

Questions Based on Venn Diagrams (Sets)

On the your test, you are likely to see questions with 2 or 3 variables.

In case of 2 variables, there are a maximum of four divisions possible:

Imagine that at a B-school, applicants can choose Marketing and Finance among other specializations, where dual major is allowed. In this case, there can be only four types of sets of people possible:

1. Students taking Marketing Only
2. Students taking Finance Only
3. Students taking both Marketing and Finance
4. Students taking neither Marketing nor Finance

These types of questions are best solved by making a 2-way-matrix (table). Just remember that if one row has “Marketing”, the other has to have “Not Marketing”. If one column has “Finance”, the other has to have “Not Finance”.

	Finance	NOT FINANCE	TOTAL
Marketing	A	B	A + B
NOT Marketing	C	D	C + D
TOTAL	A + C	B + D	A + B + C + D

The table can be completed by putting all the given values in the question and thus the unknown can be found out by simple addition / subtraction of rows / columns. In case of percentage problems (where all values are mentioned in terms of percentage), it is better to take the total as 100.

Let us solve a practical example:

50% of the apartments in a certain building have windows and hardwood floors. 25% of the apartments without windows have hardwood floors. If 40% of the apartments do not have hardwood floors, what percent of the apartments with windows have hardwood floors?

10% 16.66% 40% 50% 83.33%

On your test, you can easily solve this problem in less than 2 minutes. But we illustrate here the entire mental process.

This problem involves two sets: Set 1: Apartments with windows / Apartments without windows. Set 2: Apartments with hardwood floors / Apartments without hardwood floors. It is easiest to organize two-set problems by using a matrix as follows:

	Windows	NO Windows	TOTAL
Hardwood Floors			
NO Hardwood Floors			
TOTAL			

Let's say that there are 100 total apartments in the building. This is the first number we can put into our matrix. The absolute total is placed in the lower right hand corner of the matrix as follows:

	Windows	NO Windows	TOTAL
Hardwood Floors			
NO Hardwood Floors			
TOTAL			100

Next, we will attack the complex wording by reading each piece of information separately, and filling in the matrix accordingly.

50% of the apartments in a certain building have windows and hardwood floors. Thus, 50 of the 100 apartments have BOTH windows and hardwood floors. This number is now added to the matrix:

	Windows	NO Windows	TOTAL
Hardwood Floors	50		
NO Hardwood Floors			
TOTAL			100

25% of the apartments without windows have hardwood floors. Here's where the subtlety of the wording is very important. This does NOT say that 25% of ALL the apartments have no windows and have hardwood floors. Instead it says that OF the apartments without windows, 25% have hardwood floors. Since we do not yet know the number of apartments without windows, let's call this number x . Thus the number of apartments without windows and with hardwood floors is $.25x$. These figures are now added to the matrix:

	Windows	NO Windows	TOTAL
Hardwood Floors	50	$.25x$	
NO Hardwood Floors			
TOTAL		x	100

40% of the apartments do not have hardwood floors. Thus, 40 of the 100 apartments do not have hardwood floors. This number is put in the Total box at the end of the “No Hardwood Floors” row of the matrix:

	Windows	NO Windows	TOTAL
Hardwood Floors	50	.25x	
NO Hardwood Floors			40
TOTAL		x	

Before answering the question, we must complete the matrix. To do this, fill in the numbers that yield the given totals. First, we see that there must be 60 total apartments with Hardwood Floors (since $60 + 40 = 100$) Using this information, we can solve for **x** by creating an equation for the first row of the matrix:

$$50 + 0.25X = 60 \text{ so } X = 40$$

Now we put these numbers in the matrix:

	Windows	NO Windows	TOTAL
Hardwood Floors	50	10	60
NO Hardwood Floors			40
TOTAL		40	100

Finally, we can fill in the rest of the matrix:

	Windows	NO Windows	TOTAL
Hardwood Floors	50	10	60
NO Hardwood Floors	10	30	40
TOTAL	60	40	100

Note: You don’t have to create 7 tables separately. You need to create only one.

We now return to the question: What percent of the apartments with windows have hardwood floors?

Again, pay very careful attention to the subtle wording. The question does NOT ask for the percentage of TOTAL apartments that have windows and hardwood floors. It asks what percent OF the apartments with windows have hardwood floors. Since there are 60 apartments with windows, and 50 of these have hardwood floors, the percentage is calculated as follows: $50/60 = 83.33\%$. Ans. E

In case of 3 variables, there are a maximum of eight divisions possible. So a table will become very complicated. So we will deal with such questions by drawing 3 overlapping circles:

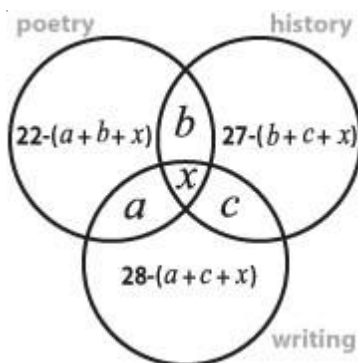
Each of the 59 members in a high school class is required to sign up for a minimum of one and a maximum of three academic clubs. The three clubs to choose from are the poetry club, the history club, and the writing club. A total of 22 students sign up for the poetry club, 27 students for the history club, and 28 students for the writing club. If 6 students sign up for exactly two clubs, how many students sign up for all three clubs?

2 5 6 8 9

This is a three-set overlapping sets problem. When given three sets, a Venn diagram can be used. The first step in constructing a Venn diagram is to identify the three sets given. In this case, we have students signing up for the poetry club, the history club, and the writing club. The shell of the Venn diagram will look like this:



We are told that the total number of poetry club members is 22, the total number of history club members is 27, and the total number of writing club members is 28. We can use this information to fill in the rest of the diagram:



We can now derive an expression for the total number of students by adding up all the individual segments of the diagram. The first bracketed item represents the students taking two or three courses. The second bracketed item represents the number of students in only the poetry club, since it's derived by adding in the total number of poetry students and subtracting out the poetry students in multiple clubs. The third and fourth bracketed items represent the students in only the history or writing clubs respectively.

$$59 = [a + b + c + x] + [22 - (a + b + x)] + [27 - (b + c + x)] + [28 - (a + c + x)] \quad \text{OR}$$

$$59 = a + b + c + x + 22 - a - b - x + 27 - b - c - x + 28 - a - c - x \quad \text{OR}$$

$$59 = 77 - 2x - a - b - c \quad \text{OR}$$

$$59 = 77 - 2x - (a + b + c)$$

By examining the diagram, we can see that $(a + b + c)$ represents the total number of students who sign up for two clubs. We are told that 6 students sign up for exactly two clubs. Consequently: $59 = 77 - 2x - 6$ or $x = 6$.

So, the number of students who sign up for all three clubs is 6. The correct answer is C.

PERCENTAGES

Definitions & Formulae:

Percent : The term percent means “for every hundred”. A fraction whose denominator is 100 is called percentage and the numerator of the fraction is called the rate percent. Thus, when we say a man made a profit of 20 percent we mean to say that he gained \$20 for every hundred dollars he invested in the business, i.e. 20/100 dollars for each dollar.

1. To express a% as a fraction divide it by 100 $\Rightarrow a\% = a/100$
2. To express a fraction as a percent multiply it by 100 $\Rightarrow a/b = [(a/b) \times 100]\%$
3. Increase % = $[\text{Increase} / \text{Original value}] \times 100\%$
Decrease % = $[\text{Decrease} / \text{Original value}] \times 100\%$
Change % = $[\text{Change} / \text{Original value}] \times 100\%$
4. Let the present population of a town be p and let there be an increase of X% per annum. Then :
(i) population after n years = $p [1 + (X/100)]^n$
(ii) population n years ago = $p / [1 + (X/100)]^n$
5. If the population of a town (or value of a machine) decreases at R% per annum, then :
(i) population (or value of machine) after n years = $p [1 - (R/100)]^n$
(ii) population (or value of machine) n years ago = $p / [1 - (R/100)]^n$

GENERAL CONCEPTS IN PERCENTAGES:

Let's start with a number X (= 1 X)

1. X increased by 10% would become $X + 0.1 X = 1.1 X$
2. X increased by 1% would become $X + 0.01 X = 1.01 X$
3. X increased by 0.1% would become $X + 0.001 X = 1.001 X$
4. X decreased by 10% would become $X - 0.1 X = 0.9 X$
5. X decreased by 1% would become $X - 0.01 X = 0.99 X$
6. X decreased by 0.1% would become $X - 0.001 X = 0.999 X$
7. X increased by 200 % would become $X + 2 X = 3 X$
8. X decreased by 300 % would become $X - 3 X = - 2X$
9. Gain % = $(\text{Gain} / \text{CP}) \times 100\%$
10. Loss % = $(\text{Loss} / \text{CP}) \times 100\%$

NOTE : In the problems on **DISCOUNT** remember the following:

1. Marked price is the price listed on the article (called list price).
2. Discount is calculated on Marked price and NOT on Cost price.

So Marked Price - Discount = Sale Price. Also Cost Price + Profit = Sale Price.

Examples (For Concept Building):

1. A child spends 30% of his pocket money, and has Rs 126 left. How much had he at first ?
Sol. The child spends 30% of the whole $\Rightarrow 70\%$ of the whole = Rs 126
 $\Rightarrow$ The child's original pocket money = $\text{Rs } 126 \times 100/70 = \text{Rs } 180$ Ans.

2. When the cost of petroleum increases by 40%, a man reduces his annual consumption by 20%. Find the percentage change in his annual expenditure on petroleum.

Sol.

First expenditure : Suppose 10 litres of petroleum at 10 units of money per litre, then total expenditure = 10×10 units of money = 100 units of money

Second expenditure : Now, 8 litres of petroleum at 14 units of money per litre, total expenditure = 8×14 units of money = 112 units $\Rightarrow$ Expenditure increases by 12% Ans.

Alt. Sol. : $\text{Exp}_1 = PX$, $\text{Exp}_2 = 1.4P(0.8X) = 1.12PX \Rightarrow$ Directly we see, answer = 12%

3. The number of students in a school increases at a certain rate per cent. The number at present is 1323 and the number two years ago was 1200; find the rate per cent of the increase.

Sol. By formula, we have $1200 \times (1 + R/100)^2 = 1323 \Rightarrow R = 5\%$

4. A tradesman marks his goods at 25 percent above cost price and allows discount of 12.5 percent for cash payment. What profit percent does he make ?

Sol. If the CP is 100, Marked Price = 125 But discount to the cash purchaser = 12.5 % of Rs 125 = Rs $15\frac{5}{8}$
Now the reduced price he gets from the cash purchaser = $125 - 15\frac{5}{8} = 109\frac{3}{8}$ i.e. Gain percent = $109\frac{3}{8} - 100 = 9\frac{3}{8}$
8

5. A hawker has a certain number of oranges, of which 2% are bad. He sells 95% of the remainder, and then has 49 oranges left. How many had he originally ?

Sol. Suppose he has X originally, then $(X - 0.02X - 0.95 \times 0.98X) = 49$ or $X = 1000$.

6. A reduction of 20% in the price of apples could enable a man to get 120 more for Rs 1,440. Find the first price of one apple.

Sol. We have $1440 = X \times Y$ (1) X = no. of apples Y = price of one apple.
Now $1440 = (X + 120) \times 0.8Y$ (2) From (1) and (2), $X = 480$ and $Y = \text{Rs } 3$

7. A man's working hours a day were increased by 25%, and his wages per hour were increased by 20%. By how much percent were his daily earnings increased ?

Sol. Let initially X be number of hours & Y = wages/hour
Later these become 1.25 X & 1.2 X respectively. Daily earnings initially = $X \times Y$
Now Daily earnings = $1.25X \times 1.2Y = 1.5XY$ Hence 50% increase.

8. A tradesman allows a discount of 15% on the marked price. How much above the C.P. must he mark his goods to make a profit of 19 % ?

Sol. Let CP = 100, Gain = 19, SP = $100 + 19 = 119$
Now Marked price should be such that Marked price reduced by 15% is equal to 119
or 85% of M.P. = 119 or $MP = 119 \times 100/85 = \text{Rs } 140 \Rightarrow$ Answer = 40 % above the C.P.

Percentages

The term **percentage** means **parts per 100** or "for every hundred". A fraction whose denominator is 100 is called percentage and the numerator of the fraction is called the rate percent. Thus, when we say a man made a profit of 20 percent we mean to say that he gained Rs.20 for every hundred rupees he invested in the business, i.e., 20/100 rupees for each Rupee. The abbreviation of percent is p.c. and it is generally denoted by %.

Let's start with a number X (= 1 X)

X increased by 10% would become $X + 0.1 X = 1.1 X$

X increased by 1% would become $X + 0.01 X = 1.01 X$

X increased by 0.1% would become $X + 0.001 X = 1.001 X$

X decreased by 10% would become $X - 0.1 X = 0.9 X$

X decreased by 1% would become $X - 0.01 X = 0.99 X$

X decreased by 0.1% would become $X - 0.001 X = 0.999 X$

X increased by 200% would become $X + 2X = 3X$

X decreased by 300% would become $X - 3X = -2X$

Similarly, you can work mentally with any specifically chosen number (say 500) and work out different answers.

Complete all entries in the following table.

Increased by Number	10 %	25 %	50 %	100 %	300 %
50	55				
75		93.75			
150			225		
500				1000	
600					2400
1000				2000	
2000			3000		
2500		3125			

Increased by Number	10%	25%	50%	100%	300%
50	55	62.5	75	100	200
75	82.5	93.75	112.5	150	300
150	165	187.5	225	300	600
500	550	625	750	1000	2000
600	660	750	900	1200	2400
1000	1100	1250	1500	2000	4000
2000	2200	2500	3000	4000	8000
2500	2750	3125	3750	5000	10000

A Percentage can be expressed as a Fraction. 10% can be expressed as $10/100$ or $1/10$. To express a percentage as a fraction divide it by 100

$$a \% = a/100.$$

To express a fraction as a percent multiply it by 100 $\Rightarrow \frac{a}{b} = \frac{a}{b} \times 100\%$

To express percentage as a decimal we remove the symbol % and shift the decimal point by two places to the left. For example, 10% can be expressed as 0.1. 6.5% = 0.065 etc.

To express decimal as a percentage we shift the decimal point by two places to the right and write the number obtained with the symbol % or simply we multiply the decimal with 100. Similarly $0.7 = 70\%$.

$$\text{Increase \%} = \frac{\text{Increase}}{\text{Original Value}} \times 100$$

$$\text{Decrease \%} = \frac{\text{Decrease}}{\text{Original Value}} \times 100$$

In increase %, the denominator is smaller, whereas in decrease %, the denominator is larger.

$$\text{Change \%} = \frac{\text{Change}}{\text{Original Value}} \times 100$$

Successive change in percentage. If a number A is increased successively by X% followed by Y%,

and then by Z%, then the final value of A will be $A \left(1 + \frac{X}{100}\right) \left(1 + \frac{Y}{100}\right) \left(1 + \frac{Z}{100}\right)$.

In a similar way, at any point or stage, if the value is decreased by any percentage, then we can replace the same by a negative sign. The same formula can be used for two or more successive changes. The final value of A in this case will be

$$A \left(1 - \frac{X}{100}\right) \left(1 - \frac{Y}{100}\right) \left(1 - \frac{Z}{100}\right) \dots \text{etc.}$$

Also, let us remember that:

2 = 200% (or 100% increase), 3 = 300% (or 200% increase), 3.26 = 326% (means 226% increase), fourfold (4 times) = 400 % of original = 300% increase, 10 times means 1000% means 900% increase, 0.6 means 60% of the original means 40% decrease, 0.31 times means 31% of the original means 69% decrease etc.

$$1/2 = 50\%, 3/2 = 1 + 1/2 = 100 + 50 = 150\%, 5/2 = 2 + 1/2 = 200 + 50 = 250\% \text{ etc.,}$$

$$2/3 = 1 - 1/3 = 100 - 33.33 = 66.66\%, 4/3 = 1 + 1/3 = 100 + 33.33 = 133.33\%, \\ 5/3 = 1 + 2/3 = 100 + 66.66 \% = 166.66\%, 7/3 = 2 + 1/3 = 200 + 33.33 = 233.33\%, \\ 8/3 = 2 + 2/3 = 200 + 66.66 = 3 - 1/3 = 300 - 33.33 = 266.66\% \text{ etc.}$$

$$1/4 = 25\%, 3/4 = 75\%, 5/4 (1 + 1/4) = 125\% (= 25 \text{ increase}), 7/4 (1 + 3/4 = 2 - 1/4) = 175\% (= 75\% \text{ increase}), 9/4 (2 + 1/4) = 225\% (= 125\% \text{ increase}), 11/4 = 275\% (= 175\% \text{ increase}).$$

$$1/5 = 20\%, 2/5 = 40\%, 3/5 = 60\%, 4/5 = 80\%, 6/5 = 120\%, 7/5 (1 + 2/5) = 140\% \text{ etc.}$$

$$1/6 = 16.66\%, 5/6 = 83.33\%, 7/6 (1 + 1/6) = 116.66\%, 11/6 = 183.33\%$$

$$1/8 = 12.5\%, 3/8 = 37.5\%, 5/8 = 62.5\%, 7/8 = 87.5\%, 9/8 = (1 + 1/8) = 112.5\%, 11/8 = (1 + 3/8) = 137.5\%, 13/8 = 162.5\%, 15/8 = 187.5\% \text{ etc.}$$

$$1/9 = 11.11\%, 2/9 = 22.22\%, 4/9 = 44.44\%, 5/9 = 55.55\%, 7/9 = 77.77\%, 8/9 = 88.88\%, 10/9 = 111.11\%, \\ 11/9 = (1 + 2/9) = 122.22\% \text{ etc.}$$

Problems

1. The production of a firm increases from 340 MT to 500 MT. What is the percent increase?

Sol. $160/340 = 8/17 = 47.05\%$ (direct).

2. The production of a firm decreases from 500 MT to 340 MT. What is the percent decrease?

Sol. $160/500 = 32\%$.

Note the answers to the above 2 questions are different.

$$\% \text{ increase} = \frac{\text{Change}}{\text{smaller value}} \times 100$$

$$\% \text{ decrease} = \frac{\text{Change}}{\text{larger value}} \times 100$$

3. The production of a firm increases by 20%, 25% and 50% in 3 successive years over the previous year. If the production is 160 MT in the first year, find the production at the end of 3 years.

$$\text{Sol. } 160 \times 1.2 \times 1.25 \times 1.5 = 160 \times \frac{6}{5} \times \frac{5}{4} \times \frac{3}{2} = 360$$

4. The production of a firm decreases by 20% in the first year, then decreases by 25% in the next year and then increases by 50% the next year and then increases by 10% in the next year. All percentage changes being consecutive (over the previous year). If at the end of the changes, the value is 990 MT, what was the value initially?

$$\text{Sol. } A \times 0.8 \times 0.75 \times 1.5 \times 1.1 = 990 \text{ or } A \times \frac{4}{5} \times \frac{3}{4} \times \frac{3}{2} \times \frac{11}{10} = 990 \Rightarrow A = 1000$$

5. Which is bigger: 0.004% of 25000 or 25000% of 0.004?

Sol. Both equal. **A % of B = B % of A.**

6. If price decreases by 25%, by what % should consumption increase so that the expenditure does not increase?

Sol. $25\% = 1/4$, so 4 was made 3. Now 3 has to be made 4, so $1/3 = 33.33\%$.

7. If speed increases by 33.33%, what is the percent reduction in the time taken to travel the same distance?

Sol. $33.33\% = 1/3$, so 3 was made 4. Now for 4 to be made 3, drop = $1/4 = 25\%$.

8. The price of a product increased by 20% but the turnover increased by only 12%. What is the % drop in quantity sales?

Sol. $100 \times 100 = 10000$ **(10 × 10 Funda)**

$$120 \times X = 11200.$$

So we have $X = 1120/12 = 280/3$. So we have $X = 93.33$, so drop = 6.66%.

The 10 × 10 Funda. Let us take a very simple question and try to work it out by different methods.

The length of a rectangle increases by 25%. Find the percent drop in its width for area to remain same.

Sol.

Method 1: 25% increase means 5/4. So we have to do 4/5 means 80%, so 20 drop.

Method 2: $a + b + \frac{ab}{100}$ gives $25 + (-b) + \frac{25(-b)}{100} = 0$ so $b = 20\%$.

Concept: The net percentage change when two variables are increased / decreased by given percentages, say a% and b% will be **$a + b + \frac{a \times b}{100}$**

Like in the example: If price and consumption each are increased by 20% and 25% respectively, then the total increase in expenditure will be

$$\text{New E} = 1.2P \times 1.25C = 1.5PC = 1.5E = 50\% \text{ increase.}$$

For the same data given above, applying the formula, we get the net percentage change in expenditure

$$= 20 + 25 + \frac{20 \times 25}{100} = 20 + 25 + 5 = 50\%.$$

In case a given value decreases by any percentage, we will use a negative sign before that.

In the same way: If the length of a rectangle is decreased by 40% and the breadth is increased by 30%, then find the percentage change in the area of the rectangle.

Area of rectangle = length × breadth. Here, both length and breadth are changed. So, using the

$$\text{formula, net percentage change in area} = -40 + 30 + \frac{(-40)(30)}{100} = -22, (-)\text{ve sign signifies decrease.}$$

Hence the area of the rectangle decreases by 22%.

The above formula can be used to find out the following:

Percentage effect on expenditure.

$$\text{Expenditure} = \text{Price} \times \text{Quantity (or Consumption)}$$

Percentage effect on area of Rectangle / Square / Circle etc.

$$\text{Area} = \text{Length} \times \text{Breadth}$$

Revenue calculation

$$\text{Revenue} = \text{Unit Sale} \times \text{Price per unit}$$

Distance

$$\text{Distance} = \text{Speed} \times \text{Time}$$

Anything that involves product of two quantities.

Method 3: 100 to 125. Now 125 to 100, so we have $25/125 = 1/5 = 20\%$.

Method 4: Using formula $\frac{100r}{100+r}\%$, we have $\frac{100 \times 25}{100+25} = 20\%$

Method 5: Suppose the length is L and width is B, we have $A = L \times B$ and also $A = 1.25L \times B_{\text{new}}$, so we get

$$B_{\text{new}} = \frac{L \times B}{1.25L} = 0.8B, \text{ so } 20\% \text{ decrease.}$$

Method 6: (10×10) . In this assume length and breadth are both 10. So we have

$$10 \times 10 = 100$$

$$12.5 \times B = 100 \text{ or } B = 8, \text{ so } 20\% \text{ drop.}$$

Of all the methods, 10×10 method gives you the best and the fastest results.

One more Example (of 10×10 method). The price of petrol increases by 40% but the final expenditure increases by 20% only. By what percent does the consumption decrease?

Sol. $10 \times 10 = 100$

$$14 \times X = 120.$$

So we get $X = 120/14 = 60/7$. So the drop = $10 - (60/7) = 10/7$. Now $1/7 = 14.28\%$ Answer.

OR Using fractions

40% jump means $7/5$ and 20% increase means $6/5$ so we get $7/5 \times X = 6/5$,

so

$$X = 6/7 = (1 - 1/7) = 14.28\% \text{ drop.}$$

22. The radius of a sphere (or edge of a cube or regular tetrahedron etc.) increases by 20%. Find the % increase in its area and volume.

Sol.

Percentage concepts in Geometry: For similar figures, on doubling the length, the area becomes 4 times and the volume 8 times. So we can say,

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 \text{ and } \frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3. \text{ So if there are questions of this kind, they can be very easily calculated.}$$

For area: $10 \times 10 = 100$

$$12 \times 12 = 144, \text{ so area increases by } 44\%.$$

For volume $10 \times 10 \times 10 = 1000$

$$12 \times 12 \times 12 = 1728, \text{ so } 72.8\% \text{ increase.}$$

23. The side of a square (or radius of a circle or side of an equilateral triangle) decreases by 30%. Find the % decrease in its perimeter and area.

Sol. Perimeter changes by the same percent as length so 30% decrease.

For area: $10 \times 10 = 100$

$$7 \times 7 = 49, \text{ so area decreases by } 51\%.$$

RATIO: The comparison between two quantities of the same kind of unit is the Ratio of one quantity to another. The ratio of a and b is usually written as $a : b$ or a/b , where a is called the antecedent (numerator) and b the consequent (denominator).

- $a : b = ka : kb$ where k is a constant
- $a : b = a/k : b/k$
- $a : b : c = X : Y : Z$ is equivalent to $a/X = b/Y = c/Z$
- $a : b > X : Y$ if $aY > bX$
- $a : b < X : Y$ if $aY < bX$
- If $a/b = c/d$, then
 - $(a + b)/b = (c + d)/d$
 - $(a - b)/b = (c - d)/d$
 - $(a + b)/(a - b) = (c + d)/(c - d)$

7. If $a/b = c/d = e/f = \dots = k$, then $\frac{a + c + e + \dots}{b + d + f + \dots} = k$.

Also note that : $k = a/b = Xc/Xd = -5e/-5f$

$\Rightarrow$ Each ratio $= (a + Xc - 5e)/(b + Xd - 5f) = k$.

Here, we have randomly taken X and -5. You can take any factor.

8. If $a/b > 1$ or $a > b$ then $(a + X)/(b + X) < a/b$ a, b, X are natural numbers

9. If $a/b < 1$ or $a < b$ then $(a + X)/(b + X) > a/b$ a, b, X are natural numbers

VARIATION :

1. Direct proportion :

If two quantities X & Y are related such that any increase or decrease in 'Y' produces a proportionate increase or decrease in 'X' or vice versa, then the two quantities are said to be in direct proportion.

In other words
or

$$X : Y = X/Y = k \text{ (a constant)}$$

$$X = KY \text{ or } Y = K'X \quad (\text{where K and K' are constants})$$

X is directly proportional to Y is written as $X \propto Y$ or $X = K Y$

2. Inverse proportion :

Here two quantities X & Y are related such that, any increase in X would lead to a decrease in Y or any decrease in X would lead to an increase in Y. Thus the quantities X & Y are said to be inversely related and X is inversely proportional to Y is written as

$$X \propto 1/Y \text{ or } X = k/Y \text{ or } XY = k \text{ (constant) or the product of two quantities remains constant.}$$

MIXTURES FORMULA

This rule enables us to find the proportion in which two or more ingredients at the given price must be mixed to produce a mixture at a given price.

The C.P. of unit quantity of the mixture is called the MEAN PRICE.

The rule says : If two ingredients are mixed in a ratio, then

$$\frac{\text{Quantity of Cheaper}}{\text{Quantity of Dearer}} = \frac{\text{CP of Dearer} - \text{Mean Price}}{\text{Mean Price} - \text{CP of Cheaper}}$$

1. In what ratio should tea @ 35/kg be mixed with tea @ 27/kg so that mixture may cost Rs. 30/kg ?

Sol. Quantity of cheaper / Quantity of dearer = $(35 - 30) / (30 - 27) = 5/3$

Hence the two should be mixed in the ratio 5 : 3.

2. Find $a : b : c$ if $6a = 9b = 10c$.

Sol. $a/b = 9/6 = 3 : 2$ $= 15 : 10$, $b/c = 10/9 = 10 : 9$. Hence $a : b : c = 15 : 10 : 9$.

3. A's present age is to be B's as 8 : 5; and 20 years ago it was as 12 : 5. Find the present age of each.

Sol. Let A's present age = $8X$ $\Rightarrow$ B's present age = $5X$

$\Rightarrow (8X - 20) / (5X - 20) = 12 / 5$ Solving this, we get : $X = 7$

$\Rightarrow$ A's age = $8X = 56$ yrs, B's age = $5X = 35$ yrs

4. An alloy contains 24% of tin by weight. How much more tin to the nearest kg must be added to 100 kg of the alloy so that the % of tin may be doubled ?

Sol. Let X kg of tin be added to the alloy.

Tin (kg)	Alloy (kg)
24	100
24 + X	100 + X

$$\Rightarrow (24 + X) / (100 + X) = 2 (24/100) \Rightarrow X = 46 \quad \text{Hence 46 kg of tin must be added to the alloy.} \quad \text{Ans.}$$

5. The expenses of a hotel consist of two parts. One part varies with the number of inmates, while the other is constant. When the number of inmates is 200 & 250, the expenses are respectively Rs. 1300 & Rs. 1600. Then find the expenses for 300 inmates.

Sol. Let $E = K_1 X + K_2$, where K_1 & K_2 are constants, E stands for expenses, X for the number of inmates. When $X = 200$ & $E = 1300$ we have $\rightarrow [200 K_1 + K_2 = 1300]$. When $X = 250$ & $E = 1600$ we have $\rightarrow [250 K_1 + K_2 = 1600]$. Solving the equations we have $K_1 = 6$ & $K_2 = 100 \Rightarrow E = 6X + 100$. Now when $X = 300$, $E = 6 \times 300 + 100 = \text{Rs } 1900$ Ans.

6. Two tins A and B contain mixtures of wheat and rice. In A, the weights of wheat and rice are in the ratio 2 : 3 and in B they are in the ratio 3 : 7. What quantities must be taken from A and B to form a mixture containing 5 kg of wheat and 11 kg of rice?

Sol. Let X kg of mixture be taken from A, then $(16 - X)$ kg is taken from B

$\Rightarrow 2X/5$ kg of wheat from A and $3(16 - X)/10$ kg of wheat from B is to be taken.

Now we have, $2X/5 + 3(16 - X)/10 = 5$ or $X = 2$ kg $\Rightarrow 2$ kg from A and 14 kg from B.

7. Two vessels contain mixtures of spirit and water. In the first vessel the ratio of spirit to water is 8 : 3 and in the second vessel the ratio is 5 : 1. A 35-litre cask is filled from these vessels so as to contain a mixture of spirit and water in the ratio of 4 : 1. How many litres are taken from the first vessel ?

Sol. Let X litres be taken from the first vessel; then $(35 - X)$ litres are taken from the second. In the first vessel $8/11$ of the mixture, and in the second vessel $5/6$ of the mixture, is spirit $\Rightarrow$ The spirit in the 35-litre cask is $4/5$ of the mixture $\Rightarrow 8/11 X + 5/6 (35 - X) = 4/5 \times 35 \Rightarrow X = 11 \Rightarrow 11$ litres are taken from the first vessel Ans.

8. A bag contains \$ 600 in the form of one-dollar, 50 cents & 25-cents coins in the ratio 3 : 4 : 12. Find the number of 25 cents coins.

Sol. Ratio of values of coins = $3/1 : 4/2 : 12/4 = 3 : 2 : 3$. Value of 25 cents coins = $\text{Rs } 600 \times 3/(3 + 2 + 3) = 225$. $\Rightarrow$ No. of 25 cents coins = $225 \times 4 = 900$ Ans.

Alternate method :

Assume that the number of 1 \$ coins is 3X. Then the value equation would be $3X + 4X(0.50) + 12X(0.25) = 600$. Find X and get answer = 12 X.

9. A mixture contains milk & water in the ratio 5 : 1. on adding 5 litres of water, the ratio of milk and water becomes 5 : 2. Find the quantity of milk in the original mixture.

Sol. Let the quantity of milk be 5X & that of water X. Then $5X / (X + 5) = 5/2$ or $X = 5 \Rightarrow$ Quantity of milk = $5X = 25$ litres

10. The ratio of the number of boys to the number of girls in a school of 546 is 4 : 3. If the number of girls increases by 6, what must be the increase in the number of boys to make the new ratio of boys to girls 3 : 2 ?

Sol. Original no. of boys = $546 \times 4/7 = 312$. Original no. of girls = $78 \times 3 = 234$. Final no. of girls = $234 + 6 = 240$
 $\Rightarrow$ No. of boys reqd. to make the new ratio = $240 \times 3/2 = 360 \Rightarrow$ The reqd. increase in the no. of boys = $360 - 312 = 48$ Ans.

11. In what ratio should 30% milk be mixed with 50% milk to get a 35% milk strength?

Sol. From mixture rule we have :

$$\text{Qty. of 30\%} / \text{Qty. of 50\%} = (50 - 35) / (35 - 30) = 3 : 1 \quad \text{Ans.}$$

12. Two numbers are in the ratio of 3 : 4. If 5 is subtracted from each, the resulting numbers are in the ratio 2 : 3. Find the numbers.

Sol. Let $3X$ and $4X$ be the numbers $\Rightarrow (3X - 5) / (4X - 5) = 2/3 \Rightarrow 9X - 15 = 8X - 10 \Rightarrow X = 5 \Rightarrow$ The reqd numbers are 15 and 20 Ans.

Work / Rate:

FUNDAMENTAL CONCEPTS:

1. If a man can do a piece of work in N days (or hours or any other unit of time), then the work done by him in one day will be $1/N$ of the total work.
2. If A is twice as good a workman as B , then A will take half the time B takes to finish a piece of work.
3. If A and B can do a piece of work in X & Y days respectively while working alone, they will together take $XY / (X + Y)$ days to complete it.
4. If A, B, C can do a piece of work in X, Y, Z days respectively while working alone, they will together take $XYZ / [XY + YZ + ZX]$ days to finish it.
5. If an inlet pipe can fill a cistern in X hours, the part filled in 1 hour = $1/X$
6. If an inlet pipe can fill a tank in X hours and an outlet pipe empties the full tank in Y hours, then the net part filled in 1 hour when both the pipes are opened = $(1/X) - (1/Y) \Rightarrow$ In 1 hour, the part filled (or emptied) = $1/X - 1/Y \Rightarrow$ Time required to fill or empty the tank = $XY / (X \sim Y)$ hours. $X \sim Y$ indicates $[X - Y]$ or $[Y - X]$, whichever is positive).

1. If A and B together finish a piece of work in 10 days & B alone can finish it in 20 days. In how many days can A alone finish the work ?

Sol. Let X and Y be the number of days required by A and B respectively.

$\Rightarrow$ By the standard formula, $XY / (X + Y) = 10$ & $Y = 20$

$\Rightarrow X \times 20 / (X + 20) = 10$ or $X = 20$ days Ans.

2. Four men working together all day, can finish a piece of work in 11 days; but two of them having other engagements can work only one half-time and quarter time respectively. How long will it take them to complete the work ?

Sol. Each man will take $11 \times 4 = 44$ days to complete the work. If one man works half day/day he will take $44 \times 2 = 88$ days to finish the work. Similarly, a man working quarter day/day will take $44 \times 4 = 176$ days to finish the work. When these work together they will require $1 / [(1/44) + (1/44) + (1/88) + (1/176)] = 16$ days.

3. 20 men can complete a piece of work in 10 days, but after every 4 days 5 men are called off, in what time will the work be finished ?

Total work = 20×10	=	200 monday
1. First 4 days' work	=	$20 \times 4 = 80$ md
2. Next 4 days' work	=	$15 \times 4 = 60$ md
3. Next 4 days' work	=	$10 \times 4 = 40$ md
4. Next 4 days' work	=	<u>$5 \times 4 = 20$</u>
		$\Sigma = 200$ md
Hence, days reqd	=	$4 + 4 + 4 + 4 = 16$

4. A vessel can be filled by one pipe A in 10 minutes, by a second pipe B in 15 minutes. It can be emptied by a waste pipe C in 9 minutes. In what time will the vessel be filled if all the three were turned on at once ?

Sol. We will follow exactly the same method as in time & work. The part of vessel filled in 1 minute when all three are on = $1/10 + 1/15 - 1/9 = 1/18 \Rightarrow$ Total vessel will be filled in 18 minutes Ans.

5. Three pipes A, B and C can fill a cistern in 15, 20 and 30 min resp. They were all turned on at the same time. After 5 minutes the first two pipes were turned off. In what time will the cistern be filled ?

Sol. A, B and C can fill $(1/15 + 1/20 + 1/30)$ or $3/20$ of the cistern in 1 minute $\Rightarrow A, B$ and C filled $(3/20 \times 5)$ or $3/4$ of the cistern in 5 min. Now A and B are turned off $\Rightarrow (1 - 3/4)$ or $1/4$ of the cistern will be filled by $C \Rightarrow C$ will fill $1/4$ of the cistern in $(30 \times 1/4)$ or 7.5 minutes $\Rightarrow$ The cistern will be filled in $7.5 + 5$ or 12.5 min. Ans.

6. A cistern can be filled by two taps A and B in 12 minutes and 14 minutes respectively and can be emptied by a third in 8 minutes. If all the taps are turned on at the same moment, what part of the cistern will remain unfilled at the end of 7 minutes ?

Sol. We have $(7/12) + (7/14) - 7/8 = 5/24$ part filled in 7 minutes. Hence $1 - 5/24 = 19/24$ th of the tank is unfilled.

Time / Speed / Distance

1. Speed:

The rate at which anything covers a particular distance is called its speed

$\Rightarrow \text{Speed} = \text{Distance Travelled} / \text{Time Taken}$

Generally speed is expressed in the following units : miles / hr, km/hr, m/sec, m/min, etc.

- If A goes from X to Y at U km/hr and comes back from Y to X at V km/hr, then Average speed during the whole journey = $2UV / (U + V)$ km/hr.
- If a man changes his speed in the ratio m : n then the ratio of times taken becomes n : m.
- If three men cover the same distance with speeds in the ratio a : b : c, the times taken by these three will be resp. in the ratio $1/a : 1/b : 1/c$.
- When two objects travel in the same direction, relative speed = difference of speeds and time to catch / overtake = lead distance / difference of speeds
- When two objects travel in the Opposite directions, relative speed = sum of speeds and time to meet = lead distance / sum of speeds.
- If the speed of a boat (or man) in still water be X km/hr, and the speed of the stream (or current) be Y km/hr, then
 - Speed of boat with the stream (or Downstream or D/S) = $(X + Y)$ km/hr
 - Speed of boat against the stream (or upstream or U/S) = $(X - Y)$ km/hr

We have $X = [(X + Y) + (X - Y)] / 2$ and $Y = [(X + Y) - (X - Y)] / 2$
 $\Rightarrow \text{Boat's speed in still water} = [\text{Speed downstream} + \text{Speed upstream}] / 2$
 $\text{Speed of current} = [\text{Speed downstream} - \text{Speed upstream}] / 2$

1. A policeman goes after a thief who is 176 m ahead of him. When and where will the policeman catch the thief when they run at the rates of 11440 and 10560 meters per hour respectively?

Sol. Time to catch / overtake = lead distance / difference of speeds = $176 / (11440 - 10560) = 176 / 880 = 1/5$ hours = 12 minutes $\Rightarrow$ The time required to overtake the thief = 12 min. Ans.

- (b) The distance from the starting point = $11440 \times 12/60$ kms = 2288 meters

2. If I walk at the rate of 4 kms an hour, I reach my destination 30 min too late; If I walk at the rate of 5 kms an hour I reach 30 minutes too soon. How far is my destination ?

Sol. Let time taken be T hrs for the distance to be covered at the normal speed (neither fast nor slow). Then we have $4(T + 0.5) = 5(T - 0.5)$ {Note : 0.5 here is 30 min} $\Rightarrow T = 4.5$ hours $\Rightarrow \text{Distance} = 4(T + 0.5) = 4 \times 5 = 20$ kms. Ans.

3. A man rows 18 kms down a river in 4 hours with the stream and returns in 12 hours; find his speed and also the velocity of the stream.

Sol. Speed with the stream = $18/4 = 4.5$ kms an hour. $\Rightarrow$ Speed against the stream = $18/12 = 1.5$ kms an hour. $\Rightarrow$ Speed of the stream = $1/2 (4.5 - 1.5) = 1.5$ kms an hour and his speed = $4.5 - 1.5 = 3$ kms an hour Ans.

4. A, B and C can walk at the rates of 3, 4, 5 kms an hour. They start from X at 1, 2, 3 o'clock respectively; when B catches up with A, B sends him back with a message to C; when will C get the message ?

Sol. In one hour A covers 3 kms Now $(B - A) = 1$ kmph hence B catches up A after 3 hours i.e. at 5 o'clock. Now upto 5 o'clock A has covered 12 kms and C has covered 10 kms. Hence Distance between A & C = 2 kms and their relative speed $(3 + 5) = 8$ kmph. To cover 2 kms at 8 kmph, Time = $2/8$ hours = 15 min. Hence C gets the message at 15 minutes past 5 o'clock. Ans.

5. A student walks to school at the rate of 2.5 kms an hour and reaches 6 minutes too late. Next day he increases his speed by 2 kms an hour and then reaches there 10 minutes too soon. Find the distance of the school from his home.
Sol. Let t be the usual time We have $\Rightarrow 2.5 \times (t + 1/10) = 4.5 (t - 1/6)$, or $t = 1/2$ hours. Hence distance = $2.5 (1/2 + 1/10) = 2.5 \times 6/10 = 1.5$ kms.
6. A man can row in still water a distance of 4 kms in 20 minutes and 4 kms with the current in 16 min. How long will it take him to row the same distance against the current ?
Sol. $X = 4 / (20/60) = 12$, $X + Y = 4 \times 60 / 16 = 15$ or $Y = 3$.
 or Time = $4 / (X - Y) = 4 \times 60 / 9 = 80/3$ minutes Ans.

PROBLEMS ON DIGITS

In these problems the student should carefully understand the difference between the **DIGIT VALUE** (or **FACE VALUE** or **ABSOLUTE VALUE**) and **LOCAL VALUE** (or **PLACE VALUE**) of the digits forming a number.

Thus the face value of the digits of the number 789 are 7, 8 and 9 resp. But the local values are 700, 80 and 9 respectively.

The following table will make the difference clear :

100's digit	10's digit	Unit's digit	Number Formed
	5	4	$5 \times 10 + 4 = 54$
9	7	8	$9 \times 100 + 7 \times 10 + 8 = 978$
	X	Y	$X \times 10 + Y = 10X + Y$
	Y	X	$Y \times 10 + X = 10Y + X$
X	Y	Z	$100X + 10Y + Z$

Translating Word Problems into equations:

1. Find two consecutive odd numbers the difference of whose squares is 296.
Sol. Let the numbers be $2X + 1$ and $2X + 3$ Then $(2X + 3)^2 - (2X + 1)^2 = 296 \Rightarrow X = 36$ Hence $2X + 1 = 2 \times 36 + 1 = 73$ and $2X + 3 = 75 \Rightarrow$ The required numbers are 73 and 75.
 [Verification. $(75)^2 - (73)^2 = 5625 - 5329 = 296$]
2. A is 29 years older than B, B is 3 years older than C and D is 2 years younger than C. Two years hence A's age will be twice the united ages of B, C and D. Find their present ages.
Sol. Let D's age be = X years Then C's age = $(X + 2)$ years, B's age = $(X + 5)$ years and A's age = $(X + 34)$ years. Two yrs hence A's, B's, C's and D's ages will be $X + 36$, $X + 7$, $X + 4$ and $X + 2$ years respectively. $\Rightarrow 2 (X + 2 + X + 4 + X + 7) = X + 36 \Rightarrow X = 2 \Rightarrow$ A's age = 36 yrs; B's age = 7 yrs, C's age = 4 yrs; D's age = 2 yrs.
3. A number consists of three consecutive digits, that in the unit's place being the greatest of the three. The number formed by reversing the digits exceeds the original number by 22 times the sum of the digit. Find the number.
Sol. Let the hundred's digit be X . Then the ten's digit = $X + 1$ and the unit's digit = $X + 2$
 $\Rightarrow$ The number = $100 \times X + 10(X + 1) + X + 2 = 111X + 12$.
 The number formed by reversing the digits = $100(X + 2) + 10(X + 1) + X$
 $= 111X + 210 \Rightarrow 111X + 210 - 111X - 12 = 22 (X + 2 + X + 1 + X) \Rightarrow X = 2$. Hence the required number = 234.
4. The crew of a boat can row at the rate of 5 miles an hour in still water. If to row 24 miles, they take 4 times as long as to row the same distance down the river, find the speed at which the river flows.
Sol. Let X miles per hour be the speed of the river. Hence, on equating the times, we get: $24/(5 - X) = 4 \times 24/(X + 5)$
 $\Rightarrow X = 3$ Thus, the river flows at the rate of 3 miles an hour.

5. The area of a rectangle remains the same if the length is increased by 7 metre and the breadth is decreased by 3 metres. The area remains unaffected if the length is decreased by 7 m and breadth is increased by 5 m. Find the dimensions of the rectangle.

Sol. Let the length of the rectangle be X m and breadth of the rectangle = Y m. Area = XY sq. m.

I Case : Length = $(X + 7)$ m and breadth = $(Y - 3)$ m

$$\Rightarrow \text{Area} = (X + 7)(Y - 3) \text{ sq. m.} \Rightarrow (X + 7)(Y - 3) = XY \text{ or } -3X + 7Y - 21 = 0 \dots (1)$$

II Case : Length = $(X - 7)$ m and breadth = $(Y + 5)$ m $\Rightarrow (X - 7)(Y + 5) = XY$

$$\text{or } 5X - 7Y - 35 = 0 \dots (2) \Rightarrow Y = 15 \text{ and } X = 28$$

Hence length = 28 m and breadth = 15 m Answer.

Note : Assume $L = 28$ and $B = 15$ as an option and try checking the conditions given in the problem. You will see that working backwards is exceptionally fast in such cases.

6. The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them saves Rs. 200 per month, find their monthly incomes.

Sol. Let the monthly income of first person be Rs $9X$ and the monthly income of second person be Rs $7X$. Let the expenditure of first person be $4Y$ and the expenditure of second person be $3Y$. $\Rightarrow$ Saving of the first person = Rs $(9X - 4Y)$ and solving of second person = Rs $(7X - 3Y)$. Using the given informations, we have : $9X - 4Y = 200$ (1) and $7X - 3Y = 200$ (2) $X = 200$

Hence, the monthly income of first person = Rs. $9 \times 200 =$ Rs. 1800 and the monthly income of second person = Rs. $7 \times 200 =$ Rs. 1400 Ans.

7. Find two consecutive even numbers such that $1/6$ th of the greater exceeds $1/10$ th of the smaller by 29.

Sol. Let the numbers be $2X$ and $2X + 2$ Then $(2X + 2)/6 - 2X/10 = 29$.

$$\Rightarrow X = 21 \text{ Hence } 2X = 42 \text{ and } 2X + 2 = 44. \Rightarrow \text{The required numbers are 42 and 44.}$$

[Verification. $44/6 - 42/10 = 7\frac{2}{3} - 4\frac{1}{5} = 29$]

8. A number consists of two digits whose sum is 12. The ten's digit is three times the unit's digit. What is the number?

Sol. Let the unit's digit be X , Then the ten's digit is $12 - X$. $\Rightarrow 3X = 12 - X \Rightarrow X = 3$

Hence the number is 93. [Verification. $9 = 3 \times 3$; and $9 + 3 = 12$]

9. A train travelled a certain distance at a uniform rate. Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less; and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance.

Sol. Let us suppose that X miles per hour is the speed of the train and Y hours is the time taken for the journey. $\Rightarrow$ Distance travelled = $XY = (X + 6)(Y - 4) = (X - 6)(Y + 6)$ This gives two simultaneous equations. Solving, we get : $X = 30$, $Y = 24 \Rightarrow$ Distance = $XY = 720$ miles Ans.

10. A sum of money was divided equally among a certain number of persons; had there been six more, each would have received a rupee less, and had there been four fewer, each would have received a rupee more than he did; find the sum of money and the number of men.

Sol. Let X be the number of persons and Rs Y be the share of each. Then by the conditions of the problem, we have $(X + 6)(Y - 1) = XY$ (1) $(X - 4)(Y + 1) = XY$ (2). Thus the number of person is $X = 24$ and the share of each is $Y =$ Rs 5. The sum of money = $5 \times$ Rs 24 = Rs 120.

MEAN

1. Avg or mean or AM = Sum of n quantities (or numbers) / number of them (n) OR
2. Arithmetic Mean (A.M) is given by $\bar{X} = \frac{\sum x}{N}$
3. Mean of the Combined Series If n_1 and n_2 are the sizes and $\bar{X}_1, \bar{X}_2$ are the respective means of two series then the mean M of the combined series of size $n_1 + n_2$ given by $M = \frac{M_1 N_1 + M_2 N_2}{N_1 + N_2}$ or we can write:

$$\frac{N_1}{N_2} = \frac{M_2 - M}{M - M_1} = \frac{D_2}{D_1} \dots \text{this is the most important result.}$$

4. If each one of the given numbers is increased (or decreased) by K, their average is increased (or decreased) by k.
5. If each one of some given numbers is multiplied by K, their average is multiplied by K.
6. If a man (or train or boat or bus) covers some journey from A to B at X km/hr (or m/sec) and returns to A at a uniform speed for Y km/hr, then the average speed during the whole journey is $[2XY / (X + Y)]$ km/hr.

TIP: The average speed in such a case will be a bit less than the simple average.

7. If there are r series of observations $N_1, N_2, \dots, N_r$, the mean M of the whole series is related to the mean $M_1, M_2, \dots$ of the component series by the equation $NM = N_1 M_1 + N_2 M_2 + \dots + N_r M_r$.
8. Weighted Arithmetic Average is given as $\bar{X}_w = \frac{\sum wx}{\sum w}$
9. The sum of first “n” natural numbers is given by $n(n + 1)/2$.

10. For consecutive integers or for equally spaced numbers (AP), Mean = (First term + Last term) / 2.

11. Count of consecutive numbers inclusive = last term - first term + 1, Example 9 to 15, total = 7.
12. Count of consecutive numbers exclusive (terms greater than x but less than y) = last term - first term - 1.
Example: 9 to 15, total = 5
13. If the average of a few consecutive integers is 0, then either all numbers are zero or there will be an odd number of integers.
14. The average of an odd number of consecutive integers is an integer and the average of an even number of consecutive integers is a non-integer.
15. If in a set of numbers, the average = the highest or the lowest number, all the numbers will have to be equal.

MEDIAN

1. Median is the middle value or the average of two middle values when the **values are arranged in an order**, either ascending or descending.

2. If there are odd number of observations, median is directly the middle number.
3. If there are even number of observations, median is the average of the two middle numbers.
4. **For consecutive integers or for equally spaced numbers (AP), Median = (First term + Last term) / 2.**

So, Median = Mean in this case.

Ex. (i) The following are the marks of 9 students in a class. Find the median

34, 32, 48, 38, 24, 30, 27, 21, 35

(ii) Find the median of the daily wages of ten workers.

Rs. 20, 25, 17, 18, 8, 15, 22, 11, 9, 14

Sol. (i) Arranging the data in ascending order of magnitude, we have 21, 24, 27, 30, 32, 34, 35, 38, 48. Since, there are 9, an odd number of items, therefore median is 32.

(ii) Arranging the wages in ascending order of magnitude, we have 8, 9, 11, 14, 15, 17, 18, 20, 22, 25. Since, there are 10 observations, therefore median is the arithmetic mean of 15 and 17. So median = $(15 + 17) / 2 = 16$.

Ex. The number of students absent in a school was recorded every day for 147 days and the raw data was presented in the form of the following frequency table.

No. of students absent :	5	6	7	8	9	10	11	12	13	15	18	20
No. of days :	1	5	11	14	16	13	10	70	4	1	1	1

Obtain the median and describe what information it conveys.

Sol. Calculation of median

x_i	f_i	CF
5	1	1
6	5	6
7	11	17
8	14	31
9	16	47
10	13	60
11	10	70
12	70	140
13	4	144
15	1	145
18	1	146
20	1	147

We have $N = 147$. So $N/2 = 147/2 = 73.5$. The cumulative frequency just greater than $N/2$ is 140 and the corresponding value of x is 12. Hence, median = 12. This means that for about half the number of days, more than 12 students were absent.

Mode is the value which occurs most frequently in a set of observations

In case more than one values occurs most frequently, all those values are called the modes.

Range. It is defined as the difference between the two extreme observations of the distribution.

$$\text{Range} = X_{\max} - X_{\min}$$

where $X_{\max}$ is the greatest observation and $X_{\min}$ is the smallest observation of the variable value.

If Range = 0, all the observations are equal.

Standard deviation. It is defined as positive square root of the A.M. of the squares of the deviations of the given observations from their A.M. If $X_1, X_2, \dots, X_N$ is a set of N observations then its standard deviation is given by

Standard Deviation $\sigma = \sqrt{\frac{\sum d^2}{N}}$ or $\sigma = \sqrt{\frac{1}{N} \sum (x - \bar{x})^2}$. It is a measure of how much each value varies from the mean of all the values.

Less SD implies more consistency, less variation, less spread, more compactness AND vice versa.

If SD = 0, all the observations are equal.

Range is always greater than SD, except when all observations are equal, when both are equal to 0.

Precise relation between SD and range: $SD \leq \frac{\text{Range}}{2}$

	Addition	Subtraction	Sign Change	Multiplication	Division
Mean	Change	Change	Change	Change	Change
Median	Change	Change	Change	Change	Change
Mode	Change	Change	Change	Change	Change
Range	NO Change	NO Change	NO Change	Change	Change
SD	NO Change	NO Change	NO Change	Change	Change

Ex. Find the S. D. of the set of the numbers 3, 4, 9, 11, 13, 6, 8 and 10.

Sol. Here we have

Numbers (x)	$x - \bar{x}$	$(x - \bar{x})^2$
3	-5	25
4	-4	16
9	1	1
11	3	9
13	5	25
6	-2	4
8	0	0
10	2	4

$$\sum x = 64$$

$$\sum (x - \bar{x})^2 = 84$$

Here N = number of items = 8. $\therefore \bar{x}$ = arithmetic mean = $\Sigma x / N = 64/8 = 8$.

$$\therefore \sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{N}} = \sqrt{\frac{84}{8}} = \sqrt{10.5} = 3.24$$

INEQUALITIES

Let a and b be real numbers. If $a - b$ is negative we say that a is less than b and write $a < b$. If $a - b$ is positive then a is greater than b , i.e., $a > b$.

Properties of Inequalities.

- For any two real numbers a and b , we have $a > b$ or $a = b$ or $a < b$.
- If $a > b$ and $b > c$, then $a > c$. If $a > b$ then $(a + c) > (b + c)$ and $(a - c) > (b - c)$, however, $ac > bc$ and $(a/c) > (b/c)$ (not sure) (is true only when c is positive)
- If $a > b$, then $a + m > b + m$, for any real number m .
- If $a \neq 0$, $b \neq 0$ and $a > b$, then $1/a < 1/b$.
- If $a > b$, then $am > bm$ for $m > 0$ and $am < bm$ for $m < 0$, **that is, when we multiply both sides of inequality by a negative quantity, the sign of inequality is reversed.**
- If $a > X$, $b > Y$, $c > Z$ then
 (1) $a + b + c + \dots > X + Y + Z + \dots$ (2) $abc \dots > XYZ \dots$ (Provided none is negative)
- If $x > 0$ and $a > b > 0$, then $a^x > b^x$.
- If $a > 1$ and $x > y > 0$, then $a^x > a^y$.
- If $0 < a < 1$ and $x > y > 0$, then $a^x < a^y$.
- Do not cancel anything from both sides of an inequality unless you are sure that the cancelled quantity is positive, so $ax > ay$ does not necessarily mean $x > y$, etc.
- The concept of number line is very useful in checking inequalities. The common values to check are $x = 0, 1, -1, >1$ (preferred value = 2), between 0 and 1 (preferred value = 1/2), between -1 and 0 (preferred value = -1/2), and less than -1 (preferred value = -2). So in short, there are 7 points: -2, -1, -1/2, 0, 1/2, 1, 2.
- $|x|$ is defined as the non-negative value of x and hence is never negative. $\sqrt{x^2} = |x|$, that means, the square root of any quantity is defined to be non-negative, so $\sqrt{36} = 6$ and not -6 . BUT if $x^2 = 36 \Rightarrow x = 6$ or -6 both. So $\sqrt{x^2} = x$ or $-x$ both are possible. If x is negative, then $\sqrt{x^2} = -x$ as it has to be +ve eventually. In this case x is negative and $-x$ is positive.**
- $|5| = 5$, $|-5| = 5$, so $|x| = x$, if x is positive or 0 and $|x| = -x$ if x is negative.
- If $|x| > x$, then x is negative.
- If $|x| = a$, then $x = a$ or $x = -a$.
- If $|x| > a$, then $x > a$ or $x < -a$.
- If $|x| < a$, then $x < a$ or $x > -a$.
- If $|x - a| > b$, then either $x - a > b$ or $x - a < -b$

19. If $|x - a| < b$, then either $x - a < b$ or $x - a > -b$.
20. If $|x| = x$, then x is either positive or 0.
21. $|a + b| \leq |a| + |b|$, $|a - b| \geq ||a| - |b||$, $|ab| = |a| |b|$, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$, $|a^2| = a^2$.
22. If $(x - a)(x - b) < 0$, then x lies between a and b . OR $a < x < b$.
23. If $(x - a)(x - b) > 0$, then x lies outside a and b . OR $x < a$, $x > b$.
24. If $x^2 > x$, then either $x > 1$ or x is negative ($x < 0$)
25. If $x^2 < x$, then x lies between 0 and 1. ($0 < x < 1$)
26. If $x^2 = x$, then $x = 0$ or $x = 1$.
27. If $x^3 > x$, then either $x > 1$ or x is between -1 and 0 (either $x > 1$ or $-1 < x < 0$).
28. If $x^3 < x$, then either x lies between 0 and 1 or x is less than -1 . (either $0 < x < 1$ or $x < -1$)
29. If $x^3 = x$, then $x = 0$ or $x = 1$ or $x = -1$.
30. If $x^3 = x$, then $x = 0$ or $x = 1$ or $x = -1$.
31. If $x > y$, it is not necessary that $x^2 > y^2$ or $\sqrt{x} > \sqrt{y}$ etc. So even powers can't be predicted.
32. If $x > y$, it is necessarily true that $x^3 > y^3$ or $\sqrt[3]{x} > \sqrt[3]{y}$ etc. So odd powers and roots don't change sign.
33. $ab > 0$ means $a/b > 0$ and vice versa. The two are of the same sign.
34. $ab < 0$ means $a/b < 0$ and vice versa. The two are of the opposite sign.
35. If x is positive, $x + \frac{1}{x} \geq 2$.
36. If X is positive, then

(1) $(a + X) / (b + X) > a/b$ if $a < b$	(2) $(a + X) / (b + X) < a/b$ if $a > b$
--	--
37. If X is negative, then

(1) $(a + X) / (b + X) > a/b$ if $a > b$	(2) $(a + X) / (b + X) < a/b$ if $a < b$
--	--
38. $(a + c + e + \dots) / (b + d + f + \dots)$ is less than the greatest and greater than the least of the fractions a/b , c/d , e/f ,

NUMBERS

Numbers are basically of two types : 1. Real and 2. Imaginary

For the purpose of the your test, all numbers are real.

REAL numbers are basically of two types :

1. **Rational numbers :** A rational number can always be represented by a fraction of the form p/q where p and q are integers and $q \neq 0$. Examples : finite decimal numbers, infinite repeating decimals, whole numbers, integers, fractions i.e.
 $3/5, 16/9, 0.666 \dots \infty = 2/3$ etc.
2. **Irrational numbers :** Any number which can not be represented in the form p/q where p and q are integers and $q \neq 0$ is an irrational number. AN INFINITE NON-RECURRING DECIMAL IS AN IRRATIONAL NUMBER. Examples – $\sqrt{2}, \pi, \sqrt{5}, \sqrt{7}$.
3. **Prime numbers :** A natural number which has no other factors besides itself and unity is a prime number. Examples – 2,3,5,7,11,13,17,19
If a number has no factor equal to or less than its square root, then the number is prime. This is a test to judge whether a number is prime or not.

NOTE :

1. The only even prime number is 2
2. 1 is neither prime nor composite (by definition)

Composite numbers : A composite number has other factors besides itself and unity. e.g. 8, 72, 39 etc. Alternatively, we might say that a natural natural number that is not prime is a composite number.

NATURAL NUMBERS : The numbers 1,2,3,4,5 are known as natural numbers. The set of natural numbers is denoted by N . Hence $N = \{1, 2, 3, 4 \dots\}$

WHOLE NUMBERS : The numbers 0,1,2, 3,4, $W = \{0, 1, 2, 3, \dots\}$

INTEGERS : The set of Integers $I = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

EVEN NUMBERS : The numbers divisible by 2 are even numbers. Eg. 0, 2, 4, 6, 8, 10 Even numbers are expressible in the form $2n$ where n is an integer other than zero. **Thus -2, -6 etc. are also even numbers.**

ODD NUMBERS : The numbers not divisible by 2 are odd numbers e.g. 1, 3, 5, 7, 9 Odd numbers are expressible in the form $(2n + 1)$ where n is an integer other than zero (not necessarily prime). Thus, -1, -3, -9 etc. are all odd numbers.

TESTS FOR DIVISIBILITY:

1. A number is divisible by 2 if its unit's digit is even or zero e.g. 128, 146, 34 etc.
2. A number is divisible by 3 if the sum of its digits is divisible by 3 e.g. 102, 192, 99 etc.
3. A number is divisible by 4 when the number formed by last two right hand digits is divisible by '4' e.g. 576, 328, 144 etc.
4. A number is divisible by 5 when its unit's digit is either five or zero : e.g. 111535, 3970, 145 etc.
5. A number is divisible by 6 when it's divisible by 2 and 3 both. e.g. 714, 509796, 1728 etc.

6. Divisibility by 7 : No test upto three digits.
The rule which holds good for numbers with more than 3 digits is as follows :
(a) Group the numbers in three from unit digit
(b) Add the odd groups and even groups separately
(c) The difference of the odd and even groups should be divisible by 7
e.g. 85437954 The groups are 85, 437, 954
Sum of odd groups = $954 + 85 = 1039$
Sum of even groups = 437
Difference = 602 which is divisible by 7 hence the number is divisible by 7.
7. A number is divisible by 8 when the number formed by the last three right hand digits is divisible by '8'. e.g. 512, 4096, 1304 etc.
8. A number is divisible by 9 when the sum of its digits is divisible by 9 e.g. 1287, 11583, 2304 etc.
9. A number is divisible by 10 when its units digit is zero. e.g. 100, 170, 10590 etc.
10. A number is divisible by 11 when the difference between the sums of digits in the odd and even places is either zero or a multiple of 11. e.g. 17259, 62468252, 12221 etc. For the number 17259 : Sum of digits in even places = $7 + 5 = 12$, Sum of digits in the odd places = $1 + 2 + 9 = 12$ Hence $12 - 12 = 0$.
11. A number is divisible by 12 when it is divisible by 3 & 4 both. e.g. 672, 8064 etc.
12. Divisibility by 13 : The rules are same as that of 7 with 13 replacing 7.
Ex. Test the divisibility of the following numbers by 13 –
(i) 17653792 Ans. (i) divisible (ii) 46343323 Ans. (ii) divisible.
13. A number is divisible by 25 when the number formed by the last two Right hand digits is divisible by 25. e.g. 1025, 3475, 55550 etc.
14. A number is divisible by 125, when the number formed by last three right hand digits is divisible by 125. e.g. 2125, 4250, 6375 etc.

NOTE :

- When any number with even number of digits is added to its reverse, the sum is always divisible by 11. e.g. $2341 + 1432 = 3773$ which is divisible by 11.
- If X is a prime number then for any whole number "a" $(a^X - a)$ is divisible by X e.g.
Let $X = 3$ and $a = 5$. Then according to our rule $5^3 - 5$ should be divisible by 3.
Now $(5^3 - 5) = 120$ which is divisible by 3.

HCF (GCD / GCF) & LCM OF NUMBERS

HCF : It is the greatest factor common to two or more given numbers. It is also called GCF OR GCD (greatest common factor or greatest common divisor). e.g. HCF of 10 & 15 = 5, HCF of 55 & 200 = 5, HCF of 64 & 36 = 4

To find the HCF of given numbers, resolve the numbers into their prime factors and then pick the common term(s) from them and multiply them if more than one. This is the required HCF.

LCM : Lowest common multiple of two or more numbers is the smallest number which is exactly divisible by all of them.

e.g. LCM of 5, 7, 10 = 70, LCM of 2, 4, 5 = 20, LCM of 11, 10, 3 = 330

To find the LCM resolve all the numbers into their prime factors and then pick all the quantities (prime factors) but not more than once and multiply them. This is the LCM.

NOTE:

1. $\text{LCM} \times \text{HCF} = \text{Product of two numbers}$ (valid only for “two”)
 2. $\text{HCF of fractions} = \text{HCF of numerators} \div \text{LCM of denominators}$
 3. $\text{LCM of fractions} = \text{LCM of numerators} \div \text{HCF of denominators}$
- Q. Find the LCM of 25 and 35 if their HCF is 5. A. $\text{LCM} = 25 \times 35/5 = 175$

Calculating LCM :**Method 1 : Factorization Method**

Rule – After expressing the numbers in terms of prime factors, the LCM is the product of highest powers of all factors.

Q. Find the LCM of 40, 120, 380.

A. $40 = 4 \times 10 = 2 \times 2 \times 2 \times 5 = 2^3 \times 5^1$,
 $120 = 4 \times 30 = 2 \times 2 \times 2 \times 5 \times 3 = 2^3 \times 5^1 \times 3^1$
 $380 = 2 \times 190 = 2 \times 2 \times 95 = 2 \times 2 \times 5 \times 19 = 2^2 \times 5^1 \times 19^1$
 $\Rightarrow \text{Required LCM} = 2^3 \times 5^1 \times 3^1 \times 19^1 = 2280.$

Calculating HCF :**Factorization**

After expressing the numbers in term of the prime factors, the HCF is product of COMMON factors.

Ex. Find HCF of 88, 24, 124

$$\begin{aligned} 88 &= 2 \times 44 = 2 \times 2 \times 22 = 2 \times 2 \times 2 \times 11 = 2^3 \times 11^1 \\ 24 &= 2 \times 12 = 2 \times 2 \times 6 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3^1 \\ 124 &= 2 \times 62 = 2 \times 2^1 \times 31^1 = 2^2 \times 31^1 \end{aligned} \Rightarrow \text{HCF} = 2^2$$

DECIMAL FRACTIONS

Recurring Decimals (Conversion to a Rational Number): If in a decimal fractions a figure or a set of figures is repeated continually, then such a number is called a recurring decimal.

If a single figure is repeated, it is shown by putting a dot on it. Also, if a set of figures is repeated, we express it by putting on dot at the starting digit and one dot at the last digit of the repeated set.

(i) $2/3 = 0.6666 \dots = 0.\dot{6} = 0.\overline{6}$ (ii) $22/7 = 3.142857142857 = 3.\dot{1}4285\dot{7}$

Rule : Write the recurring figures only one in the numerator and take as many nines in the denominator as the number of repeating figures.

Ex. (1) $0.\dot{6} = 6/9 = 2/3$ (2) $0.\dot{2}3\dot{4} = 234/999$ (3) $3.\dot{5} = 3 \frac{5}{9}$ (4) $2.\dot{0}3\dot{5} = 2 \frac{35}{999}$

Examples of Rounding Decimals:	Number	Nearest tenth	Nearest hundredth	Nearest thousandth
	1.2346	1.2	1.23	1.235
	31.6479	31.6	31.65	31.648
	9.7462	9.7	9.75	9.746

Whether a fraction will result in a terminating decimal or not? To determine this, express the fraction in the lowest form and then express the denominator in terms of Prime Factors. If the denominator contains powers of only 2 and 5, it is finite (terminating). If the denominator contains any power of any other prime number, it is infinite (non-terminating).

If p is a prime number, then $1 + (p - 1)!$ is divisible by p .

Example. $16! + 1$ i.e., $(17-1)! + 1$ is divisible by 17.

21. The fifth power of any single digit number has the same right hand digit as the number itself.
22. For any integer n , $n^3 - n$ is divisible by 3, $n^5 - n$ is divisible by 5, $n^{11} - n$ is divisible by 11, $n^{13} - n$ is divisible by 13. **In general, if p is a prime number then for any whole number a , $a^p - a$ is divisible by p .**

Factor Theorem : If $f(x)$ is completely divisible by $(x - a)$, then $f(a) = 0$. So, $(x - a)$ is a factor of $f(x)$, $f(a) = 0$

Check whether $(x + 1)$ is a factor of $f(x) = 4x^2 + 3x - 1$. Putting $x + 1 = 0$, i.e., $x = -1$ in the given expression we get $f(-1) = 0$. So, $(x + 1)$ is a factor of $f(x)$.

Remainder Theorem : If an expression $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$.

Let $f(x) = x^3 + 3x^2 - 5x + 4$ be divided by $(x - 1)$. Find the remainder.

Remainder = $f(1) = 1^3 + 3 \times 1^2 - 5 \times 1 + 4 = 3$.

Some properties of square numbers:

- A square number cannot end with 2, 3, 7, 8 or an odd number of zeroes.
 - The square of an odd number is odd.
 - The square of an even number is even.
 - Every square number is a multiple of 3, or exceeds a multiple of 3 by unity.
 - Every square number is a multiple of 4 or exceeds a multiple of 4 by unity.
 - If a square number ends in 9, the preceding digit is even.
11. When any number with even number of digits is added to its reverse, the sum is always divisible by 11.
e.g. $2341 + 1432 = 3773$ which is divisible by 11.
 12. If X is a prime number then for any whole number " a " $(a^X - a)$ is divisible by X e.g. Let $X = 3$ and $a = 5$. Then according to our rule $5^3 - 5$ should be divisible by 3. Now $(5^3 - 5) = 120$ which is divisible by 3.
- The product of r consecutive integers is divisible by $r!$
 - If p is a prime number and n is prime to p , then $n^{p-1} - 1$ is divisible by p .
Example. $(99^6 - 1)$ i.e., $(99^{7-1} - 1)$ is divisible by 7.
 - If p is a prime number, then $1 + (p - 1)!$ is divisible by p .
Example. $16! + 1$ i.e., $(17-1)! + 1$ is divisible by 17.
 - Whenever an even power of $(n - 1)$ is divided by n the remainder is always 1 and whenever an odd power of $(n - 1)$ is divided by n the remainder is always $n - 1$.

Power of a Prime Number in a Factorial. If we have to find the power of a prime number p in $n!$, it is found using a general rule, which is

$$\left[\frac{n}{p} \right] + \left[\frac{n}{p^2} \right] + \left[\frac{n}{p^3} \right] + \dots, \text{ where } \left[\frac{n}{p} \right] \text{ denotes the greatest integer less than or equal to } \left[\frac{n}{p} \right] \text{ etc.}$$

$$\begin{aligned} \text{For example power of 3 in } 100! &= \left[\frac{100}{3} \right] + \left[\frac{100}{3^2} \right] + \left[\frac{100}{3^3} \right] + \left[\frac{100}{3^4} \right] + \left[\frac{100}{3^5} \right] + \dots \\ &= 33 + 11 + 3 + 1 + 0 = 48. \end{aligned}$$

$$\begin{aligned} \text{For example power of 5 in } 200! &= \left[\frac{200}{5} \right] + \left[\frac{200}{5^2} \right] + \left[\frac{200}{5^3} \right] + \dots \\ &= 40 + 8 + 1 + 0 = 49. \end{aligned}$$

Number of Zeroes at the end of a Factorial. It is given by the power of 5 in the number.

Actually, the number of zeroes will be decided by the power of 10, but 10 is not a prime number, we have $10 = 5 \times 2$, and hence we check power of 5.

$$\text{For example, the number of zeroes at the end of } 100! = \left[\frac{100}{5} \right] + \left[\frac{100}{5^2} \right] + \dots = 20 + 4 = 24.$$

$$\text{The number of zeroes at the end of } 500! = \left[\frac{500}{5} \right] + \left[\frac{500}{5^2} \right] + \left[\frac{500}{5^3} \right] + \dots = 100 + 20 + 4 = 124.$$

$$\begin{aligned} \text{The number of zeroes at the end of } 1000! &= \left[\frac{1000}{5} \right] + \left[\frac{1000}{5^2} \right] + \left[\frac{1000}{5^3} \right] + \left[\frac{1000}{5^4} \right] + \dots = 200 + 40 + 8 + 1 \\ &= 249. \end{aligned}$$

Indices

- We define,

$$a^m = \underbrace{a.a.a \dots a}_{(m \text{ times})}$$

Here a is called the base and m the exponent, index or power.

Laws of Indices

- If a, b are two positive real numbers and m, n are two real numbers then
 - $a^m \cdot a^n = a^{m+n}$
 - $a^m / a^n = a^{m-n}$
 - $(a^m)^n = a^{mn}$
 - $(ab)^m = a^m \cdot b^m$
 - $(a/b)^m = a^m / b^m$
 - $a^0 = 1, a \neq 0$
 - $a^{-m} = 1/a^m, a \neq 0$
 - $a^m = a^n \Rightarrow a = 0 \text{ or } 1 \text{ or } -1 \text{ or } m = n$
 - $a^m = b^m \Rightarrow m = 0 \text{ or } a = b$
 - $\sqrt[m]{ab} = \sqrt[m]{a} \sqrt[m]{b}$
 - $\sqrt[m]{a} = a^{1/m}$
 - $a^{p/q} = \sqrt[q]{a^p}$

Ex 1. What will be the unit's digit in 128^{96} ?

Sol. The unit's digit in $(128)^1 = 8$
 The unit's digit in $(128)^2 = 4$ ($8 \times 8 = 64$)
 The unit's digit in $(128)^3 = 2$ ($4 \times 8 = 32$)
 The unit's digit in $(128)^4 = 6$ ($2 \times 8 = 16$)
 The unit's digit in $(128)^5 = 8$ ($6 \times 8 = 48$)
 The unit's digit in $(128)^{10} = \text{Unit's digit in } (128)^5 \times \text{Unit's digit in } (128)^5 = 4$ ($8 \times 8 = 64$)
 The unit's digit in $(128)^{20} = \text{Unit's digit in } (128)^{10} \times \text{Unit's digit in } (128)^{10} = 6$ ($4 \times 4 = 16$)
 The unit's digit in $(128)^{40} = \text{Unit's digit in } (128)^{20} \times \text{Unit's digit in } (128)^{20} = 6$ ($6 \times 6 = 36$)
 The unit's digit in $(128)^{80} = \text{Unit's digit in } (128)^{40} \times \text{Unit's digit in } (128)^{40} = 6$ ($6 \times 6 = 36$)
 The unit's digit in $(128)^{90} = \text{Unit's digit in } (128)^{80} \times \text{Unit's digit in } (128)^{10} = 4$ ($6 \times 4 = 24$)
 The unit's digit in $(128)^{95} = \text{Unit's digit in } (128)^{90} \times \text{Unit's digit in } (128)^5 = 2$ ($4 \times 8 = 32$)
 The unit's digit in $(128)^{96} = \text{Unit's digit in } (128)^{95} \times \text{Unit's digit in } (128)^1 = 6$ ($2 \times 8 = 16$)
 $\Rightarrow$ Required answer = 6

Ex 2. A number X when divided by 289 leaves 18 as the remainder. The same number when divided by 17 leaves Y as the remainder. Find Y .

Sol. We know that : $\text{DIVIDEND} = (\text{DIVISOR} \times \text{QUOTIENT}) + \text{REMAINDER}$
 $\Rightarrow X = (289 \times K) + 18 = (17 \times 17 \times K) + 18$
 $= (17 \times 17 \times K) + (17 \times 1) + 1 = 17(17 \times K + 1) + 1$
 $= 17(\text{New Quotient}) + 1 \Rightarrow Y = 1$ Answer.

Ex 14. Find the value of :

$$\frac{1}{(216)^{-2/3}} + \frac{1}{(256)^{-3/4}} + \frac{1}{(243)^{-1/5}}$$

Sol. We have $216 = 6^3$, $256 = 4^4$, $243 = 3^5$
 Hence the above expression $= (6^3)^{2/3} + (4^4)^{3/4} + (3^5)^{1/5}$ $[1/a^{-m} = a^m] = 6^2 + 4^3 + 3^1 = 103$ Ans.

Ex 17. Find the largest number in $\sqrt[4]{4}$, $\sqrt[3]{3}$ and $\sqrt{2}$

Sol. In this we can compare by taking the LCM of roots and we see that $\text{LCM} = 4 \times 3 = 12$
 Hence we have $\sqrt[4]{4} = \sqrt[12]{4^3} = \sqrt[12]{64}$, $\sqrt[3]{3} = \sqrt[12]{3^4} = \sqrt[12]{81}$
 $\sqrt{2} = \sqrt[12]{2^6} = \sqrt[12]{64} \Rightarrow$ Obviously $\sqrt[3]{3}$ is the largest.

Ex 7. An orderly has a number of pegs to peg in a row. At first he tried to peg 5 in each row, then 6, then 8 and then 12, but had always 1 left. On trying 13 he had none left. What is the smallest number of pegs that he could have had ?

Sol. LCM of 5, 6, 8, and 12 = 120 $\Rightarrow$ Let the number be $120P + 1$ (because 1 was always left)
 $\Rightarrow \text{Number} = (13 \times 9 + 3)P + 1 = (13 \times 9)P + 3P + 1$ which is divisible by 13
 $\Rightarrow 3P + 1$ has to be divisible by 13. The least value of P that makes it divisible by 13 is $P = 4$
 $\Rightarrow \text{Number} = 120P + 1 = 120 \times 4 + 1 = 481$

Ex 13. A number $1568X35Y$ is divisible by 88. Find X & Y .

Sol. As the no. is divisible by 8×11 , the last 3 digits must be divisible by 8 or $35Y$ should be 352 or $Y = 2$. To be divisible by 11, the differences of the sum of its digits in the even and odd places must be either zero or a multiple of 11 $\Rightarrow$ Adding the digits of even and odd places, we get $2 + 3 + 8 + 5 = 18$ & $5 + X + 6 + 1 = 12 + X$. To make the difference zero $X + 12 = 18$ or $X = 6$. Hence $X = 6$, $Y = 2$ Ans.

Ex 18. A heap of stones can be made up exactly into groups of 25. But when made into groups of 18, 27 and 32, there are always 11 left. Find the least number of stones that may be contained in such a heap.

Sol. The LCM of 18, 27, 32 = 864. Hence the required number must be of the form $(864K + 11)$ such that it is divisible by 25. The least such value of $K = 1$. Hence the required number = 875.

Ex 20. Solve for X : $|3X - 6| < 9$

Sol. Since the absolute value of a number denotes both +ve & -ve values, then either

(a) $3X - 6 < 9$ or (b) $-(3X - 6) < 9$ or $-3X + 6 < 9$ or $-3X < 3$

$\Rightarrow$ (a) $3X < 15 \Rightarrow X < 5$

(b) $-3X < 3 \Rightarrow X > -1 \Rightarrow$ Combining, we get : $-1 < X < 5$ Ans.

Ex 22. Find the greatest fraction out of $2/5, 5/6, 11/12, 7/8$

Sol. We have LCM of 5, 6, 12, 8 = 120

Hence the numbers are $48/120, 100/120, 110/120, 105/120$

Hence $110/120$ or $11/12$ is the largest Ans.

Ex 24. Find the value of :

$$[1 / (\sqrt{9} - \sqrt{8})] - [1 / (\sqrt{8} - \sqrt{7})] + [1 / (\sqrt{7} - \sqrt{6})] - [1 / (\sqrt{6} - \sqrt{5})] + [1 / (\sqrt{5} - \sqrt{4})]$$

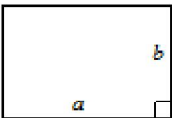
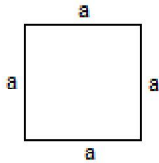
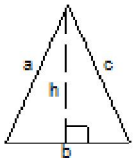
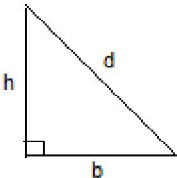
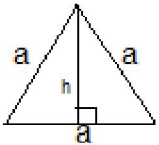
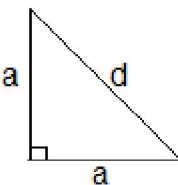
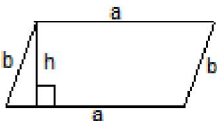
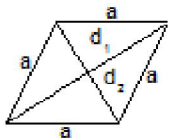
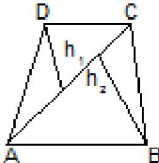
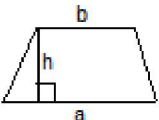
Sol. By rationalization we have :

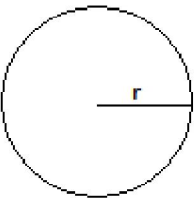
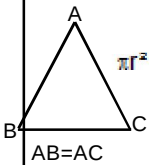
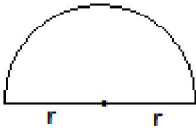
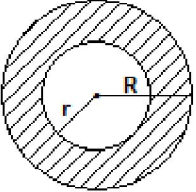
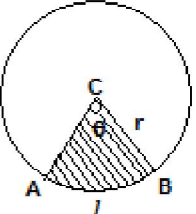
$$1/(\sqrt{9} - \sqrt{8}) = [1/(\sqrt{9} - \sqrt{8})] \times [(\sqrt{9} + \sqrt{8}) / (\sqrt{9} + \sqrt{8})] = (\sqrt{9} + \sqrt{8}) / (9 - 8) = \sqrt{9} + \sqrt{8}$$

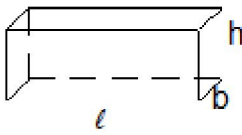
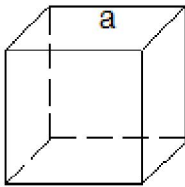
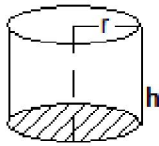
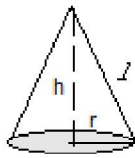
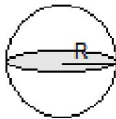
Similarly $1/(\sqrt{8} - \sqrt{7}) = (\sqrt{8} + \sqrt{7})$ etc. Hence the given expression

$$= (\sqrt{9} + \sqrt{8}) - (\sqrt{8} + \sqrt{7}) + (\sqrt{7} + \sqrt{6}) - (\sqrt{6} + \sqrt{5}) + (\sqrt{5} + \sqrt{4}) = \sqrt{9} + \sqrt{4} = 3 + 2 = 5 \quad \text{Ans}$$

GEOMETRY (FROMULA SHEET)

Name	Figure	Perimeter	Area	Nomenclature
Rectangle		$2(a + b)$	ab	a = length b = breadth
Square		$4a$	a^2	a = side
Triangle		$a + b + c = 2s$	$\frac{1}{2} b \times h$ $\sqrt{s(s-a)(s-b)(s-c)}$	b is the base and h is the altitude. a, b, c are three sides of Δ . s is the semiperimeter
Right triangle		$b + h + d$	$\frac{1}{2} bh$	d (hypotenuse) $= \sqrt{b^2 + h^2}$
Equilateral triangle		$3a$	$\frac{1}{2} ah$ $\frac{\sqrt{3}}{4} a^2$	a = side h = altitude $= \frac{\sqrt{3}}{2} a$
Isosceles right triangle		$2a + d$	$\frac{1}{2} a^2$	d (hypotenuse) $= a\sqrt{2}$, a = Each of equal sides.
Parallelogram		$2(a + b)$	ah	a = side b = side adjacent to a h = distance between the parallel sides (a) and (b)
Rhombus		$4a$	$\frac{1}{2} d_1 d_2$	a = side of rhombus d_1, d_2 are the two diagonals.
Quadrilateral		Sum of its four sides	$\frac{1}{2} (AC) (h_1 + h_2)$	AC is one of its diagonals and h_1, h_2 are the altitudes on AC from D, B respectively.
Trapezium		Sum of its four sides	$\frac{1}{2} h (a + b)$	a, b are parallel sides and h is the distance between parallel sides.

Circle		$2\pi r$	 AB=AC	$r = \text{radius of the circle}$ $\pi = 22/7 \text{ or } 3.1416$
Semicircle		$\pi r + 2r$	$\frac{1}{2} \pi r^2$	$r = \text{radius of the circle}$
Ring (shaded region)		---	$\pi (R^2 - r^2)$	$R = \text{outer radius}$ $r = \text{inner radius}$
Sector of a circle		$l + 2r$ where $l = \theta/360 \times 2\pi r$	$\theta/360 \times \pi r^2$	$\theta^\circ = \text{central angle of the sector}$ $r = \text{radius of the sector}$ $l = \text{length of the arc}$

S. No.	Name	Figure	Lateral/curved surface area	Total surface area	Volume	Nomenclature
1.	Cuboid		$2h(l + b)$	$2(lb + bh + lh)$	lbh	l =length b =breadth h =height
2.	Cube		$4a^2$	$6a^2$	a^3	a = edge
3.	Right circular cylinder		$2\pi rh$	$2\pi r(r+h)$	$\pi r^2 h$	r -radius of base h =height of the cylinder
	Right circular cone		πrl	$\pi r(l + r)$	$\frac{1}{3}.\pi r^2 h$	h = height r =radius l =slant height
	Sphere		$4\pi r^2$	$4\pi r^2$	$\frac{4}{3}.\pi r^3$	r = radius

Throughout this chapter –

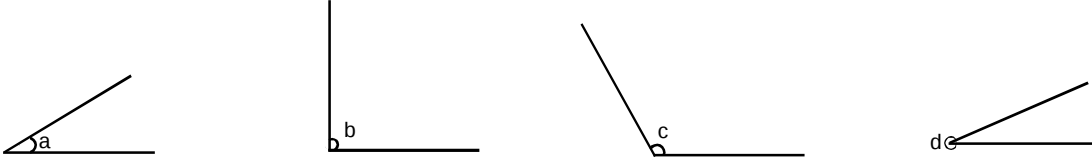
A = area,
V = volume,

P = perimeter,
H = Height,

C = circumference
hyp = Hypotenuse

ANGLES and PARALLELS

(a) Two straight lines which meet at a point form an angle between them.



Acute angle : $0^\circ < a < 90^\circ$

Right angle : $b = 90^\circ$

Obtuse angle : $90^\circ < c < 180^\circ$

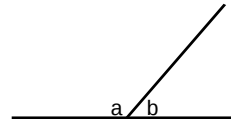
Reflex angle : $180^\circ < d < 360^\circ$

(b) Theorems :

If AOB is a straight line, then

$$a + b = 180^\circ$$

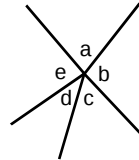
(Adjacent angles on a straight line)



The sum of all the angle at a point, each being adjacent to the next, is 4 right angles.

$$\Rightarrow a + b + c + d + e = 360^\circ$$

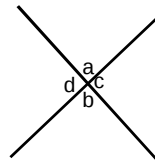
(Angles at a point)



If two straight lines intersect, the vertically opposite angles are equal.

$$a = b, \quad c = d$$

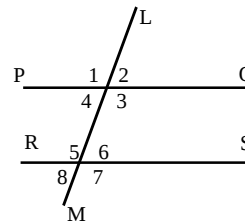
(Vertically Opposite angles)



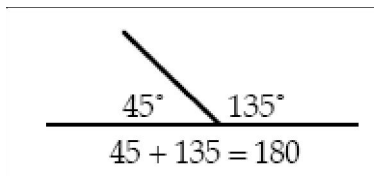
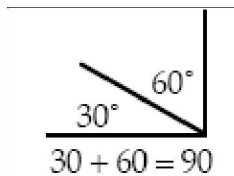
Parallel lines PQ and RS are cut by a transversal LM, then we have :

$$1 = 3 = 5 = 7$$

$$2 = 4 = 6 = 8$$



(c) (i) Two angles whose sum is 90° , are complementary. Each one is the complement of the other.

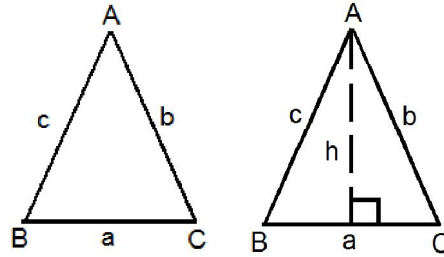


(ii) Two angles whose sum is 180° , are supplementary. Each one is the supplement of the other.

TRIANGLES :

A closed figure enclosed by 3 sides is called a Triangle.

ABC is a triangle. The sides AB, BC, AC are respectively denoted by c , a , b . Please carefully note the capital and small letters.



1. In any triangle ABC

(1) $A = \frac{1}{2} a \times h = \frac{1}{2} \text{ base} \times \text{perpendicular to base from opposite vertex}$

(2) $A = \sqrt{s(s-a)(s-b)(s-c)}$, $s = (a + b + c)/2 = \text{semi-perimeter}$

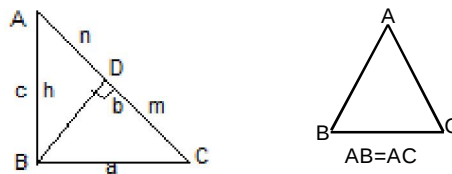
(3) $P = (a + b + c) = 2s$

TRIANGLES

PROPERTIES :

1. Sum of the three interior angles is 180°
2. When one side is extended in any direction, an angle is formed with another side. This is called the exterior angle.
3. There are six exterior angles of a triangle.
4. Interior angle + corresponding exterior angle = 180° .
5. An exterior angle = Sum of the other two interior angles not adjacent to it
6. Sum of any two sides is greater than the third side.
7. Difference of any two sides is less than the third side.
8. Side opposite to the greatest angle will be the greatest and vice versa.
9. A triangle must have at least two acute angles.
10. Triangles on equal bases and between the same parallels have equal areas.
11. If a , b , c denote the sides of a triangle then
 - (i) if $c^2 < a^2 + b^2$, Triangle is acute angled
 - (ii) if $c^2 = a^2 + b^2$, Triangle is right angled
 - (iii) if $c^2 > a^2 + b^2$, Triangle is obtuse angled

2. Right Angled Triangle :



A triangle whose one angle is 90° is called a right (angled) Triangle.

In the figure, b is the hypotenuse,
and a & c the legs, called base and height resp.

(1) $R = \frac{1}{2} \times \text{Hypotenuse}$

(2) $r = A/s = ac / (a + b + c)$

(3) $h^2 = m n$

(4) $AC^2 = AB^2 + BC^2$ (Pythagoras theorem)

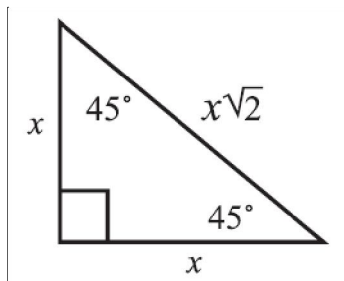
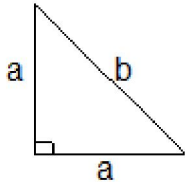
NOTE : You should remember some of the Pythagorean triplets (e.g. 3,4,5 because $5^2 = 3^2 + 4^2$). Some others are (5, 12, 13), (7, 24, 25) etc.

Exercise : Try to find out at least three other sets of Pythagorean Triplets. But be clear that we are talking about distinct triplets. Thus if 3,4,5 is a Pythagorean triplet, then it does not mean that (3 x 10), (4 x 10), (5 x 10) is a distinct triplet.

3. Right triangle :

Median to the hypotenuse = $\frac{1}{2}$ x hypotenuse = circumradius

6. Isosceles Right Triangle :



A right triangle whose two legs are equal is an isosceles right triangle.

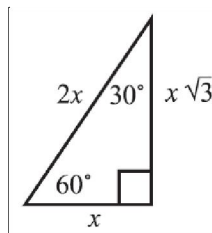
We have $b = a\sqrt{2}$ or $a = b/\sqrt{2}$

7. 30, 60, 90 triangle :

This is a special case of a right triangle whose angles are 30° , 60° , 90° .

In this triangle side opposite to angle $30^\circ = \text{HYP}/2$,

Side opposite to Angle $60^\circ = \sqrt{3}/2 \times \text{HYP}$.



Right Angled Triangle:

1. If a, b be legs and c the hypotenuse, then $c^2 = a^2 + b^2$ (Pythagoras Theorem)
2. Altitude to hypotenuse $h = \frac{ab}{c}$
3. Area = $\frac{1}{2}ab$
4. Median to Hypotenuse = half of Hypotenuse or $M_d = \frac{1}{2} \text{Hyp}$
5. Circum-radius = Median to Hypotenuse = Half of Hypotenuse or $R = M_d = \frac{1}{2} \text{Hyp}$
6. Hypotenuse = Diameter of circum-circle = $2R$.
7. Circum-center = Mid-point of Hypotenuse
8. The triangle is enclosed in the semicircle only – Angle in a semicircle is a right angle.
9. In-radius (r) = $\frac{a+b+c}{2}$
10. If x and y are the segments in which the altitude divides the hypotenuse then $h^2 = xy$
11. The only basic right angled triangle in which sides are in an AP is 3, 4, 5 (or its multiples)

Triplets

1. If three positive integers (a, b, c) in an increasing order (and in their most basic form – means at least one of them is a prime number) satisfy the Pythagoras Theorem $c^2 = a^2 + b^2$ then the ordered triplet (a, b, c) is called a Pythagorean Triplet.

How to generate triplets :

2. **Starting with an odd number – Rule** – Square the smallest side (will be odd) and write it as the sum of two consecutive numbers (always odd). Thus we obtain the triplet. Like if the first number is 3, then $3^2 = 9 = x + (x + 1)$ or $x = 4$ and $x + 1 = 5$, so the triplet is 3, 4, 5. Similarly, (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61), (13, 84, 85), (15, 112, 113) etc.

Note: (9, 12, 15), (15, 20, 25) are also triplets but not basic triplets. They are just multiples of (3, 4, 5). So the basic triplet starting with an odd number will always be unique.

3. **Starting with an even number – Rule** – Square the smallest side (a) and keep on writing it as:

$\frac{a^2}{2^n} = x + (x + 2^n)$, where n is a natural number-till we do not get a fractional number. The values of x and $(x + 2^n)$ will be the two other sides.

Let's assume the smallest side is 20, then

$$\frac{20^2}{2} = x + (x + 2) \text{ Or } x = 99, \text{ so the triplet is } (20, 99, 101)$$

$$\frac{20^2}{4} = x + (x + 4) \text{ Or } x = 48, \text{ so the triplet is } (20, 48, 52) \text{ which is basically } (5, 12, 13)$$

$$\frac{20^2}{8} = x + (x + 8) \text{ Or } x = 21 \text{ so the triplet is } (20, 21, 29)$$

$$\frac{20^2}{16} = x + (x + 16), \text{ i.e., } x \text{ is fractional – so we stop here.}$$

Some more possible triplets starting with an even number will be (8, 15, 17), (12, 35, 37), (28, 45, 53) etc.

4. Results for the 3, 4, 5 Right Triangle (very important –commit to memory):

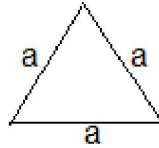
1. Median = $2.5 = \frac{5}{2}$
2. Altitude = $2.4 = \frac{12}{5}$
3. If x and y are the segments in which the altitude divides the hypotenuse then
$$x = 1.8 = \frac{9}{5} \qquad y = 3.2 = \frac{16}{5}$$
4. R = $2.5 = \frac{5}{2}$
5. r = 1
6. Ratio of Areas of Circum-circle to in-circle = $\frac{25}{4}$
7. The two triangles that are obtained when the altitude intersects the hypotenuse are also right angled and in the same ratio of sides 3:4:5.
8. One of them is 1.8, 2.4, 3 (3 : 4 : 5) and the other is 2.4, 3.2, 4 (3 : 4 : 5).

Some more facts about different Right Triangles:

1. For 3, 4, 5: Smallest side = $\frac{1}{4}$ Perimeter
$$\text{Perimeter} = 3 \times 4$$
2. For 5, 12, 13 Smallest Side = $\frac{1}{6}$ Perimeter
$$\text{Perimeter} = \frac{1}{2} \times 5 \times 12$$
3. For 7, 24, 25 Smallest side = $\frac{1}{8}$ Perimeter
$$\text{Perimeter} = \frac{1}{3} \times 7 \times 24$$
4. For 9, 40, 41 Smallest side = $\frac{1}{10}$ Perimeter
$$\text{Perimeter} = \frac{1}{4} \times 9 \times 40$$

And so on....

4. Equilateral Triangle :



A triangle whose all sides are equal is called an equilateral triangle. If a be the side of an equilateral triangle, then

$$H = \frac{\sqrt{3}}{2} a \quad A = \frac{\sqrt{3}}{4} a^2$$

$$r = \frac{1}{3} \times H = \frac{1}{3} \times \frac{\sqrt{3}}{2} \times a = \frac{a}{2\sqrt{3}}$$

$$R = \frac{2}{3} H = \frac{a}{\sqrt{3}}$$

2. **Equilateral Triangle :**

- (a) All the four points viz. centroid, circumcentre, incentre, orthocentre coincide.
- (b) Medians, angle bisectors, altitudes, perpendicular bisectors of sides are all represented by same straight lines.
- (c) Given the perimeter, equilateral triangle has the maximum area.
- (d) Of all the triangles that can be inscribed in a circle, the equilateral triangle has the greatest area.

Points (c) and (d) give you a hint regarding the nature of symmetry.

1. Area = $\frac{\sqrt{3}}{4} a^2$

2. Altitude = $h = \frac{\sqrt{3}}{2} a$

3. Inradius (r) = $\frac{h}{3} = \frac{a}{2\sqrt{3}}$

4. Circumradius (R) = $\frac{2h}{3} = \frac{a}{\sqrt{3}}$

5. Incircle Area = $\frac{\pi a^2}{12}$

6. Circumcircle Area = $\frac{\pi a^2}{3}$

7. $R = 2r$ i.e. Circumradius = $2 \times$ Inradius

8. Circum-circle Area = $4 \times$ In-circle Area

9. **Area of Outer Circle is 4 times the area of Inner Circle**

10. The triangle formed by joining the mid-points of the sides will be half in perimeter and one-fourth in area.

11. So the Perimeter of inner triangle = $\frac{P}{2}$

12. And the area of the inner triangle = $\frac{A}{4}$

13. If we go on inscribing triangles in this fashion infinitely, total perimeter = $2P$ and total area = $\frac{4A}{3}$

14. If we draw three lines within an equilateral triangle parallel to its sides and touching its in-circle then the three triangles obtained will be one-third in length and one-ninth in area of the original triangle.

15. If we go on inscribing circles and triangles within an equilateral triangle, then each circle will be one-half in perimeter and one-fourth in area of the previous circle.

16. Also, each triangle will be one-half in perimeter and one-fourth in area of the previous triangle.

17. For similar figures and solids, $\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$ and for similar solids $\frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$

5. Isosceles Triangle :

A triangle whose two sides are equal is an isosceles triangle.

$$A = \frac{b}{4} \sqrt{4a^2 - b^2}$$

1. Isosceles triangle :

In this the base angles (or any two angles are equal).
The bisector of $\angle A$ is perpendicular bisector of the base and is also the median to the base.

QUADRILATERALS :

A closed figure (plane) bounded by four sides is called a quadrilateral. In any quadrilateral :

1. $A = \frac{1}{2} \times \text{one diagonal} \times (\text{sum of perpendiculars to it from opposite vertices}).$
2. $A = \frac{1}{2} \times \text{product of diagonals} \times \sin \text{ of angle between them}$
3. Area of a cyclic quadrilateral
 $A = \sqrt{[(s-a)(s-b)(s-c)(s-d)]}$ where a, b, c, d are sides of quadrilateral and
 $s = \text{semi perimeter} = (a + b + c + d) / 2$

Properties :

- (a) Sum of the four interior angles = 360°
- (b) If a quadrilateral can be inscribed in a circle, it is called a cyclic quadrilateral. Here opposite angles are supplementary. If one side is produced, then the exterior angle = Remote interior angle.
- (c) The figure formed by joining the mid points of the adjacent sides of a quadrilateral is a parallelogram.
- (d) If a quadrilateral is circumscribed about a circle, the sums of opposite sides are always equal.

1. Rectangle :

A quadrilateral whose opposite sides are equal and each internal angle equal to 90° , is called a rectangle.

$l = \text{length}$

$b = \text{breadth}$

$A = l \times b$

$P = 2(l + b)$

Diagonal = $\sqrt{l^2 + b^2}$, diagonals are equal and bisect each other.

RECTANGLE :

- (a) Opposite sides equal, each angle = 90°
- (b) Diagonals bisect each other (not at 90°).
- (c) Of all rectangles of given perimeter, a square has max. area
- (d) When inscribed in a circle, it will have maximum area when it's a square.
- (e) Figure formed by joining the midpoints of a rectangle is a rhombus.
- (f) If P is any point within a rectangle ABCD then $PA^2 + PC^2 = PB^2 + PD^2$
- (g) The biggest circle that can be inscribed in a rectangle will have the diameter equal to the breadth of the rectangle.
- (h) When a rectangle is inscribed in a circle, the diameter of the circle is equal to the diagonal of the rectangle.

2. Parallelogram

A quadrilateral in which opposite sides are equal and parallel is called a parallelogram.

$$A = bh$$

$$A = ab \sin \theta,$$

$$P = 2(a + b), \text{ Diagonals bisect each other.}$$

PARALLELOGRAM :

If in a quadrilateral, the opposite sides are parallel and equal, it is a $\parallel^{\text{gm}}$.

- (a) Opposite angles are equal
- (b) Diagonals bisect each other (not at 90°)
- (c) Sum of any two adjacent angles = 180°
- (d) Bisectors of four angles enclose a rectangle.
- (e) Each diagonal divides it into 2 equal triangles.
- (f) When inscribed in a circle, it becomes a rectangle
- (g) When circumscribed about a circle, it becomes a rhombus.
- (h) Diagonals divide it into 4 equal triangles.
- (i) Point of intersection of diagonals is equidistant from the four vertices.
- (j) The figure formed by joining the Mid-points of adjacent sides of a $\parallel^{\text{gm}}$ is a $\parallel^{\text{gm}}$.
- (k) Diagonals are unequal in lengths and do not bisect angles at vertices.

3. Square

A quadrilateral whose all sides are equal and all angles 90° is called a square.

$$A = (\text{side})^2$$

$$P = 4 \times \text{side}$$

$$A = \frac{1}{2} \times d^2$$

$$(d = \text{diagonal}), \text{ Diagonals are at right angles}$$

SQUARE :

- (a) All sides equal, all angles 90° .
- (b) Diagonals bisect each other at 90° and are equal
- (c) When inscribed in a circle, diagonal = diameter of circle
- (d) When circumscribed about a circle, Side of square = Diameter of circle.

Square

1. Area = a^2
2. Diagonal = $a\sqrt{2}$
3. Area = $\frac{\text{Diagonal}^2}{2}$
4. Diagonal bisects the squares into two 45–45–90 triangles which are equal.
5. Diagonals are perpendicular.
6. The figure formed by joining the mid-points of the sides of a square is also a square.

7. In this case the side will become $\frac{1}{\sqrt{2}}$ times the side of the original square, perimeter will become $\frac{1}{\sqrt{2}}$ times the perimeter of the original square and area will become $\frac{1}{2}$ times the area of the original square.

8. So $\text{side}_{\text{new}} = \frac{\text{Side}_{\text{original}}}{\sqrt{2}}$

$$\text{Perimeter}_{\text{new}} = \frac{\text{Perimeter}_{\text{original}}}{\sqrt{2}}$$

$$\text{Area}_{\text{new}} = \frac{\text{Area}_{\text{original}}}{2}$$

9. If we go on inscribing squares within the original square by joining the mid-points of successive squares then the sum of perimeters of all the squares is given by

$$\Sigma P = \frac{P}{1 - \frac{1}{\sqrt{2}}} = \frac{P\sqrt{2}}{\sqrt{2} - 1} = \frac{4a\sqrt{2}}{\sqrt{2} - 1}.$$

10. And the sum of areas of all the squares is given by $2a^2$.

11. If a square (biggest possible) is inscribed in a circle of radius r , then Diameter of circle = Diagonal of Square.

12. If the area of the circle is A then $A = \pi r^2 = \frac{\pi D^2}{4}$ or $\frac{D^2}{2} = \frac{2A}{\pi}$ or

$$\text{Area of Square} = \frac{2A}{\pi} \text{ or } 2r^2.$$

13. When another circle is inscribed in this square then

Circumference of the circle = $\frac{1}{\sqrt{2}}$ times the circumference of the original circle and Area of circle = $\frac{1}{2}$ times the area of original circle.

14. If now in this circle, if another square is inscribed, then

The perimeter of this square = $\frac{1}{\sqrt{2}}$ times the perimeter of the outer square

And the area of this square = $\frac{1}{2}$ times the area of the outer square.

15. The circumference of first circle = $2\pi r$, the circumference of the next circle = $\frac{2\pi r}{\sqrt{2}}$ and so on. The perimeter of the first square = $4r\sqrt{2}$, the perimeter of the next square = $4r$ and so on.

16. If a circle is inscribed in a square of side a , then **Side of square = Diameter of circle**. If the area of the square is S , then the area of circle = $\frac{\pi S}{4} = \frac{\pi a^2}{4}$.

17. If now in this circle, if another square is inscribed, then

The perimeter of this square = $\frac{1}{\sqrt{2}}$ times the perimeter of the outer square

And The area of this square = $\frac{1}{2}$ times the area of the outer square.

18. When another circle is inscribed in this inner square then

Circumference of the circle = $\frac{1}{\sqrt{2}}$ times the circumference of the original circle and Area of circle = $\frac{1}{2}$ times the area of original circle.

19. The perimeter of the first square will be $4a$, the perimeter of the next square will be $\frac{4a}{\sqrt{2}}$ etc. The circumference of the first circle will be πa and the circumference of the next circle will be $\frac{\pi a}{\sqrt{2}}$ and so on
20. **Square and Rectangle inscribed in a semi-circle:** If the biggest possible square is inscribed in a semi-circle of radius r , then the area of square = $\frac{4r^2}{5}$. The side of the square = $\frac{2r}{\sqrt{5}}$. If the biggest possible rectangle is inscribed in a semicircle of radius r , then the area of the rectangle is r^2 . The sides are $r\sqrt{2}$ and $\frac{r}{\sqrt{2}}$.

4. **Rhombus**

A quadrilateral whose all sides are equal and parallel is called a rhombus.

$$A = \frac{1}{2} \times X \times Y \quad (X, Y \text{ are diagonals})$$

$$P = 4a$$

Diagonals are at right angles

$$X^2 + Y^2 = 4a^2$$

RHOMBUS :

If in a parallelogram, all sides are equal, it is a Rhombus.

- Opposite angles are equal
- Diagonals bisect each other at 90° .
- Diagonals bisect angles at vertices.
- Sum of any two adjacent angles = 180° .
- Figure formed by joining the mid points of the adjacent sides of a rhombus is a rectangle.
- Diagonals are unequal.
- Point of intersection of diagonals is equidistant from the four vertices.

5. **Trapezium (Trapezoid)**

A quadrilateral in which one pair of opposite sides is parallel is a trapezium.

$$A = \frac{1}{2} \times (a + b) \times h$$

TRAPEZIUM (TRAPEZOID) :

- The median is half the sum of parallel sides.
- If inscribed in a circle, it becomes an isosceles Trapezium. The oblique sides are equal, angles made by each parallel side with oblique sides are equal. Diagonals are equal.
- Diagonals intersect proportionally and the ratio = ratio of length of parallel sides.
- Four similar triangles are obtained by joining the mid points of adjacent sides.
- In the figure ABCD is a trapezium, then $AC^2 + BD^2 = BC^2 + AD^2 + 2 AB \cdot CD$
- In the above figure if X & Y are mid points of diagonals, then $XY = \frac{1}{2} (AB - CD)$.

CYCLIC QUADRILATERAL :

- The four vertices lie on a circle.
- Opposite angles are supplementary.
- If any one side is produced,
Exterior angle = Remote interior angle
- If one pair of opposite sides are equal, diagonals are equal

5. The line joining the points of intersection of the bisectors of opposite angles of a cyclic quadrilateral with the circle is the diameter of the circle.
6. The quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.
7. Sum of products of opposite sides = product of diagonals

CIRCLES

REGULAR POLYGONS :

A many sided closed figure is called a polygon. If all the sides of a polygon are equal, it is called a regular polygon.

1. Interior Angle + Exterior angle = 180°
2. $P = n \times a$ (n = number of sides, a = side)
3. Sum of exterior angles = 360°
4. Sum of Interior angles = $(2n - 4) 90^\circ$
5. Each Interior angle = $[(2n - 4)/n] \times 90^\circ$
6. Each exterior angle = $360/n$
7. $A = \frac{1}{2} P r = \frac{1}{2} n a r$
P = perimeter (r = $\perp$ from centre to any one side = radius of incircle)
8. $A = (na^2/4) \times \cot (180/n)^\circ$

Regular Hexagon :

1. Each interior angle = 120°
2. Sum of Interior angles = 720°
3. Each exterior angle = 60°
4. Area = $(3\sqrt{3}/2) \times a^2$ (a = side)
5. $P = 6a$

Exercise : Try to get all these from the general formulae listed above by putting n = 6 etc.

NOTE:

1. In any polygon, (n – 2) triangles are formed by drawing diagonals from one vertex
2. Area of a regular octagon = $2a^2 (1 + \sqrt{2})$ (a = side)

Circle :

A circle is the path traversed by a point which moves in such a way that its distance from a fixed point always remains constant.

C is called centre and R the radius of the circle. Circumference = $2 \pi R = \pi D$

Area = $A = \pi R^2 = \pi D^2/4$

(D = Diameter = 2R) $\pi = 22/7$ or 3.14

Length of arc of a circle is given by

$$l(AB) = (\theta^\circ/360^\circ) \times 2\pi R$$

Area of Sector ABC = $(\theta/360) \times \pi R^2 = \frac{1}{2} l(AB) \times R$

Distance travelled by a wheel in n revolutions = n x circumference

CIRCLES

- (a) Tangent is perpendicular to radius.
- (b) Perpendicular from centre to a chord bisects the chord.
- (c) Tangents drawn from an external point are equal.
- (d) Equal chords of a circle are equidistant from the centre.
- (e) When two circles touch, their centres & the point of contact are collinear
- (f) If two circles touch externally, distance between centres = sum of radii
- (g) If two circles touch internally, distance between centres = difference of radii
- (h) Circles with same centre and different radii are concentric circles.
- (i) Points lying on the same circle are called concyclic points.
- (j) Only one circle can pass through three given points
- (k) Equal arcs subtend equal angles at the centre.
- (l) Measure of an arc means measure of central angle.
- (m) $m(\text{minor arc}) + m(\text{major arc}) = 360^\circ$
- (n) Angle at the centre made by an arc = twice the angle made by the arc at any point on the remaining part of the circumference.
We have $\angle APB = \frac{1}{2} \angle AOB = 30^\circ = \angle AQB$
- (o) Angle in a semicircle is a right angle.
- (p) Alternate Segment Theorem : In the fig.

if BAC is the tangent at A to a circle and if AD is any chord, then
 $\angle DAC = \angle APD$ or
 $\angle PAB = \angle PDA$ (Angles in alternate segment)

MENSURATION OF SOLIDS

CUBE :

A six-faced solid figure with all faces equal and adjacent faces mutually perpendicular is a cube.

If "a" be the edge of a cube,

- 1. The longest diagonal = $a\sqrt{3}$ The face diagonal = $a\sqrt{2}$
- 2. Volume = a^3
- 3. Total surface area = $6a^2$

For any regular solid Number of faces + Number of vertices = No. of edges + 2

CUBOID or RECTANGULAR BOX :

If a,b,c are the edges of a box,

- 1. The longest diagonal = $\sqrt{a^2 + b^2 + c^2}$
- 2. Surface area = $2(ab + bc + ac)$
- 3. Volume = abc

RIGHT CIRCULAR CYLINDER :

If r is the radius of base and h is the height, then

- 1. Volume = $\pi r^2 h$
- 2. Curved surface area = $2\pi rh$
- 3. Total surface area = $2\pi r(r + h)$

This cylinder is generated by rotating a rectangle by fixing one of its sides.

If a rectangle of length L and breadth B is rotated about its length to form a cylinder, then $L = 2\pi R$ and $B = h$.

If a rectangle of length L and breadth B is rotated about its breadth to form a cylinder, then $B = 2 \pi R$ and $L = h$.

RIGHT CIRCULAR CONE :

R = radius of base

H = Height

L = slant height = $\sqrt{(H^2 + R^2)}$

1. Volume = $\frac{1}{3} \times (\pi R^2 H)$
2. Curved surface Area = $\pi R L$
3. Total Surface Area = $\pi R (R + L)$

SPHERE :

R = Radius

1. Volume = $\frac{4}{3} \times \pi R^3$
2. Surface Area = $4 \pi R^2$

SIMILAR PLANE FIGURES & SOLIDS

1. If A_1 & A_2 denote the areas of two similar figures, and l_1 & l_2 denote their corresponding linear measures, then $A_1 / A_2 = (l_1 / l_2)^2$
2. If V_1 & V_2 denote the volumes of two similar solids, and l_1, l_2 denote their corresponding linear measures, then $V_1 / V_2 = (l_1 / l_2)^3$

Some More Important Points:

1. If with a given perimeter, different figures are formed like equilateral triangle, square, regular hexagon, regular octagon and eventually a circle (a regular polygon of infinite sides), then the triangle will have the minimum area and circle will have the maximum area.
2. n lines can divide a plane into $\frac{n(n+1)}{2} + 1$ regions (some bounded and some unbounded). Out of this $2n$ will be unbounded regions and remaining bounded.
3. If different triangles are inscribed in a circle, then the equilateral triangle will have the maximum area.
4. If the perimeter of a triangle is fixed, then the equilateral triangle will have the maximum area.
5. If the sum of two sides of a triangle is constant, then the isosceles right angled triangle will have the maximum area.
6. In a rectangular polygon (in which all corners are either 90° or 270°), if the number of corners with 90° is n , the number of corners with 270° will be $(n - 4)$.
7. With n points given in a plane, a maximum of nC_2 lines can be drawn through these points.
8. If n lines are given in a plane, then a maximum of nC_2 points of intersection can be generated.

SOLVED EXAMPLES

GROUND WORK

Try to solve each question on your own. Then check your solution with the one given here. Make sure that you solve each and every question that appears here.

Ex 1. In an engine the driving wheel is 343 cm in diameter, and makes 100 revolutions per minute. Find the rate in cm per hour at which the engine is travelling.

Sol. Circumference of wheel = diameter $\times \pi$ = $343 \times 22/7$ cm = 1078 cm

In one minute the engine travels = 1078×100 cm = 107800 cm

In 1 hour the engine travels = 107800×60 = 6468000 cm

$\Rightarrow$ The rate is 6468000 cm per hour. Ans.

Ex 2. Water flows at 3 metres per sec through a pipe of radius 4 cm. How many hours will it take to fill a tank 40 metres long, 30 metres broad and 8 metres deep, if the pipe remains full ?

Sol. Radius of the pipe (r) = 4 cm = 0.04 metre

Volume of water flowing out per sec = $\pi r^2 h$ = $22/7 \times 0.04^2 \times 3$ cu metres = 0.0151 cu m

Time taken to fill the tank = $40 \times 30 \times 8 / 0.0151$ sec.

= $40 \times 30 \times 8 / 0.0151 \times 1/3600$ hours = 176.6 hours Ans.

Ex 3. A solid iron bar of circular cross-section 20 cm in diameter, is melted down and recast into a hollow pipe of the same length as the bar and of internal diameter 14 cm. Find its external radius.

Sol. Let r cm be the internal radius of the pipe, R cm be the external radius of the pipe,

X cm be the radius of the bar, h cm the length of the pipe and the bar.

Volume of the bar = $\pi X^2 h$. Volume of the pipe = $\pi R^2 h - \pi r^2 h = \pi h (R^2 - r^2)$ $\pi h (R^2 - r^2) = \pi X^2 h$

$R^2 - 7^2 = 10^2$ $R^2 = 100 + 49 = 149$ $R = 12.21$

$\Rightarrow$ The external radius of the pipe is 12.21 cms Ans.

Ex 4. A playground 100 m $\times$ 50 m has two roads, each 10 m wide running in the middle of it. One is parallel to the breadth and the other is parallel to the length. Find the cost of preparing the ground for a cricket match if the per square cm cost of preparing the ground is Rs 1.25.

Sol. Area of playground abcd = $100 \times 50 = 5000$ m²

Area of two roads = $100 \times 10 + 50 \times 10 = 1500$ m²

Common Area = $10 \times 10 = 100$ m²

(this has to be subtracted from 1500 m²)

Net playground area = $5000 - (1500 - 100) = 3600$ m²

$\Rightarrow$ Total Cost of preparing the ground

= $3600 \times 1.25 \times 100 \times 100$ = Rs 450 lakhs

Ex 5. The length, breadth, and height of a room are X, Y, and Z feet resp. The cost of whitewashing the four walls of this room is Rs 2500. Find the cost of preparing another room whose length, breadth, and height are double that of the previous room respectively.

Sol. Surface area of the walls of the first room = $2 (XZ + YZ)$

Surface area of the walls of the second room = $2 (4XZ + 4YZ)$

= $8 (XZ + YZ)$ = 4 times the first area $\Rightarrow$ Cost required = 4×2500 = Rs 10,000

Ex 6. If each side of a cube is trebled, calculate the percentage change in its volume and surface area.

Sol. New Side = 3 (Old Side) $\Rightarrow$ New Volume = 27 (Old Volume)

$\Rightarrow$ Percent change = $(\text{New} - \text{Old}) / \text{Old} \times 100\%$ = 2600 %

New Surface Area = $(6 \times 9 \times \text{Old Side}^2) = 54$ Old Surface Area

$\Rightarrow$ Percent change = $(54 - 6) / 6 \times 100\%$ = 800 % Ans.

Ex.7. Prepare a detailed table listing the properties of the diagonals of all standard quadrilaterals.

Sol.

Quadrilateral	90°	Bisect	Equal
Square	◀	◀	◀
Rectangle		◀	◀
Parallelogram		◀	
Rhombus	◀	◀	
Trapezium			

Ex 8. In the figure, ABCD is a square. Arcs AC are drawn with centres B and D. (take $\pi = 22/7$)

- (a) What fraction of the square is the sector ACD ?
 (b) What fraction of the square is shaded ?

Sol. Let r be the side of square Area of square = r^2

Area of sector ACD = $\frac{1}{4} \pi r^2 = \frac{1}{4} \times \frac{22}{7} \times r^2$

(a) The fraction of the square is the sector ACD = $\frac{1}{4} \times \frac{22}{7} \times \frac{r^2}{r^2} = \frac{22}{28} = \frac{11}{14}$ Ans.

(b) Shaded area = 2(area of sector – area of triangle) = $2(\frac{1}{4} \times \frac{22}{7} \times r^2 - \frac{1}{2} r^2)$
 $= 2(\frac{11}{14} r^2 - \frac{1}{2} r^2)$

$= 2(\frac{11r^2}{14} - \frac{7r^2}{14}) = \frac{4}{7} r^2$ The fraction = $\frac{4/7 r^2}{r^2} = \frac{4}{7}$ Ans.

Ex 9. A piece of paper in the form of a circular sector is made to form a right circular cone. The radius of the sector is 5 cm. The angle at the centre is 216° as shown in the figure.

- (a) Find the radius of the base of the cone.
 (b) Find the capacity of the cone.
 (c) If the cone with vertex downwards contains water to a depth equal to one half of its height, find the area of the paper in contact with the water.

Sol. (a) Circumference of the base of cone = length of the arc

$\Rightarrow$ length of arc = $2 \times 5\pi \times \frac{216}{360}$ cm = 6π cm Circumference of the cone = $2\pi r$

$\Rightarrow 6\pi = 2\pi r \Rightarrow r = 3$ cm Ans.

(b) Ht. of cone = $\sqrt{(5^2 - 3^2)} = 4$ cm $\Rightarrow$ Vol. of cone = $\frac{1}{3} \pi (3)^2 (4) = 12\pi$ cm³ Ans.

(c) Area of paper in contact = slant surface area of cone = $\pi r l$

At $h = \frac{4}{2}$ cm = half height, $r = \frac{3}{2}$ cm, $l = \frac{5}{2}$ cm $\Rightarrow$ Area = $\pi (\frac{3}{2}) (\frac{5}{2}) = \frac{15}{4} \pi$ cm²

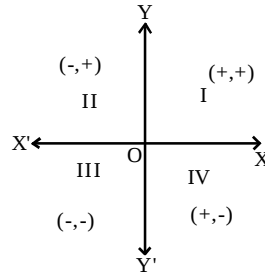
Ex 10. The capacities of two spherical tanks are in the ratio 27 : 64. If 72 kg of paint is required to paint the outer surface of the smaller tank, then how many kg of paint would be required to paint the outer surface of the bigger tank ?

Sol. The ratio of surface areas = $[\sqrt[3]{(27/64)}]^2 = 9/16$

$\Rightarrow$ The quantity of paint reqd to paint the outer surface of the bigger tank = $\frac{16}{9} \times 72$ kg
 $= 128$ kg

Co-ordinate Geometry Basics

Co-ordinate System: Sign Convention and Quadrants:



1. **DISTANCE BETWEEN TWO POINTS :** If there are two points A (X_1, Y_1) and B (X_2, Y_2) on the XY plane, the distance between them is given by $AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Distance of the point P(x, y) from the origin O is $OP = \sqrt{x^2 + y^2}$
2. The co-ordinates of P(X,Y) such that P divides the line joining A (X_1, Y_1) and B (X_2, Y_2) internally in the ratio $m : n$ will be :
$$X = \frac{mX_2 + nX_1}{m + n}, \quad Y = \frac{mY_2 + nY_1}{m + n}$$
3. If P (X,Y) is the midpoint between A & B then $X = (X_1 + X_2) / 2$, $Y = (Y_1 + Y_2) / 2$

EQUATION OF A CURVE :

An equation in two variables X and Y (with or without a constant term) is called the equation of a particular curve if the graph of that equation plotted on the XY cartesian plane give that particular curve.

e.g. $X^2 + Y^2 = 36$. If we plot this curve, by taking different values of Y (and thereby different values of X), we get a circle. Hence $X^2 + Y^2 = 36$ is the equation of a circle.

STRAIGHT LINE :

An equation of the form $AX + BY + C = 0$ is called the general equation of a straight line, where X and Y are variables and A, B, C are constants.

Any point lying on this line will satisfy the equation of the line. i.e. the coordinates of the point when substituted by X & Y resp. in the above equation will make the LHS vanish.

GENERAL CONCEPTS

If AB is a straight line on the XY plane, then the ratio of y intercept to x intercept (with signs) is called its slope and is denoted by 'm'. The lengths OP and OQ are respectively called the intercepts on X and Y axes, made by the line.

So slope = RISE / RUN

Also, slope = DIFFERENCE OF Y COORDINATES / DIFFERENCE OF X COORDINATES

For convenience, same straight line can be represented by many different forms of the same equation. The different forms are given below one by one.

1. If 'm' is the slope of the line and 'c' the intercept made by the line on Y axis, the equation is
 $Y = mX + c$
2. If the slope of the line is m and it passes through (X_1, Y_1) , the equation is $(Y - Y_1) = m(X - X_1)$
3. If the line passes through two points (X_1, Y_1) and (X_2, Y_2) , the equation is
$$(Y - Y_1) = \frac{(Y_2 - Y_1)}{(X_2 - X_1)} (X - X_1)$$

Hence the slope of the line through (X_1, Y_1) , (X_2, Y_2) is given by $m = (Y_2 - Y_1)/(X_2 - X_1)$
or $(Y_2 - Y_1)/(X_2 - X_1)$
4. If the X intercept of the line is a and Y intercept is b, the equation is : $X/a + Y/b = 1$
5. General form : $AX + BY + C = 0$ In this slope = $m = -A/B$
X intercept = $-C/A$ Y intercept = $-C/B$
 - (a) If two lines are parallel then their slopes are equal ($m_1 = m_2$). If two lines do not intersect, they are parallel.
 - (b) If two lines are perpendicular to each other, the product of their slopes is -1 . ($m_1 m_2 = -1$).
 - (c) $a_1X + b_1Y + c_1 = 0$ and $a_2X + b_2Y + c_2 = 0$ will represent the same straight lines if $a_1/a_2 = b_1/b_2 = c_1/c_2$. In this case, the lines are coincident and theoretically intersect at infinite points.
7. The point of intersection of two lines (X, Y) is obtained by simultaneously solving both the equations.
8. The equation of a line parallel to a given line $AX + BY + C = 0$, will be $AX + BY + K = 0$ where K is a constant which can be found by any additional given condition.
9. The equation of a line perpendicular to a given line $AX + BY + C = 0$ will be $BX - AY + K = 0$, where K is a constant which can be found by an additional given conditions.
10. The length of perpendicular (p) from (X_1, Y_1) on the line $AX + BY + C = 0$ is :
$$p = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$
11. Equation of a line parallel to X axis is $Y = b$ (b is a constant)
12. Equation of a line parallel to Y axis is $X = a$ (a is a constant)
13. Equation of X and Y axes are $Y = 0$ and $X = 0$ respectively
14. Any point on the X axis can be taken as (a,0)
15. Any point on the Y axis can be taken as (0,b)

16. In order to find the X–intercept of a line, put $Y = 0$ in the equation of the line and find X
17. In order to find the Y intercept of a line, put $X = 0$ and find Y
18. The image of the point (a, b) in x axis is (a, -b)
19. The image of the point (a, b) in y axis is (-a, b)
20. The image of the point (a, b) in the line $y = x$ is (b, a)
21. To plot a line, first put $y = 0$, find the point on x axis; then put $x = 0$, find the point on y axis. Join the two points to get the desired graph.

Note :

The equation of a circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

The equation $y = Ax^2 + Bx + C$ is the equation of the quadratic graph which is a parabola with axis parallel to y axis. If $A > 0$, the parabola opens upwards. If $A < 0$, the parabola opens downwards.

If $B^2 > 4AC$, the parabola cuts the x-axis at 2 different points.

If $B^2 = 4AC$, the parabola touches the x-axis at one point (the two points become co-incident).

If $B^2 < 4AC$, the parabola does not cut the x-axis at all.

Examples:

1. Find the ratio in which the line segment joining the points $P(-2,3)$ and $Q(3,5)$ is divided by the Y–axis. Find also the Y–coordinate of the point of division.

Sol. Let the line segment cut at the point $R(0,Y)$, then $0 = (-2 + 3r)/(1 + r)$ $r = 2/3$

Also $Y = (3 + 2/3 (5)) / (1 + 2/3) = 3^{4/5} \Rightarrow$ The required ratio is 2 : 3. Ans.

The Y–coordinate of the point of division is $3^{4/5}$ Ans.

2. Given three points $A(-1,0)$, $B(2,-1)$, $C(3,2)$, show that AB is perpendicular to BC.

Sol. Slope of AB = $(-1 - 0) / (2 - (-1)) = -1/3$ Slope of BC = $(2 - (-1)) / (3 - 2) = 3$

$\Rightarrow$ Product of slope = $(-1/3) (3) = -1 \Rightarrow AB \perp BC$ Ans.

3. If the points $A(-1,1)$, $B(5/2,3/2)$, $C(2,5)$ are the three vertices of a parallelogram :
(a) find the fourth one. (b) prove that ABCD is a square.

Sol. (a) Let $D(X,Y)$ be the required point.

Coordinates of the mid–point $M(X_1, Y_1)$ of AC is given by

$X_1 = (-1 + 2) / 2 = 1/2$ $Y_1 = (1 + 5) / 2 = 3$. Thus the mid–point of BD = $M(X_1, Y_1)$

$\Rightarrow X_1 = 1/2 = (5/2 + X) / 2 \Rightarrow X = -2.5$ $Y_1 = 3 = (3/2 + Y) / 2 \Rightarrow Y = 4.5$

Thus the fourth point is $(-2.5, 4.5)$

(b) For square just check that 1. Sides are equal and 2. Diagonals are equal Ans.

4. The equation of the straight line is $-2X + 5Y + 7 = 0$. Find :
- (a) the slope of the line, (b) the X-intercept of the line and
(c) the y-intercept of the line.

Sol. By general form $AX + BY + C = 0$ where $A = -2$, $B = 5$, $C = 7$

- (a) The slope $m = -A/B = -(-2/5) = 2/5$ Ans.
(b) The X-intercept $= -C/A = -(7/-2) = 7/2$ Ans.
(c) The Y-intercept $= -C/B = -(7/5) = -7/5$ Ans.

5. (a) Find the equation of the line L passing through the point $P_1(3, -2)$ and parallel to the line $L_1: 2X - 3Y - 4 = 0$.

(b) If $(2, p)$ is on the line L , find the value of p .

Sol. (a) Convert L_1 to slope-intercept form $\Rightarrow 2X - 3Y - 4 = 0$, $3Y = 2X - 4$, $Y = 2/3 X - 4/3$
 $\Rightarrow$ Slope of $L_1 = 2/3$ Slope of $L =$ slope of $L_1 = 2/3$

By point-slope form, equation of L is $Y + 2 = 2/3 (X - 3) \Rightarrow 2X - 3Y - 12 = 0$ Ans.

- (b) Since $(2, p)$ is on the line L , $(2, p)$ satisfies the equation of L .
 $\Rightarrow 2(2) - 3p - 12 = 0 \quad -3p = 8 \quad p = -8/3$ Ans.

6. Find the equation of the line L passing through the point $P_1(-1, 3)$ and perpendicular to the line $L_1: 5X - 2Y + 3 = 0$

Sol. $L_1: 5X - 2Y + 3 = 0 \quad Y = 5/2 X - 3/2 \Rightarrow$ slope of $L_1 = m_1 = 5/2$

$\Rightarrow$ Slope of $L \cdot m_1 = -1$ ($L \perp L_1$) $\Rightarrow$ Slope of $L = -2/5 \Rightarrow P_1(-1, 3)$ is on the line L .

$\Rightarrow$ By point-slope form, equation of L is $(Y - 3) = -2/5 (X + 1)$. i.e. $2X + 5Y - 13 = 0$ Ans.

7. Find the equation of the line L perpendicular to the line $L_1: X - 3Y - 1 = 0$ and having X-intercept same as L_1 .

Sol. $L_1: X - 3Y - 1 = 0 \Rightarrow Y = 1/3 X - 1/3 \Rightarrow$ slope of $L_1 = m_1 = 1/3$

$\Rightarrow$ slope of $L \cdot m_1 = -1$ ($L \perp L_1$) $\Rightarrow$ slope of $L = -3$ Put $Y = 0$ into L_1 ,

$X - 3(0) - 1 = 0 \quad X = 1 \Rightarrow$ X-intercept of $L_1 = 1$

Hence, eqn of L is $Y - 0 = -3(X - 1) \Rightarrow 3X + Y - 3 = 0$ Ans.

Permutations and Combinations, Probability

Definitions :

1. Each of the different orders of arrangements, obtained by taking some, or all, of a number of things, is called a **Permutation**.
2. Each of the different groups, or collections, that can be formed by taking some, or all, of a number of things, irrespective of the order in which the things appear in the group, is called a **Combination**.

Example : Suppose, there are four quantities A,B,C,D. The different orders of arrangements of these four quantities by taking three at a time, are :

ABC, ACB, BAC, BCA, CAB, CBA, ... (1)

ABD, ADB, BAD, BDA, DAB, DBA, ... (2)

ACD, ADC, CAD, CDA, DAC, DCA, ... (3)

BCD, BDC, CDB, CBD, DBC, DCB. ... (4)

Thus, each of the 24 arrangements, of the four quantities A,B,C,D by taking three at a time, are each called a permutation. Hence, it is clear that the number of permutations of four things taken three at a time is 24.

Again, it may be easily seen, from the above that out of these 24 permutations, the six, given in (1), are all formed of the same three quantities A,B,C in different orders; hence, they all belong to the same group. Similarly, the permutations, given in (2), all belong to a second group; those given in (3), belong to a third and those in (4), belong to a fourth. Hence, we see that **there are only four different groups** that can be formed of four quantities A,B,C,D by taking three at a time.

Thus, the number of combinations of four things taken three at a time is only four.

Note:

It may be observed that the total number of permutations, as given by (1), (2), (3), (4), may be supposed to have been obtained by either :

(1) forming all possible different groups and then re-arranging the constituents of each of these groups in different orders in all possible ways; or

(2) filling up three places by means of the four quantities A,B,C,D in all possible ways.

Before proceeding to discuss the subject-matter of the present chapter, we propose to enumerate an important Lemma, a clear idea of which is necessary for a comprehensive grasp of the subject under discussion.

Lemma : (an evident truth that does not require any proof)

If there are **m** ways of doing a thing and **n** ways of doing a second thing and **p** ways of doing a third thing, then the total number of “distinct” ways of doing all these together is **m x n x p**.

Ex 1. Suppose, there are five routes for going from a place A to another place B and six routes for going from the place B to a third place C. Find the numbers of different ways through which a person can go from A to C via B.

Sol. Since there are five different routes from A to B, person can go from A to B in five different ways. After reaching B, he has six different ways of finishing the second part of his journey (i.e. going from B to C). Thus for one way of going from A to B there are six different ways of completing the journey from A to C via B. Hence, the total number of different ways of finishing both parts of the journey (i.e. A to B and then from B to C) = 5 times six different ways = $5 \times 6 =$ no. of ways from the first part to the second point x number of ways from the second part to the third point Answer.

Ex 2. Find the number of different ways in which four persons can be accommodated in three different chairs.

Sol. Let's assume that the four persons are P, Q, R, and S. Since all the three different chairs are vacant, any one of the four persons can occupy the 1st chair. Thus, there are four ways of filling up the 1st chair. When the 1st chair has been filled up by any one of the four person, say P, the 2nd chair can be filled up by any one of the remaining three persons Q, R and S. Thus, for each way of filling up the 1st chair, the 2nd chair can be filled up in three different ways. Hence, total no. of ways in which the first two chairs can be filled up is equal to $4 \times 3 = 12$ ways. Again, when the 1st and 2nd chairs are filled up in any one way (i.e. the 1st by P and the 2nd by Q), the 3rd chair can be filled up by any one of the two remaining persons, R and S. Thus, for each way of filling up the first two chairs, there are $4 \times 3 \times 2$ i.e. 24 ways of filling up the third chair along with the first two chairs. Hence, the total no. of ways in which four persons can be accommodated in the three given chairs is equal to $4 \times 3 \times 2 = 24$. We therefore conclude that, the total number of different orders of arrangements of 4 different things, taken 3 at a time, is the same as the total number of different ways in which 3 places can be filled up by 4 different things.

Note :- The statement holds true generally.

Meaning of factorial :

Factorial of a number (whole number only) is equal to the product of all the natural numbers upto that number.

Factorial of n is written as ${}_n$ or $n!$ and is read as factorial n.

$$\text{Hence } 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$n! = n(n-1)(n-2) \times \dots \times 3 \times 2 \times 1$$

- NOTE:**
1. $0! = 1$ (by definition)
 2. ${}^n P_r = n! / (n-r)!$ where $r \leq n$
 3. ${}^n C_r = n! / [(n-r)! r!]$ where $r \leq n$

PERMUTATIONS

1. Permutations of n different things taken 'r' at a time is denoted by ${}^n P_r$ and is given by ${}^n P_r = n! / (n-r)!$
2. The total number of arrangements of n things taken r at a time, in which a particular things always occurs = $r \times {}^{n-1} P_{r-1}$
3. The total number of permutations of n different things taken r at a time in which a particular thing never occurs = ${}^{n-1} P_r$
4. The total number of permutations of n dissimilar things taken r at a time with repetitions = n^r
5. (a) No. of circular permutations of n things taken all at a time = $(n-1)!$
(b) No. of circular permutations of n different things taken r at a time = ${}^n P_r / r$
7. The number of permutations when things are not all different : If there be n things, p of them of one kind, q of another kind, r of still another kind and so on, then the total number of permutations is given by $n! / (p! q! r! \dots)$

COMBINATIONS

1. Number of combinations of n dissimilar things taken ' r ' at a time is denoted by nC_r & is given by ${}^nC_r = n! / [(n - r)! r!]$
2. Number of combinations of n different things taken r at a time in which p particular things will always occur is ${}^{n-p}C_{r-p}$
3. No. of combinations of n dissimilar things taken ' r ' at a time in which ' p ' particular things will never occur is ${}^{n-p}C_r$
4. ${}^nC_r = {}^nC_{n-r}$

PROBABILITY

THE MEANING OF PROBABILITY :

In our daily life, we often come across events which are concerned with the idea of the likelihood or the chance of occurrence of future events.

Probability is the measure of the chance of occurrence of a future event. It tells us how likely we expect the event to happen.

Now, we may define the probability of an event occurring as follows :

Probability of an event occurring = $\frac{\text{Number of favourable outcomes}}{\text{Number of all possible outcomes}}$

Note :

1. If an event E is sure to occur, we say that the probability of the event E is equal to 1 and we write $P(E) = 1$.
2. If an event E is sure not to occur, we say that the probability of the event E is equal to 0 and we write $P(E) = 0$.

Therefore for any event E , $0 \leq P(E) \leq 1$

Mathematical definition of probability :

- (A) If the outcome of an operation can occur in n equally like ways, and if m of these ways are favourable to an event E , the probability of E , denoted by $P(E)$ is given by $P(E) = m/n$
- (B) As $0 \leq m \leq n$, therefore for any event E , we have $0 \leq P(E) \leq 1$
- (C) The probability of E not occurring, denoted by $P(\text{not } E)$, is given by $P(\text{not } E) \text{ or } P(\bar{E}) = 1 - P(E)$
- (D) Odds in favour = No. of favourable cases / No. of unfavourable cases
- (E) Odds against = No. of unfavourable cases / No. of favourable cases

Mutually Exclusive Events and Addition Law

(A) Mutually Exclusive Events :

Two events are mutually exclusive if one happens, the other can't happen & vice versa. In other words, the events have no common outcomes. For example

1. In rolling a die
E :- The event that the no. is odd
F :- The event that the no. is even
G :- The event that the no. is a multiple of three
2. In drawing a card from a deck of 52 cards
E :- The event that it is a spade
F :- The event that it is a club
G :- The event that it is a king

In the above 2 cases events E & F are mutually exclusive but the events E & G are not mutually exclusive or disjoint since they may have common outcomes.

(B) ADDITION LAW OF PROBABILITY :

If E & F are two mutually exclusive events, then the probability that either event E or event F will occur in a single trial is given by : **$P(E \text{ or } F) = P(E) + P(F)$**

If the events are not mutually exclusive, then **$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ \& } F \text{ together})$** .

Note : Compare this with $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ of set theory.

Similarly **$P(\text{neither E nor F}) = 1 - P(E \text{ or } F)$** .

Independent Events And Multiplication Law :

(A) Two events are independent if the happening of one has no effect on the happening of the other.

For ex :

1. On rolling a die & tossing a coin together
E :- The event that no. 6 turns up.
F :- The event that head turns up.
2. In shooting a target
E :- Event that the first trial is missed.
F :- Event that the second trial is missed.

In both these cases events E & F are independent.

BUT 3. In drawing a card from a well shuffled pack

- E :- Event that first card is drawn
F :- Event that second card is drawn without replacing the first
G :- Event that second card is drawn after replacing the first

In this case E & F are not Independent but E & G are independent.

(B) MULTIPLICATION LAW OF PROBABILITY :

If the events E & F are independent then **$P(E \text{ \& } F) = P(E) \times P(F)$**

& $P(\text{not E \& F}) = 1 - P(E \text{ \& } F \text{ together})$.

SOLVED EXAMPLES

GROUND WORK

Ex 1. Find the number of ways in which the letters of the word “machine” can be arranged such that the vowels may occupy only odd positions ?

Ans. “Machine” consists of seven letters : four of them are consonants and three vowels. Let us mark out the position to be filled up as follows :

1	2	3	4	5	6	7
(a)	()	(i)	()	(e)	()	()

Since the vowels can be placed only in three out of the four positions marked 1,3,5,7, the total number of ways in which they can be made to occupy odd positions = ${}^4P_3 = 4.3.2 = 24$ (1)
Suppose one arrangement of the vowels is as shown in the diagram; then for this particular arrangement of the vowel, the number of ways in which the 4 consonants can be made to occupy the remaining positions (marked 2,4,6,7) = ${}^4P_4 = 4.3.2.1 = 24$.
Hence, for each way of placing the vowels in odd positions there are 24 arrangements of the whole set. Consequently the total number of arrangements of the given letters under the given condition = $24 \times 24 = 576$ Ans.

Ex 2. Sixteen jobs are vacant; how many different batches of men can be chosen out of twenty candidates ? How often may any particular candidate be selected ?

Sol. We have only to find out the number of different groups of 16 men that can be formed out of 20 without any reference to the appointment to be given to each.

Hence, the required number of ways = ${}^{20}C_{16} = {}^{20}C_4$
 $= 20 \times 19 \times 18 \times 17 / (1 \times 2 \times 3 \times 4) = 5 \times 19 \times 3 \times 17 = 4845$.

Let us now find out how many times a particular candidate may be chosen.

Every time that a particular candidate is selected the other 15 candidates will have to be chosen from the remaining 19 candidates.

Hence a particular man may be selected as many times as we can select a group of 15 men out of the remaining 19. Hence, the required number of times = ${}^{19}C_{15} = {}^{19}C_4$
 $= 19 \times 18 \times 17 \times 16 / (1 \times 2 \times 3 \times 4) = 19 \times 3 \times 17 \times 4 = 3876$ Ans.

Alternative Solution : Directly use the formula given in the theory for the case where a particular thing is always to be included etc.

Ex 3. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

Sol. Since, each number is to consist of not less than 7 digits, we shall have to use all the digits in forming the numbers. Now, among these 7 digits there are 2 two's and 3 three's; hence the total number of ways of arranging the digits = $7! / (2! 3!) = 420$. But out of these arrangements we have to reject those that begin with zero, for they are six-digit numbers. Now, evidently there are as many such arrangements as there are ways of arranging the remaining 6 digits among themselves, and $\Rightarrow$ Their no. = $6! / 2! 3! = 60$ Hence, the reqd number = $420 - 60 = 360$.

Ex 4. In how many ways can 3 letters be posted in four letter boxes in a village ? If all the three letters are not posted in the same letter box, find the corresponding number of ways of posting.

Sol. We can post the first letter in 4 ways. Similarly the second and third can be posted in 4 ways each. So the total number of ways = $4 \times 4 \times 4 = 64$. Now all the three letters together can be posted in any letterbox. In this case there will be four ways and when all the letters are not posted together, the number of ways = $64 - 4 = 60$.

Ex 5. In rolling two dice, find the probability that

- (1) there is at least one '6' (2) the sum is 5.

Sol. The total possible outcomes are 36 as shown below.

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) ; (2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6) ; (4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6) ; (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

The outcomes with at least one '6' are (1,6), (2,6), (6,6). There are 11 such pairs.

(1) $\Rightarrow P(\text{at least one '6'}) = 11/36$ Ans.

(2) The pairs with a sum of 5 are (1,4), (2,3), (3,2), (4,1).

$\Rightarrow P(\text{the sum is 5}) = 4/36 = 1/9$ Ans.

Ex 6. A single card is selected from a deck of 52 bridge cards. What is the probability that

(1) it is not a heart,

(2) it is an ace or a spade ?

Sol. A deck of bridge cards has 4 suits – spade, heart, diamond and club. Each suit has 13 cards.–

Ace, two, three,, ten, jack, Queen, King.

(1) $P(\text{not a heart}) = 1 - P(\text{a heart}) = 1 - 13/52 = 39/52 = 3/4$ Ans.

(2) There are 4 aces and 12 spades besides the ace of spades

$\Rightarrow P(\text{an ace or a spade}) = 16/52 = 4/13$ Ans.

Ex 7. A box contains 2 red, 3 yellow and 4 blue balls. Three balls are drawn in succession with replacement. Find the probability that

(1) all are yellow,

(2) the first is red, the second is yellow, the third is blue,

(3) none are yellow,

(4) all three are of the same colour.

Sol. (1) In a draw, $P(\text{red}) = 2/9$, $P(\text{yellow}) = 3/9$, $P(\text{blue}) = 4/9$.

In 3 draws, $\Rightarrow \text{Prob of all are yellow} = (3/9) \cdot (3/9) \cdot (3/9) = 1/27$ Ans.

(2) Required probability = $P(\text{1st red}) \cdot P(\text{2nd yellow}) \cdot P(\text{3rd blue})$

$= 2/9 \cdot 3/9 \cdot 4/9 = 8/243$ Ans.

(3) Probability that none are yellow = $P(\text{1st not yellow}) \cdot P(\text{2nd not yellow}) \cdot P(\text{3rd not yellow})$
 $= (1 - 3/9) \times (1 - 3/9) \times (1 - 3/9) = 8/27$ Ans.

(4) Probability that all three are of the same colour

$= P(\text{all red}) + P(\text{all yellow}) + P(\text{all blue})$ {mutually exclusive}

$= (2/9)^3 + (3/9)^3 + (4/9)^3 = 11/81$ Ans. A standard doubt : If we have only 2 red balls, how can we get 3 in succession ? Answer : Because we're replacing !

Ex 8. With the data in Example 7, answer those questions when the balls are drawn in succession without replacement.

Sol. (1) Prob of all yellow = $P(\text{1st yellow}) \cdot P(\text{2nd yellow}) \cdot P(\text{3rd yellow})$

$= 3/9 \cdot 2/8 \cdot 1/7 = 1/84$ Ans.

since when the first yellow ball has been drawn, there are 8 balls remaining in the bag of which 2 are yellow.

(2) Required probability = $P(\text{1st red}) \cdot P(\text{2nd yellow}) \cdot P(\text{3rd blue})$

$= 2/9 \cdot 3/8 \cdot 4/7 = 1/21$ Ans.

(3) Probability that none are yellow

$= P(\text{1st not yellow}) \cdot P(\text{2nd not yellow}) \cdot P(\text{3rd not yellow})$

$= (1 - 3/9) (1 - 3/8) (1 - 3/7) = 6/9 \cdot 5/8 \cdot 4/7 = 5/21$ Ans.

(4) Probability that all three are of the same colour

$= P(\text{all red}) + P(\text{all yellow}) + P(\text{all blue})$

$= 2/9 \cdot 1/8 \cdot 0/7 + 3/9 \cdot 2/8 \cdot 1/7 + 4/9 \cdot 3/8 \cdot 2/7 = 5/84$ Ans.

Note : The probability of all are red = 0.

Ex 9. There are 7 Physics and 1 Chemistry book in shelf A. There are 5 Physics books in shelf B. One book is moved from shelf A to shelf B. A student picks up a book from shelf B. Find the probability that the Chemistry book :

- (1) is still in shelf A,
- (2) is in shelf B,
- (3) is taken by the student.

Sol. (1) The probability that it is in shelf A = $\frac{7}{8}$ Ans.
(this means that Physics book was picked up)

(2) The probability that it is in shelf B =

$$P(\text{it is moved from A to B}) \cdot P(\text{it is not taken by the student}) = \frac{1}{8} \cdot \frac{5}{6} = \frac{5}{48} \quad \text{Ans.}$$

(3) The probability that it is taken by the student

$$= P(\text{it is moved from A to B}) \cdot P(\text{it is taken by the student}) = \frac{1}{8} \cdot \frac{1}{6} = \frac{1}{48} \quad \text{Ans.}$$

Note : (1) + (2) + (3) = 1 (What does it signify ?)

Ex 10. The ratios of number of boys and girls in X A and X B are 3 : 1 & 2 : 5 respectively. A student is selected to be the chairman of the students' association. The chance that the student is selected from X A is $\frac{2}{3}$. Find the probability that the chairman will be a boy.

Sol. Probability that the boy comes from X A = $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$

Probability that the boy comes from X B = $\frac{1}{3} \cdot \frac{2}{7} = \frac{2}{21}$

$\Rightarrow$ The required probability = $\frac{1}{2} + \frac{2}{21} = \frac{25}{42}$ Ans.

Ex 15. The probability that a man will be alive in 25 years is $\frac{3}{5}$ and the probability that his wife will be alive in 25 years is $\frac{2}{3}$. Find the probability that :

- (1) both will be alive,
- (2) only the man will be alive,
- (3) only the wife will be alive,
- (4) at least one will be alive.

Sol. (1) $P(\text{both alive}) = P(\text{man alive}) \times P(\text{wife alive}) = \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$

(2) $P(\text{only man alive}) = P(\text{man alive}) \times P(\text{wife dead}) = \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$

(3) $P(\text{only wife alive}) = P(\text{man dead}) \times P(\text{wife alive}) = \frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$ Ans.

(4) $P(\text{at least one will be alive}) = 1 - P(\text{both dead}) = 1 - (\frac{2}{5} \times \frac{1}{3}) = \frac{13}{15}$ Ans.

