Solving Word Problems

1. Identify desired value
2. Identify unknown values and label with variables.
3. Identify relationships and translate into equations.
4. Use equations to solve for the desired value.

Units & Relationships

Additive Relationships: The units of every term are the same. Adding terms with the same units does not change the units.

Multiplicative Relationships: Treat units like numerators and denominators. Units that are multiplied together do chance.

$$Total\ Cost = Unit\ Price \times Quantity$$

$$5\left(\frac{\text{dollars}}{\text{pound}}\right) \times p\ (\text{pounds}) = 5p\ (\text{dollars})$$

The pounds in the denominator cancel out pounds in the denominator of the second term, leaving dollars as final unit.

Units Conversion

When values with units are multiplied or divided, the units change.

A conversion factor is a fraction whose numerator and denominator have different units but the same value.

$$\frac{60\ \text{seconds}}{1\ \text{minute}}$$

Since the numerator and denominator are the same, multiplying by a conversion factor is multiplying by 1.

$$7\ \text{minutes} \times \frac{60\ \text{seconds}}{1\ \text{minute}} = 420\ \text{seconds}$$

Integer Constraints

Be aware of limitations placed upon variables. Some word problems will, by nature, restrict the possible values of variables.

The most common restriction is that variables must be integers. E.g., any variable that represents the number of cars, people, marbles, etc., must be an integer.

Common Relationships

Total Cost (\$) = Unit Price (\$/unit) × Quantity Purchased (units)

Profit (\$) = Revenue (\$) − Cost (\$)

Total Earnings (\$) = Wage Rate (\$/hr) × Hours Worked (hrs)

Miles = Miles per Hour × Hours

Miles = Miles per Gallon × Gallons

Basic Motion Problems

Rate × Time = Distance   Rate × Time = Work

Rate: expressed as ratio of distance and time. Always expressed as *distance over time*.

Time: expressed as unit of time.

Distance: expressed as unit of distance.

Rate Problems

Rate × Time = Distance *or* $RT = D$
Rate × Time = Work *or* $RT = W$

Rate: expressed as ratio of distance over time
Time: expressed as unit of time
Distance: expressed as unit of distance

RTD Chart:

	Rate (miles/hr)	×	Time (hr)	=	Distance (miles)
Car	30mi/hr	×		=	75mi

Multiple Rates

To solve questions with multiple $RT = D$ relationships:

X runs 30 mi at constant rate of 4 mph. Y goes the same distance at constant rate and finishes in 90 min fewer. How fast is X?

	Rate (miles/hr)	×	Time (hr)	=	Distance (miles)
X	4		t	=	30
Y	r		$t - 1.5$		30

Solve for t:

$$4t = 30 \rightarrow t = 7.5$$
$$r \times 6 = 30 \rightarrow r = 5$$

Relative Rates

Relative rate problems have two bodies traveling *at the same time*. Three scenarios: moving towards each other, away from each other, or moving in same direction on same path.

X is 40 miles west of Y, both traveling east. X is going 50% faster than Y. If both travel at a constant rate and it takes X 2 hrs and 40 min to catch up to Y, how fast is Y going?

	Rate (miles/hr)	×	Time (hr)	=	Distance (miles)
X	$1.5r$		$8/3$		d
Y	r		$8/3$		$d-40$

Create 3rd equation for rate that the distance between them changes: $0.5r \times {}^{8}/_{3} = 40 \quad \rightarrow \quad {}^{4}/_{3}\,r = 40 \quad \rightarrow \quad r = 30.$

Average Rate

If an object moves a distance at different rates, the average rate will <u>never</u> be the average of the rates. It will be closer to the slower of the rates because the object spends more time traveling slower.

X walks 4 mph, then walks the same route at 6 mph, what is X's average walking rate, round trip?

To find the average rate (weighted by time), find total combined *time* and *distance* and pick any number for the distance.

	Rate (miles/hr)	×	Time (hr)	=	Distance (miles)
Going	4	×	**3**	=	12
Return	6	×	**2**	=	12
Total	**?**	×	**5**	=	24

$$RT = D \quad \rightarrow \quad r(5) = 24 \quad \rightarrow \quad 4 = 4.8 \text{ mph}$$

Work Problems

Work problems are another type of rate problems: $RT = W$.

	Rate (miles/hr)	×	Time (hr)	=	Distance (miles)
Going	4	×	**3**	=	12
Return	6	×	**2**	=	12
Total	**?**	×	**5**	=	24

Averages

The average or (arithmetic) mean of a set is given by the formula:

$$\text{Average} = \frac{\text{Sum}}{\#\text{ of terms}}, \textbf{ abbreviated } A = \frac{S}{n}$$

Variation: $(\text{Average}) \times (\#\text{ of terms}) = (\text{Sum})$, or $A \cdot n = S$

Every GMAT average problem can be solved with one of these two formulas.

If asked to use or find average of a set, the individual terms do not matter. All that matters is the <u>sum</u> of the terms.

Complex Average Problems

Some average problems involve setting up two average formulas.

> $2,000 commission earned, raising average commission by $100. New average commission is $900, how many sales?

Use RTD-style table to keep track of two average formulas:

	Average	$\times$	Number	$=$	Sum
Old Total	800	$\times$	n	$=$	$800n$
This Sale	2000	$\times$	1	$=$	2000
New Total	900	$\times$	$n+1$	$=$	$900(n+1)$

Right column provides equation:

$$800n + 2000 = 900(n+1) \rightarrow 1100 = 100n \rightarrow 11 = n$$

Weighted Averages

Averages are the sum of weighted numbers. With standard averages, all terms are equally weighted:

$$\frac{20+30}{2} = \frac{20}{2} + \frac{30}{2} = \frac{1}{2}(20) + \frac{1}{2}(30) = 25$$

Different weights that sum to 1:

$$\frac{4}{5}(20) + \frac{1}{5}(30) = \frac{4 \times 20}{5} + \frac{30}{5} = 16 + 6 = 22$$

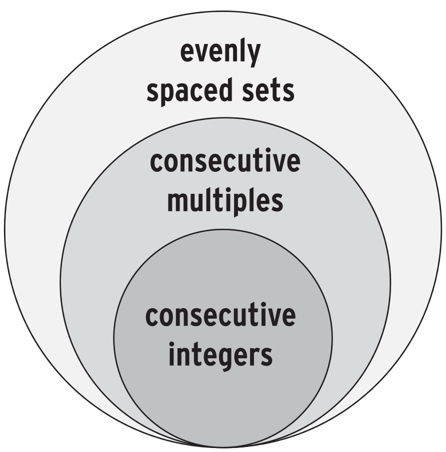
Essentially, the average of four 20's and one 30.

Evenly Spaced Sets

Evenly spaced sets go up or down by the same increment.

Consecutive multiples: evenly spaced set where all values in the set are multiples of the increment, e.g., $\{12, 16, 20, 24\}$.

Consecutive integers: consecutive multiples where all values increase by 1 and are multiples of 1, e.g., $\{12, 13, 14, 15, 16\}$.



Properties of Evenly Spaced Sets

1. The arithmetic mean (average) and median are equal.
2. The mean and median of the set are equal to the average of the <u>first</u> and <u>last</u> terms: $(\text{first} + \text{last}) \div 2$
3. The sum of the elements in the set equals the arithmetic mean (average) in the set times time the number of items in the set: $A \cdot n = S$

   This applies to all sets but is important in the case of evenly spaced sets because the "average" in also the median.

Counting Integers: Add One

When counting integers, be inclusive of extremes by adding:

$$(\text{Last} - \text{First} + 1)$$

For consecutive multiples, divide by the increment and add one:

$$(\text{Last} - \text{First}) \div \text{Increment} + 1$$

The bigger the increment, the smaller the result since there is a larger gap between numbers being counted.

Sum of Consecutive Integers

What is the sum of all integers from 20 to 100, inclusive?

Using the rules of evenly spaced sets, calculate with shortcuts:

- Average first and last term to find middle of the set: $100 + 20 = 120$ and $120 \div 2 = 60$. (*Median* equals *mean* for evenly spaced sets, essentially: $A \cdot n = S$.
- Count number of terms: $100 - 20 = 80$, plus 1 yields 81.
- Multiply the median by the number of terms to find sum: $60 \times 80 = 4{,}860$.

Average of an odd number of consecutive integers will always be an integer. The average on a even will never be an integer.

Double-Set Matrix

For problems with two sets of options, construct a double-set matrix. Rows should correspond to mutually exclusive options for one decision. Likewise, the columns should correspond to the mutually exclusive options for the other.

Of 30 integers, 15 are in set A, 22 in set B, and 8 are in both A and B. How many integers are in neither A nor B?

	A	Not A	Total
B	8	**14**	22
Not B	7	**1**	**8**
Total	15	**15**	30

Overlapping Sets & Percents

Double-set matrix is effective for overlapping sets with *percents* or *fractions*, especially if you pick a Smart Number for total. For percents, pick 100. For fractions, pick a common denominator.

70% are employees. 10% are women who are not employees. If half are men, what percent are female employees?

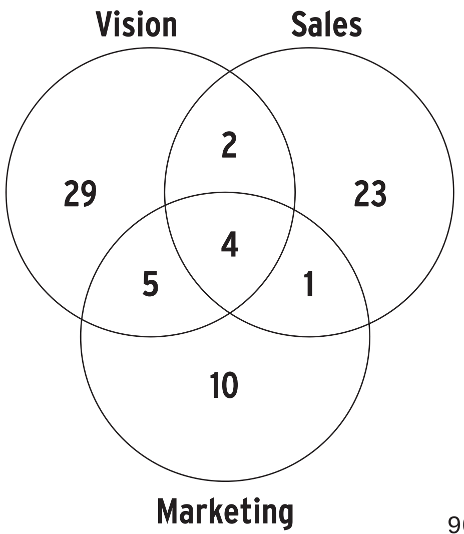
	Men	Women	Total
Employee	**30**	**40**	70
Not Employee	**20**	10	**30**
Total	50	50	100

WORD PROBLEMS > OVERLAPPING SETS

3-Set Problems: Venn Diagrams

To solve 3-set problems using a Venn Diagram, work from the inside out:

Workers are placed on a least one team. 20 on Marketing, 30 on Sales, and 40 on Vision. 5 on both Marketing and Sales, 6 on both Sales and Vision, 9 on both Marketing and Vision, and 4 on all three. How many total?



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WORD PROBLEMS > WORD PROBLEM STRATEGIES

Use Charts to Organize Variables

When a word problem involves several quantities and multiple relationships, use a chart or table to organize the information.

\$150,000 earned by selling 1,800 VIP and Standard tickets. 25% more Standard tickets than VIP. If revenue from Standard tickets represents 1/3 of total revenue, what is the price of a VIP ticket?

	Price per Ticket	×	Quantity	=	Revenue
Standard			$1.25x$		
VIP			x		
Total	-		1,800		100

Solve: $1.25x + x = 1{,}800 \rightarrow x = 800$. A VIP ticket is \$125.

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WORD PROBLEMS > EXTRA CONSECUTIVE INTEGERS

Products of Consecutive Integers & Divisibility

Factor Foundation Rule: every number is divisible by all factors of its factors.

For example, there is a multiple of 3 in any set of 3 consecutive integers, so the product of 3 consecutive integers is divisible by 3.

The product of k consecutive inters is always divisible by k factorial ($k!$).

$$1 \times 2 \times \mathbf{3} = 6 \qquad \mathbf{6} \times 7 \times 8 = 336$$

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WORD PROBLEMS > EXTRA CONSECUTIVE INTEGERS

Sums of Consecutive Integers & Divisibility

Find the sum of any 5 consecutive integers:

$$4+5+6+7+8=30 \qquad 13+14+15+16+17=75$$

Both sums are multiples of 5, i.e., divisible by 5. For any set of consecutive integers with an <u>odd</u> number of items, the sum of all integers is <u>always</u> a multiple of the number of items.

Find the sum of any 4 consecutive integers:

$$1+2+3+4=10 \qquad 8+9+10+11=38$$

For any set of consecutive integers with an <u>even</u> number of items, the sum is <u>never</u> a multiple of the number of items. I.e., the average of an even set of numbers is never an integer.

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