutions.

Same Base or Exponent

With exponential expressions on both sides, it's imperative to rewrite so either same base or exponent appears on both sides:

Solve for w: $(4^w)^3 = 32^{w-1}$

1. Rewrite bases so same base appears on both sides:

$$\left((2^2)^w\right)^3 = (2^5)^{w-1}$$

2. Simplify the equation using rules of exponents:

$$2^{6w} = 2^{5(w-1)}$$

3. Eliminate identical bases, rewrite exponents as equation, solve:

$$6w = 5w - 5 \quad \to \quad w = -5$$

NOTE: If 0, 1, or -1 is/could be the base, the outcome of raising these powers isn't unique. If $0^x = 0^y$, then you can't say $x = y$.

Exponents

Squares	Squares	Powers of 2	Powers of 3	Powers of 5
$1^2 = 1$	$13^2 = 169$	$2^1 = 2$	$3^1 = 3$	$5^1 = 5$
$2^2 = 4$	$14^2 = 196$	$2^2 = 4$	$3^2 = 9$	$5^2 = 25$
$3^2 = 9$	$15^2 = 225$	$2^3 = 8$	$3^3 = 27$	$5^3 = 125$
$4^2 = 16$	$20^2 = 400$	$2^4 = 16$	$3^4 = 81$	
$5^2 = 25$	$30^2 = 900$	$2^5 = 32$		
$6^2 = 36$		$2^6 = 64$		
$7^2 = 49$		$2^7 = 128$		
$8^2 = 64$		$2^8 = 256$		
$9^2 = 81$		$2^9 = 512$		
$10^2 = 100$		$2^{10} = 1{,}024$		
$11^2 = 121$				
$12^2 = 144$				

Cubes	Powers of 4	Powers of 10
$1^3 = 1$	$4^1 = 4$	$10^1 = 10$
$2^3 = 8$	$4^2 = 16$	$10^2 = 100$
$3^3 = 27$	$4^3 = 64$	$10^3 = 1000$
$4^3 = 64$		
$5^3 = 125$		
$10^3 = 1000$		

Square Roots Have Only One Solution

Unlike $x^2 = 16$, which has two solutions, $x = 4$ or $x = -4$, $x = \sqrt{16}$ only has one solution: $x = 4$.

If an equation contains a square root, only use the positive root.

Odd roots also have only one solution but keep the sign of the base, like odd exponents:

If $\sqrt[3]{27} = x$, what is x?

$$(-3)(-3)(-3) = -27$$

NOTE: There's no solution for an even root of a negative number. No number, multiplied an even number of times, can be negative.

Exponent Rules

$(x^r)(x^s) = x^{(r+s)}$	$(2^2)(2^3) = 2^{(2+3)} = 32$
$\frac{x^r}{x^s} = x^{(r-s)}$	$\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$
$(x^r)(y^r) = (xy)^r$	$(3^3)(4^3) = 12^3 = 1{,}728$
$(\frac{x}{y})^r = \frac{x^r}{y^r}$	$(\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}$
$(x^r)^s = x^{rs} = (x^s)^r$	$(x^3)^4 = x^{12} = (x^4)^3$
$x^{-r} = \frac{1}{x^r}$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
$x^0 = 1$	$6^0 = 1$
$x^{r/s} = (x^{1/s})^r = (x^r)^{1/s} = \sqrt[s]{x^r}$	$8^{2/3} = (8^{1/3})^2 = (8^2)^{1/3} = \sqrt[3]{8^2} = 4$

Roots & Fractional Exponents

Fractional exponents are the link between roots and exponents. The **numerator** is the power and the **denominator** is the root.

$$216^{1/3} = \sqrt[3]{216} = 6$$

When an exponent is **negative**, take reciprocal of base and change to positive equivalent:

$$\left(\frac{1}{8}\right)^{-4/3} = 8^{4/3} = \sqrt[3]{8^4} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16$$

$$\sqrt[4]{\sqrt{x}} = \left(x^{1/2}\right)^{1/4} = x^{1/8}$$

Simplifying Roots

Roots can only be combined or separated in multiplication and division, **and not addition or subtraction.**

$$\sqrt{25 \times 16} = \sqrt{25} \times \sqrt{16} = 5 \times 4 = 20$$

$$\sqrt{50} \times \sqrt{18} = \sqrt{50 \times 18} = \sqrt{900} = 30$$

$$\sqrt{144 \div 16} = \sqrt{144} \div \sqrt{16} = 12 \div 4 = 3$$

$$\sqrt{72} \div \sqrt{8} = \sqrt{72 \div 8} = \sqrt{3} = 3$$

NOTE: Only products or quotients of roots can be separate and not the sum or difference of roots.

Simplifying Roots of Imperfect Squares

Some imperfect squares can be simplified into multiples of smaller square roots.

Rewrite roots as a product of primes under radical:

$$\sqrt{52} = \sqrt{2 \times 2 \times 13} = \sqrt{2 \times 2} = \sqrt{4} = 2 \times \sqrt{13} = 2\sqrt{13}$$

$$\sqrt{72} = \sqrt{2 \times 2 \times 2 \times 3 \times 3} = 2 \times 3 \times \sqrt{2} = 6\sqrt{2}$$

Squares & Cubes & Roots

$1^2 = 1$	$\sqrt{1} = 1$
$1.4^2 \approx 2$	$\sqrt{2} \approx 1.4$
$1.7^2 \approx 3$	$\sqrt{3} \approx 1.7$
$2.25^2 \approx 5$	$\sqrt{5} \approx 2.25$
2^2	$\sqrt{4} = 2$
$3^2 = 9$	$\sqrt{9} = 3$
$4^2 = 16$	$\sqrt{16} = 4$
$5^2 = 25$	$\sqrt{25} = 5$
$6^2 = 36$	$\sqrt{36} = 6$
$7^2 = 49$	$\sqrt{49} = 7$
$8^2 = 64$	$\sqrt{64} = 8$

$9^2 = 81$	$\sqrt{81} = 9$
$10^2 = 100$	$\sqrt{100} = 10$
$11^2 = 121$	$\sqrt{121} = 11$
$12^2 = 144$	$\sqrt{144} = 12$
$13^2 = 169$	$\sqrt{169} = 13$
$14^2 = 196$	$\sqrt{196} = 14$
$15^2 = 225$	$\sqrt{225} = 15$
$16^2 = 256$	$\sqrt{256} = 16$
$20^2 = 400$	$\sqrt{400} = 20$
$25^2 = 625$	$\sqrt{625} = 25$
$30^2 = 900$	$\sqrt{900} = 30$

$1^3 = 1$	$\sqrt[3]{1} = 1$
$2^3 = 8$	$\sqrt[3]{8} = 2$
$3^3 = 27$	$\sqrt[3]{27} = 3$
$4^3 = 64$	$\sqrt[3]{64} = 4$
$5^3 = 125$	$\sqrt[3]{125} = 5$

Quadratic Equations

Quadratics have 1 unknown and 2 defining components:

1. Variable term raised to 2nd power
2. Variable term raised to 1st power

Form: $ax^2 + bx + c = 0$

Examples:

$$x^2 = 3x + 4 \qquad a = 5a^2 \qquad 6 - b = 7b^2$$

Like other even exponent equations, quadratics generally have 2 solutions, i.e., two possible values of *x*.

Factoring Quadratics

$$x^2 - 15x = -26$$

Rearrange equation so one side contains a quadratic expression and the equals 0. If the product of two numbers is 0, then at least one of the numbers is 0.

$$x^2 - 15x + 26 = 0$$

Factor the quadratic expression:

$$(x - 2)(x - 13) = 0$$

Set each factor equal to 0.

$$x = \{2, 13\}$$

Disguised Quadratics

GMAT will often disguise quadratics:

$3w^2 = 6w$ Dividing both sides by w and then by 3

$3w = 6$ yields the solution $w = 2$

Solving by factoring as a quadratic reveals another solution:

$$3w^2 = 6w \quad 3w^2 - 6w = 0 \quad w(3w - 6) = 0$$

Setting both factors to 0 yields the following solutions

$$w = \{0, 2\}$$

If a quadratic expression is equal to 0, and an x can be factored out, then $x = 0$ is a solution of the equation.

Taking the Square Root

If one side of an equation is a **perfect square** quadratic, the problem can be quickly solved by taking the square root of both sides, without setting one side equal to zero:

If $(z + 3)^2 = 25$, what is z?

$$\sqrt{(z + 3^2} = \sqrt{25}$$

$$|z + 3| = 5$$

$$z + 3 = \pm 5 = -3 \pm 5 = \{2, -8\}$$

NOTE: Square rooting a square of something is the same as taking the absolute value of that thing.

Distributing with FOIL

Solve $(a + b)(x + y)$ by multiplying every term in first equation by every term in the second, then add the products.

Use the acronym **FOIL:**

FIRST: multiply first terms in each parentheses

OUTER: multiply outer terms in each

INNER: multiply inner terms in each

LAST: multiply last terms in each

$$(x + 7)(x - 3) = x(x - 3) + 7(x - 3) = x^2 - 3x + 7x - 21$$

Using FOIL with Square Roots

What is the value of $(\sqrt{8}-\sqrt{3})(\sqrt{8}+\sqrt{3})$?

FIRST: $\sqrt{8}\times\sqrt{8}=8$

OUTER: $\sqrt{8}\times\sqrt{3}=24$

INNER: $\sqrt{8}\times(-\sqrt{3})=-\sqrt{24}$

LAST: $(\sqrt{3})\times(-\sqrt{3})=-3$

Result: $8+\sqrt{24}-\sqrt{24}-3=8-3=5.$

The two middle terms cancel out.

Perfect Square Quadratics

Not all quadratics have two solutions. Always factor quadratic equations to determine the solution(s).

$$x^2-6x+9=0$$
$$(x-3)(x-3)=0$$
$$(x-3)^2=0$$

$$x^2+8x+16=0$$
$$(x+4)(x+4)=0$$
$$(x+4)^2=0$$

Special Products of Quadratic Expressions

Square of a sum:

$$(x+y)^2=(x+y)(x+y)=x^2+2xy+y^2$$

Square of a difference:

$$(x-y)^2=(x-y)(x-y)=x^2-2xy+y^2$$

Difference of squares:

$$(x+y)(x-y)=x^2-y^2$$

These forms are often disguised:

$x^2+8x+16$ $\quad a^2-4ab+4b^2$ $\quad (1+\sqrt{2})(1-\sqrt{2})$

Solving Quadratics with No *x* Term

The fastest way to solve quadratics without an x term is to take positive and negative square roots:

$$x^2=9\rightarrow\sqrt{x^2}=\sqrt{9}\rightarrow x=\{3,-3\}$$

Alternative:

$$x^2=9$$
$$x^2-9=0$$
$$x^2+0^2-9=0$$
$$(x+3)(x-3)=0$$
$$x=\{3,-3\}$$

ALGEBRA > FORMULAS

Formula Problems

Four major types are used to test working with unknowns:

1. **Plug-in**: solve for a variable by plugging in given values for other variables.
2. **Strange Symbol**: arbitrary symbols defining certain procedures. Refer to variables as "1st number," 2nd number," etc.
3. **Unspecified Amount**: real values not provided, only how values have changed.
4. **Sequence**: collection of numbers in a set order. Each defined by a rule which can reveal the values of terms.

79

ALGEBRA > FORMULAS

Sequence Formula

A sequence is a collection of numbers in a set order, each defined by a rule that can be used to find the values of terms.

If $S_n = 15n - 7$, what is the value of $S_7 - S_5$?

I.e., what is the difference of the 7th and 5th term of the sequence:

$$S_7 = 15(7) - 7 = 105 - 7 = 98$$
$$S_5 = 15(5) - 7 = 75 - 7 = 68$$
$$S_7 - S_5 = (98) - (68) = 30$$

The GMAT will expect you to recognize sequence notation. Find the value of the nth term by plugging n into the formula.

82

ALGEBRA > INEQUALITIES

Operations on Inequalities

Many operations that can be performed on equations can be performed on inequalities:

- Constants or variable expressions can be added or subtracted on every term of the inequality.
- Positive numbers can be multiplied or divided on every term.
- **When multiplying or dividing by a <u>negative</u> number, the inequality sign flips:**

$$4 - 3x < 10 \quad \rightarrow \quad -3x < 6 \quad \rightarrow \quad x > -2$$

NOTE: You cannot multiply or divide an inequality by a variable unless you know the sign of the number the variable stands for.

91

ALGEBRA > INEQUALITIES

Combining Inequalities

Some problems require converting several inequalities to a **compound inequality**. Line up variables and combine.

To combine by **adding**, inequality signs must face same direction:

Is $a + 2b < c + 2d$? (1) $a < c$ (2) $b < d$

$$\begin{array}{cccccccc}
 & a & & & < & c & & \\
+ & & & b & < & & & d \\
\hline
 & a & + & b & < & c & + & d \\
+ & & & b & < & & & d \\
\hline
 & a & + & 2b & < & 2c & + & d
\end{array}$$

NOTE: Never subtract or divide two inequalities. Only multiply if both sides of both inequalities are positive.

92

Inequalities & Absolute Value

When $|x+b|=c$, the center point on the number line is $-b$. x must be *exactly* c units from $-b$.

Given that $|x-2|<5$, what is the range of possible values for x? For 1st scenario, remove absolute brackets and solve. For 2nd, reverse signs of terms inside brackets, solve again.

$$|x-2|<5 \qquad |x-2|<5$$
$$x-2<5 \qquad -(x-2)<5$$
$$x<7 \qquad -x+2<5$$
$$-x<3$$
$$x>-3$$

Combine: $-3<x<7$. Never change $|x-a|$ to $x+b$. Enclose entire expression in parentheses and put negative sign in front.

Square-Rooting Inequalities

As with equations, inequalities with exponents require two scenarios:

If $x^2<4$, what are possible values for x?

Recall that $\sqrt{x^2}=|x|$. Thus, when square rooting both sides of an inequality, you get:

$$\sqrt{x^2}<\sqrt{4} \quad \rightarrow \quad |x|<2$$

NOTE: Only take the square root of an inequality for which both sides are definitely not negative, since you cannot take the square root of a negative number.

Data Sufficiency Process

1. Separate *additional info* from *actual question*.
2. Determine whether the question is Value or Yes/No.
   - Value: **Sufficient** when **1 possible value; Not Sufficient** when more than 1 possible value.
   - Yes/No: **Sufficient** when a **definite Yes or No. Not Sufficient** when the answer **could be Yes or No.**

	Sufficient	Not Sufficient
Value	1 Value	More than 1 Value
Yes/No	1 Answer (Yes or No)	More than 1 Answer (Yes AND No)

3. Decide *exactly* what would be *sufficient* to answer question.
4. Use the Grid to evaluate the statements: AD
   BCE

Data Sufficiency Combo Problems

It may seem that both statements are necessary when a DS question asks for the value of a combination of variables:

What is the value of $\frac{x}{y}$? (1) $\frac{x+y}{y}=3$ (2) $y=4$

Statement 1 itself is sufficient:

$$\frac{x+y}{y}=3 \rightarrow \frac{x}{y}+\frac{y}{y}=3 \rightarrow \frac{x}{y}+1=3 \rightarrow \frac{x}{y}=2$$

Don't try to solve for the value of each variable. Manipulate the statements to **solve directly for the combination.**

Data Sufficiency Combo Problems & Equations

If a DS question has **an equation with multiple variables in the question stem**, it is probably a disguised combo problem.

If $a = 3bc$ and $abc \neq 0$, what the value of c?

(1) $a = 10 - b$ (2) $3a = 4b$

Isolate the wanted variable and create the **simplest combo**:

$$a = 3bc \rightarrow \frac{a}{3b} = c \rightarrow \frac{1}{3} \times \frac{a}{b} = c$$

(1) $a = 10 - b \rightarrow \frac{a}{b} = \frac{10-b}{b}$ INSUFFICIENT

(2) $3a = 4b \rightarrow \frac{3a}{b} = 4 \rightarrow \frac{a}{b} = \frac{4}{3}$ SUFFICIENT

Fractions, Exponents, & Roots

The effect of raising a fraction to a power varies depending upon the fraction's value, the sign, and the exponent:

EVEN EXPONENTS

$\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$	$\left(\frac{-1}{2}\right)^2 = \frac{1}{4}$	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$
result is bigger	result is bigger	result is *smaller*	result is bigger

ODD EXPONENTS

$\left(\frac{-3}{2}\right)^3 = \frac{-27}{8}$	$\left(\frac{-1}{2}\right)^3 = \frac{-1}{8}$	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$\left(\frac{3}{2}\right)^3 = \frac{27}{8}$
result is *smaller*	result is bigger	result is *smaller*	result is bigger

To raise a fraction to a negative power, raise reciprocal to equivalent positive power.

Rooting a number is the same as raising to fractional power.

Replacing Variables with Numbers

Problems with **variables in the answer choices** can almost always be answered by **replacing the variables with numbers.**

1. Identify unknowns and replace. Small primes work best.
2. Use numbers to calculate the target answer.
3. Plug into answer choices. Correct answer will equal target.

If $a \neq 2$, what equals $\frac{b(a^2-4)}{ab-2b}$? (A) ab (B) a (C) $a+2$ (D) a (E) $2b$

$$\frac{b(a^2-4)}{ab-2b} = \frac{b(a^2-4)}{b(a-2)} = \frac{(a-2)(a+2)}{(a-2)} = a+2$$

$$\frac{(5)((3)^2-4)}{(3)(5)-2(5)} = \frac{5(9-4)}{15-10} = \frac{5(5)}{5} = 5$$

Solving Absolute Value Equations with Constant(s)

To solve equations with one **variable** and at least one **constant** in more than one absolute value expression, take an **algebraic approach** by setting two cases: (1) neither expression changes sign, and (2) one expression changes sign.

$$|x-2| = |2x-3|$$

$$(x-2) = (2x-3) \qquad (x-2) = -(2x-3)$$

$$1 = x \qquad 3x = 5$$

$$x = \frac{5}{3}$$

Multiplying or Dividing Two Equations

Two complete equations can be solved by substitution, but multiplying the sides of one equation by the sides of the other can be faster. This works because both sides are being multiplied by an equivalent (equal) expression.

If $xy^2 = -96$ and $\frac{1}{xy} = \frac{1}{24}$, what is y?

$$xy^2\left(\frac{1}{xy}\right) = -96\left(\frac{1}{24}\right)$$

$$\frac{xy^2}{xy} = \frac{-96}{24}$$

$$y = 4$$

Quadratic Formula

For any quadratic equation of the form $ax^2 - bx - c = 0$, where a, b, and c are constants:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Predicting solutions based on the **discriminant**: $b^2 - 4ac$

$b^2 - 4ac > 0$ two solutions

$b^2 - 4ac = 0$ one solution

$b^2 - 4ac < 0$ no solutions (can't root negative number)

Using Conjugates to Rationalize Denominators

When a denominator is a square root, it is easy to simplify by multiplying the numerator and denominator by the square root:

$$\frac{4}{\sqrt{2}} \to \frac{4}{\sqrt{2}} \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

To simplify a denominator containing the sum or difference of a square root and another term, use **the conjugate** of the denominator, i.e., change the sign of the square root term:

$$\frac{4}{3-\sqrt{2}}\left(\frac{3+\sqrt{2}}{3+\sqrt{2}}\right) = \frac{4(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}$$

$$\frac{12+4\sqrt{2}}{9+3\sqrt{2}-3\sqrt{3}-2} = \frac{12+4\sqrt{2}}{7}$$

Functions

A **function** is a rule built on an independent variable. The value changes as the value of the variable changes. I.e., the value of a function is dependent on the value of the independent variable.

Domain indicates the possible inputs; **range**, the possible outputs.

Numerical substitution: If $f(x) = x^2 - 2$, what is $f(5)$?

$$f(5) = (5)^2 - 2 = 25 - 2 = 23$$

Variable substitution: If $f(z) = z^2 - \frac{z}{3}$, what is $f(w+6)$?

$$f(w+6) = (w+6)^2 - \frac{w+6}{3}$$

Compound Functions

Work from the *inside out* to solve compound fractions:

If $f(x) = x^3 + \sqrt{x}$ and $g(x) = 4x - 3$, what is $f(g(3))$?

$$g(3) = 4(3) - 3 = 12 - 3 = 9$$

Use the result from the *inner* function g as the new input variable for the *outer* function f:

$$f\big(g(3)\big) = f(9) = (9)^3 + \sqrt{9} = 729 + 3 = 732$$

ALGEBRA > EXTRA FUNCTIONS STRATEGIES

Functions with Unknown Constants

Solve functions with unknown constants in three steps:

If $f(x) = ax^2 - x$, and $f(4) = 28$, what is $f(-2)$?

1. Use value of input variable and corresponding output value of function to solve for unknown constant:

$$f(4) = a(4)^2 - 4 + 28 \quad \rightarrow \quad 16a - 4 = 28$$
$$16a = 32 \quad \rightarrow \quad a = 2$$

2. Rewrite, replacing constant with its numerical value:

$$f(x) = ax^2 - x = 2x^2 - x$$

3. Solve function for new input variable:

$$f(-2) = 2(-2)^2 - (-2) = 8 + 2 = 10$$

ALGEBRA > EXTRA FUNCTIONS STRATEGIES

Direct Proportionality

Direct proportionality is when two quantities always change by the same factor and in the same direction.

Such relationships are of the form $y = kx$, where x is the input value, y is the output value, and k is the **proportionality constant.**

This equation can also be written $\frac{y}{x} = k$, which means that the ratio of the output and input values is always constant.

Set up before and after ratios to solve. E.g., $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, since both **ratios** are equal to the same constant k.

ALGEBRA > EXTRA FUNCTIONS STRATEGIES

Inverse Proportionality

Inverse proportionality is when two quantities change by reciprocal factors.

Such equations are in the form $y = \frac{k}{x}$, where x is the input value, y is the output value, and k is the **proportionality constant.**

This equation can also be written $xy = k$, which means that the product of the output and input values is always constant.

To solve, set up products, not ratios, to solve the problem. E.g., $y_1x_1 = y_2x_2$, since each product equals the constant k.

Linear Growth

Linear growth (or decay) is which quantities grow at a constant rate, determined by linear function: $y = mx + b$.

The slope m is the constant rate at which the quantity grows.

The y-intercept b is the value of the quantity at time zero (the initial state).

The variable (in this case, x) stands for time. You can also use t to represent time.

Exponential Growth

Exponential growth is when a quantity is <u>multiplied</u> by the same constant each period of time (rather than <u>adding</u> the same constant, as in **linear growth**).

Any exponential growth can be written as $y(t) = y_0 \times k^t$

y the quantity as a function of time t

y_0 value of the quantity at time $t = 0$

k constant multiplier for one period.

Exponential growth multipliers often take the form of percentage multipliers. E.g., for a quantity that increases by 7% each period, $k = 1.07$.

Symmetry

Symmetry is the property that different inputs to a function yield the same output. For which function does $f(x) = f(^1/_x)$, given $x \neq -2, -1, 0,$ or 1? Two ways to see which yields the same output: (1) substitute $^1/_x$ for each x and simplify; (2) pick a number for x. For example, $x = 3$.

		$\boldsymbol{f(3)}$	$\boldsymbol{f(^1/_3)}$
(A)	$f(x) = \left\lvert\frac{x+1}{x}\right\rvert$	$\left\lvert\frac{3+1}{3}\right\rvert = \frac{4}{3}$	$\left\lvert\frac{\frac{1}{3}+1}{\frac{1}{3}}\right\rvert = \frac{\frac{4}{3}}{\frac{1}{3}} = 4$
(B)	$f(x) = \left\lvert\frac{x+1}{x}\right\rvert$	$\left\lvert\frac{3+1}{3-1}\right\rvert = \frac{4}{2} = 2$	$\left\lvert\frac{\frac{1}{3}+1}{\frac{1}{3}-1}\right\rvert = \left\lvert\frac{\frac{4}{3}}{-\frac{2}{3}}\right\rvert = 2$
(C)	$f(x) = \left\lvert\frac{x+1}{x}\right\rvert$	$\left\lvert\frac{3-1}{3}\right\rvert = \frac{2}{3}$	$\left\lvert\frac{\frac{1}{3}-1}{\frac{1}{3}}\right\rvert = \left\lvert\frac{-\frac{2}{3}}{\frac{1}{3}}\right\rvert = 2$
(D)	$f(x) = \left\lvert\frac{x+1}{x}\right\rvert$	$\left\lvert\frac{3}{3+1}\right\rvert = \frac{3}{4}$	$\left\lvert\frac{\frac{1}{3}}{\frac{1}{3}+1}\right\rvert = \left\lvert\frac{\frac{1}{3}}{\frac{4}{3}}\right\rvert = \frac{1}{4}$
(E)	$f(x) = \left\lvert\frac{x+1}{x}\right\rvert$	$\left\lvert\frac{3+1}{3+2}\right\rvert = \frac{4}{5}$	$\left\lvert\frac{\frac{1}{3}+1}{\frac{1}{3}+2}\right\rvert = \left\lvert\frac{\frac{4}{3}}{\frac{7}{3}}\right\rvert = \frac{4}{7}$

Reciprocals of Inequalities

If $x < y$, then:

- $\frac{1}{x} > \frac{1}{y}$ **when x and y are positive**. Flip inequality.

  If $3 < 5$, then $\frac{1}{3} > \frac{1}{5}$.

- $\frac{1}{x} > \frac{1}{y}$ **when x and y are negative**. Flip inequality.

  If $-4 < -2$, then $\frac{1}{-4} > \frac{1}{-2}$.

- $\frac{1}{x} < \frac{1}{y}$ **when x is negative, y is positive**. Don't flip inequality.

  If $-6 < 7$, then $\frac{1}{-6} < \frac{1}{7}$.

- **If you don't know sign of x or y, you cannot take reciprocals.**

Squaring Inequalities

- **If both sides are known to be negative, then flip the inequality sign when squaring.**

  If $x < -3$, then the left side must be negative. Hence, square and reverse the sign: $x^2 > 9$.

- **If both sides are known to be positive, the do not flip the inequality sign when squaring.**

  If $x > 3$, then both sides must be positive. Hence, square both sides: $x^2 > 9$.

- **If one side is positive and one side negative, you cannot square.**
- **If the signs are unclear, you cannot square.**