Quarter Wit_Quarter Wisdom- Part-2

1. Combinations to make group

Let's continue our discussion on combinations today. From the previous posts, we understand that combination is nothing but "selection." Today we will discuss a concept that confuses a lot of people. It is similar to making committees (that we saw last week), but with a difference. Read the two questions given below:

Question 1: In how many ways can one divide 12 different chocolate bars equally among four boys?

Question 2: In how many ways can one divide 12 different chocolate bars into four stacks of 3 bars each?

Do you think the 2 questions are the same and the answer would be the same in both cases? After all, once you divide the chocolates into four stacks, it doesn't matter who you give them to! Actually, it does! The two questions are different. Since the chocolates are different, the four stacks will be different. So how you distribute the stacks among the 4 boys is material.

Let us take a simple case first.

Say, there are just 4 chocolate bars: A, B, C, D
We want to split them in 2 groups containing 2 chocolate bars each. There are two ways of doing this:

Method I
In group 1, we can put any 2 chocolate bars and we will put the remaining 2 chocolate bars in group 2.

We could put them in two distinct groups in the following 6 ways:
1. Group1: A and B, Group2: C and D
2. Group1: C and D, Group2: A and B (If you notice, this is the same as above. The only difference is that A and B is group 2 and C and D is group 1 here)
3. Group1: A and C, Group2: B and D
4. Group1: B and D, Group2: A and C (This is the same as above. The only difference is that A and C is group 2 and B and D is group 1 here)
5. Group1: A and D, Group2: B and C
6. Group1: B and C, Group2: A and D (Again, this is the same as above. Here, B and C is group 2 and A and D is group 1)

We have to put the four chocolates in two different groups, group 1 and group 2. It is similar to distributing 4 chocolates between 2 boys equally. Boy 1 could get (A and B) or (C and D) or (A and C) or (B and D) or (A and D) or (B and C). Boy 2 gets the other 2 chocolates in each case.

Method II
The two groups can be made in the following three ways:

A and B, C and D

A and C, B and D

A and D, B and C

In this case, the groups are not named/distinct. You have 4 chocolates in front of you and you just split them in 2 groups. (A and B, C and D) is the same as (C and D, A and B). There are a total of 3 ways of doing this i.e. half of the number of ways we saw in method 1. It is logical, isn't it? You divide the answer you get above by 2! because the two groups are not distinct in this case.

Let's look at the original two questions now:

Question 1: In how many ways can one divide twelve different chocolate bars equally among four boys?

You need to divide 12 chocolate bars among four boys i.e. you have to make four *distinct* groups. To boy 1, you can give the first chocolate in 12 ways, the second chocolate in 11 ways and the third chocolate in 10 ways. But we don't want to arrange the chocolates so you can select 3 chocolates for boy 1 in 12*11*10/3! ways (this is equivalent to $^{12}C_3$ if you follow the formula). Similarly, you can select 3 chocolates for the second boy in 9*8*7/3! ways (i.e. $^{9}C_3$), for the third boy in 6*5*4/3! ways (i.e. $^{6}C_3$) and for the fourth boy in 3*2*1/3! ways (i.e. $^{3}C_3$)

Therefore, you can distribute 12 chocolates among 4 boys equally in (12*11*10/3!) * (9*8*7/3!) * (6*5*4/3!) * (3*2*1/3!) = 12!/(3!*3!*3!*3!) ways

Alternatively, you can visualize putting the 12 chocolates in a row in 12! ways and drawing a line after every three chocolates to demarcate the groups.

OOO l OOO l OOO l OOO

Since the chocolates within the groups are not arranged, we divide 12! by 3! for every group. Since there are 4 groups, number of ways of making 4 distinct groups = 12!/(3!*3!*3!*3!)

Question 2: In how many ways can one divide 12 different chocolate bars into four stacks of 3 bars each?

What do you think will the answer be here? Will it be the same as above? No. Here the 4 stacks are not distinct. You need to divide the answer you obtained above by 4! (similar to the simple example with just 4 chocolates we saw above).

In this case, the required number of ways = 12!/(3!*3!*3!*3!*4!)

Since the groups are not distinct here, your answer changes. When the question says that you need to make *n*groups/bundles/teams that are not distinct, you need to divide by (n!). If the groups/bundles/teams are distinct then you do not divide by (n!).

Let's look at another question that uses the same concept.

Question 3: 8 friends want to play doubles tennis. In how many different ways can the group be divided to make 4 teams of 2 people each?

(A) 420
(B) 2520
(C) 168
(D) 90
(E) 105

Solution: It is quite clear here that the teams are not distinct i.e. we don't have team 1, team 2 etc. But let's solve this question by first making team 1, team 2, team 3 and team 4. Later we will adjust the answer.

Out of 8 people, in how many ways can we make team 1? In 8*7/2! ways (i.e. $^{8}C_2$).
Out of 6 people, in how many ways can we make team 2? In 6*5/2! ways (i.e. $^{6}C_2$).
Out of 4 people, in how many ways can we make team 3? In 4*3/2! ways (i.e. $^{4}C_2$).
Out of 2 people, in how many ways can we make team 4? In 2*1/2! ways (i.e. $^{2}C_2$).

In how many ways can we make the 4 teams? In 8*7*6*5*4*3*2*1/(2!*2!*2!*2!) = 8!/(2!*2!*2!*2!) ways. But here, we have considered the 4 teams to be distinct. Since the teams are not distinct, we will just divide by 4!

We get 8!/(2!*2!*2!*2!*4!) = 105

Answer (E)

I hope the explanation makes sense. We will continue with Combinatorics next week. I told you that once we get into it, it takes a long time to get out of it!

2. Tackling the beast together

Now that we have discussed both permutations and combinations independently, it's time to look at questions that involve both. Mind you, these questions are not difficult — they just involve both concepts. The first one is a circular arrangement question with a tiny twist. The second one requires us to make some cases. It takes a fair bit of patience to work out one case at a time and I doubt that GMAT will give you such a question since it is a little bit of a bore. (Actual GMAT questions have more entertainment value for the test maker and the test taker. They make you think and are FUN to solve) That said, it is a great question to bind together everything that we have learned till now and strengthen your understanding. Let's start.

Question 1: Seven women and four men have to sit around a circular table so that no two men are together. In how many different ways can this be done?

Solution: Try and think about it for a while. We did a very similar question while working on circular arrangements. In that question, number of women and number of men were equal so we just had to place them in alternate positions. Here, we have fewer men. What do we do now?

Two men cannot sit together but some women will sit together since there aren't enough men. So, let's make the 7 women sit around the round table in (7-1)! = 6! ways (We covered the (n-1)! concept in the post oncircular arrangements)

Now, how many places do we have for the men? A man can sit between any two women sitting next to each other. How many such pairs of women are there? Since there are 7 women, we have 7 such pairs and hence 7 possible spaces for men. There are two different approaches you can take from here:

Approach 1:

We have 4 men but 7 possible spaces for them. For the first man, we can select a space in 7 ways. For the second man, we can select a space in 6 ways. For the third one, in 5 ways and for the fourth one in 4 ways. So we can arrange the men in 7*6*5*4 ways. This is just our basic counting principle in action.

Approach 2:

Some people like to split up the task into two steps – make the selection, then arrange. Out of 7 spaces, we need to select any 4 for the 4 men. How do you select 4 out of 7? Using basic counting principle and un-arranging concept, we can do it in 7*6*5*4/4! ways (or we can use the formula ${}^7C_4$). We have selected 4 spaces so now we just want to arrange the 4 men in the 4 spaces. We can do this in 4! ways.

It doesn't matter which approach you use. The first one uses just the basic counting principle. The second one is used more often by people who are very comfortable with the combinations formula.

The total number of arrangements we get = 6! * 7*6*5*4 or we can write this as 6!*7!/3! to make it a little compact.

Let's look at the second question now.

Question 2: How many words of 4 letters can be formed from the word "INFINITY"? (They may or may not be actual words in the English language.)

The word INFINITY has 5 distinct letters – I, N, F, T, Y

Repetitions – I, I, I, N, N

The question doesn't say that all letters of the words have to be distinct. So, you can make a word using all three Is and another letter or two Ns and two Is etc. So you cannot just select any four letters and arrange them. The number of arrangements will vary depending on whether the letters are all distinct or have some repetitions. Let's look at all possible cases:

Case 1: All letters are distinct (Form: abcd)

From the 5 distinct letters, we can select any 4 (or drop any 1 letter) in 5 ways (you can also use ${}^5C_4$ or 5*4*3*2/4! to arrive at the figure of 5)

We can arrange these 4 selected letters in 4! ways.

Number of ways in which you can make a 4 letter word with all distinct letters = 5*4! = 120 ways

Case 2: Two letters same, others different (Form: aabc)

Only I and N are repeated so we have to select one of them and we have to select 2 of the other 4 (F, T, Y and whatever is not selected out of N and I) letters.

Select one of N and I in 2 ways. Then to choose 2 other letters, pick two from the other 4 letters in 4*3/2 ways (or $4C_2$) = 6 ways.

Now we have 3 letters and one of them is repeated so in all we have 4 letters. We can arrange 4 letters (with a repetition) in 4!/2! ways (we divide by 2! because one letter is repeated).

Number of ways in which you can make a 4 letter word with one repetition = 2 * 6 * 4!/2! = 144 ways

Case 3: 2 letters, both repeated (Form: aabb)

We have only two letters that are repeated, N and I. We will need to select both of them so the selection can be done in only 1 way.

Since both the letters are repeated, the 4 letter word can be formed in 4!/(2!*2!) = 6 ways

Case 4: 3 letters same, fourth different (Form: aaab)

Only I appears 3 times so it must be selected. We have to select one letter from the other four. We can choose the fourth letter in 4 ways.

Since I is repeated 3 times, the four letters can be arranged in 4!/3! = 4 ways.

Number of ways in which you can make a 4 letter word 4 * 4 = 16 ways

All four letters cannot be the same since no letter appears four times.

Total number of 4 letter words that can be formed using the letters of the word 'INFINITY' are 120 + 144 + 6 + 16 = 286 words

The solution is long but very methodical. If you go one step at a time, it is not complicated at all. I will see you next week with some tricky questions. Till then, keep practicing!

3. Unfair Distributions in Combinatorics- I

Today, using some examples, let's look at different ways of distributing identical/distinct objects among people or in groups. There are some formulas which can be used in some of these cases but I will only discuss how to use the concepts we have learned so far to deal with these questions. I am not a fan of unintuitive formulas since the probability (we will come to this topic soon) that we will get to use even one of them in GMAT is quite low while the effort involved in cramming all of them is humongous. Therefore, I only want to focus on our core concepts which we can apply in various situations. Let's start with our first example.

Question 1: In how many ways can 5 different fruits be distributed among four children? (Some children may get more than one fruit and some may get no fruits.)

(A) 4^5
(B) 5^4
(C) 5!
(D) 4!
(E) 4!*5!

Solution 1: This is the simplest case – 5 different things are to be distributed among 4 different people. Say, the five different fruits are: an apple, a banana, a strawberry, an orange and a blueberry. In how many ways can you give an apple to someone? In 4 ways. You can give the apple to any one of the 4 children. Similarly, in how many ways can you give a banana to someone? Again in 4 ways. It is acceptable to give more than one fruit to the same child so you again have 4 possibilities for the banana. Similarly, all the remaining fruits can also be distributed in 4 ways each.

Total number of ways of distributing 5 different fruits among 4 children = 4*4*4*4*4 = 4^5 ways.

Answer (A)

Another popular variation on this type of question is this:

Question 1a: In how many ways can 5 different rings be worn in four particular fingers? (Some fingers may get more than one ring and some may get no rings.)

Here, the answer will be a little different. Think why.

Question 2: In how many ways can 5 apples (identical) be distributed among 4 children? (Some children may get no apples.)

(A) 56
(B) 144
(C) 200
(D) 256
(E) 312

Solution: We have 5 identical apples and 4 children. We want to find the number of ways in which these apples can be distributed among the children.

Method I

5 apples can be distributed in various ways: {5, 0, 0, 0}, {4, 1, 0, 0}, {3, 2, 0, 0}, {3, 1, 1, 0}, {2, 2, 1, 0}, {2, 1, 1, 1}.

{5, 0, 0, 0} means that one child gets all 5 apples and all others get none. Similarly, {4, 1, 0, 0} means that one child gets 4 apples and another child gets 1 apple. No one else gets any apples and so on…

In each one of these cases, various arrangements are possible e.g. take the case of {5, 0, 0, 0}. The first child could get all 5 apples OR the second child could get all 5 apples OR the third child could get all 5 apples OR the fourth child could get all 5 apples. Basically, the number of ways in which these 4 objects – 5, 0, 0 and 0 can be distributed in 4 different spots (i.e. 4 children) is 4!/3! = 4 arrangements (we divide by 3! because three of the objects – 0, 0 and 0 – are identical). This is just our beloved basic counting principle in action.

Similarly, in the case of {4, 1, 0, 0}, we will get 4!/2! = 12 arrangements (since 2 objects are identical) i.e. 5 apples can be distributed among 4 children by giving 4 apples to one child and 1 apple to another child in 12 ways. The first child could get 4 apples and the second child could get 1 apple OR the third child could get 4 apples and the first child could get 1 apple etc.

In the same way, we will get 12 arrangements in each one of these cases: {3, 2, 0, 0}, {3, 1, 1, 0} and {2, 2, 1, 0}. In the case of {2, 1, 1, 1}, we will get 4!/3! = 4 arrangements.

In all, 5 identical apples can be distributed among 4 children in 4 + 12 + 12 + 12 + 12 + 4 = 56 ways

Here, we have just counted out the ways in which 5 things can be distributed in 4 groups. If we miss even one of these cases, all our effort would go waste. Therefore, let's look at a more analytical method of solving this question.

Method II

Let's put the 5 apples in a row: A A A A A

We have to split them in 4 groups. The 4 groups will have a one-to-one relation with the 4 children – Apples in the first group will be given to the first child, those in the second group will be given to the second child and so on…

Say we split the apples in 4 groups in the following way: A A l A l A l A

The vertical lines separate one group from the other. The first group has 2 apples and the rest of the three groups have 1 apple each. This means, the first child gets 2 apples and each of the other 3 children get 1 apple each.

The split can also be made in the following way: A l A A l A l A

Here, the second group has 2 apples and the rest of the three groups have 1 apple each. So the second child gets 2 apples and the rest of the 3 children get 1 apple each.

The split can also be made in the following way: l A A A l A l A

Here, the first group has no apples, the second group has 3 apples and the third and the fourth groups have one apple each. The first child gets no apples, the second child gets 3 apples and the other 2 children get 1 apple each.

What I am trying to show here is that arranging these 5 identical As and the 3 identical vertical lines in as many different ways as possible will give us all the ways in which we can distribute 5 identical apples among 4 different children.

In how many different ways can we arrange these 8 objects i.e. 5 identical As and 3 identical vertical lines? In 8!/(5! * 3!) = 56 ways

Answer (A).

We have seen how to solve this question in two different ways. A point to note here is that method II cannot be used in question 1 above. Think why.

Now try to solve these two questions:

Question 3: In how many ways can 5 different fruits be distributed among 4 identical baskets?

Question 4: In how many ways can 5 apples (identical) be distributed among 4 identical baskets?

We will look at their solutions and the solution of the variation question next week!

4. Unfair Distributions in Combinatorics- II

Today's post is a continuation of last week's post and heavily refers back to it. I would suggest you to take a quick look at last week's post again to make sense of this post. Let's start with the variation question 1a we saw in the last post.
Question 1a: In how many ways can 5 different rings be worn in four particular fingers? (Some fingers may get more than one ring and some may get no rings.)
Solution: The first ring can be worn in 4 ways i.e. in any one of the four fingers. The second ring can be worn in 5 ways (it can go on any one of the four fingers and it can also go below the first ring so there are 5 distinct places for the second ring). The third ring can be worn in 6 ways (any one of the four fingers or below the second ring or below the first ring). The fourth ring can be worn in 7 ways (any one of the four fingers or below the third ring or below the second ring or below the first ring). The fifth ring can be worn in 8 ways (any one of the four fingers or below the fourth ring or below the third ring or below the second ring or below the first ring).

Total number of ways in which 5 different rings can be worn in 4 particular fingers = 4*5*6*7*8.
Compare this with question 1 of last post: In how many ways can 5 different fruits be distributed among four children? The answer in this case was 4^5 = 4*4*4*4*4.
Why are these two questions different? After all, we are distributing 5 different things among 4 children/fingers in both the cases. The difference lies in the fact that when a child gets 2 fruits, the fruits are not arranged but when a finger gets two rings, it gives us 2 different arrangements since the rings can be arranged in 2 ways. You can wear 2 rings on your

fingers in 2 different ways (A on top, B at the bottom or B on top and A at the bottom). When you get 2 fruits, there is no arrangement involved. Whether you got fruit A first or fruit B first doesn't matter. At the end of it, you have 2 fruits A and B and that's all that matters. In fact this is the reason we cannot solve question 1 using method 2 of question 2 (discussed in the last post). Let's still try to use it and see why it doesn't work.

Say, we have 5 different things in a row: A B C D E and 3 identical vertical lines to split these 5 objects into 4 groups. We can arrange these 8 objects, 3 of which are identical, in 8!/3! ways. Notice that 8!/3! = 8*7*6*5*4 i.e. the numbers of ways in which 5 different rings can be worn in 4 fingers. It is not the same as the number of ways in which 5 different fruits can be distributed among 4 children.

We see that:

AB l C l DE

and

BA l C l DE

are two different arrangements. Since how you wear the rings gives you different arrangements, the vertical lines split method can be used to get the answer in the rings question (question 1a). Since these two should not be two different arrangements in case we are talking about distributing fruits among children, this method is not suitable for question 1.I hope I haven't already confused you. We still have a long way to go!

We should now pay attention to question numbers 3 and 4 from the previous post.

Question 3: In how many ways can 5 different fruits be distributed among 4 identical baskets?

Solution: Let's use the same format as that used in the previous post. 5 fruits can be split into 4 groups in the following ways: {5, 0, 0, 0}, {4, 1, 0, 0}, {3, 2, 0, 0}, {3, 1, 1, 0}, {2, 2, 1, 0}, {2, 1, 1, 1}

Does it concern us that the baskets are identical? It does. Let's see how.

{5, 0, 0, 0} means that one basket has all 5 fruits and the rest of the 3 baskets are empty. It doesn't matter which basket has the fruits because all the baskets are identical. So, this gives us 1 way of distributing the fruits.

{4, 1, 0, 0} means that one basket has 4 fruits, another has the leftover 1 fruit and the other 2 baskets have no fruit. The lone fruit can be chosen in 5 ways. The rest of the 4 fruits will be together in another basket and 2 baskets will be empty. This gives us 5 different ways of distributing the fruits.

{3, 2, 0, 0} means that one basket has 3 fruits, another has the leftover 2 fruits and the other 2 baskets have no fruit. The 3 fruits can be chosen in 5*4*3/3! = 10 ways (= $^5C_3$). The rest of the 2 fruits will be together in another basket in one way and 2 baskets will be empty. This gives us 10 different ways of distributing the fruits.

{3, 1, 1, 0} means that one basket has 3 fruits, another two have a fruit each and the leftover basket has no fruit. The 3 fruits can be chosen in 5*4*3/3! = 10 ways (= $^5C_3$). The rest of the 2 fruits will be in two baskets in one way (since the baskets are all identical) and the last basket will be empty. This gives us 10 different ways of distributing the fruits.

{2, 2, 1, 0} means that 2 baskets have 2 fruits each, one basket has one fruit and the last basket is empty. We can select 1 fruit out of 5 in 5 ways. Now we are left with 4 fruits which have to be split into 2 groups of 2 each. This can be done in 4!/2!*2!*2! = 3 ways (We have already discussed this concept in the post on Groups. Check out the initial theory and question no. 2) This gives us 5*3 = 15 different ways of distributing the fruits.

{2, 1, 1, 1} means that one basket has 2 fruits and the rest of the 3 baskets have a fruit each. We can select 2 fruits out of 5 in 5*4/2! = 10 ways (= $^5C_2$). This gives us 10 different ways of distributing the fruits.

Total number of different ways of distributing the fruits = 1 + 5 + 10 + 10 + 15 + 10 = 51 ways

Something for you to think about: We used the brute force method here. Can we use some more analytical and direct method to solve this question?

Meanwhile, let's look at question number 4 now.

Question 4: In how many ways can 5 apples (identical) be distributed among 4 identical baskets?

Solution: As we have seen previously, 5 fruits can be split into 4 groups in the following ways: {5, 0, 0, 0}, {4, 1, 0, 0}, {3, 2, 0, 0}, {3, 1, 1, 0}, {2, 2, 1, 0}, {2, 1, 1, 1}

Here, {5, 0, 0, 0} means that one basket has all 5 apples and the rest of the baskets are empty. Since the baskets are all identical, there is only 1 way of doing this.

{4, 1, 0, 0} means that one basket has 4 apples, another one has 1 apple and the rest of the baskets are empty. Since the fruits and the baskets are all identical, there is again only 1 way of doing this.

You have to select neither the fruits nor the baskets since they are all identical. You only have to decide how to distribute the apples in 4 groups. Therefore, each one of these cases will give us only one different way of distributing the fruits. Since there are 6 such cases, there are only 6 different ways of distributing the fruits.

I wouldn't be surprised if you are a little confused at this point since the different variations change the thought process completely. That is the whole fun of combinatorics. You change one word and we have to start thinking from the scratch. You miss one word and you either don't realize at all that your answer is wrong (if the options are cunning) or you realize after you have solved the entire question. Thankfully, GMAT has only one to two questions based on this topic!

5. Of Letters & Envelopes

Another popular combinatorics concept deals with letters and envelopes. Let's look at it today in some detail.

Question 1: Robin wrote 3 different letters to send to 3 different addresses. For each letter, she prepared one envelope with its correct address. If the 3 letters are to be put into the 3 envelopes at random, in how many ways can she put

(i) all three letters into the envelopes correctly?
(ii) only two letters into the envelopes correctly?
(iii) only one letter into the envelope correctly?
(iv) no letter into the envelope correctly?

Solution: Let's say we have 3 letters: La, Lb and Lc and 3 envelopes: Ea, Eb and Ec

Total number of ways of assigning 3 different letters to 3 different envelopes = 3*2*1 = 6 (using our basic counting principle). These total 6 ways of randomly putting letters in envelopes includes all the above given 4 cases. The number of ways in each case should add up to give us 6.

(i) all three letters into the envelopes correctly?

For all three letters to be put correctly, each letter must be put in its corresponding envelope only i.e. La must go into Ea, Lb must go into Eb and Lc must go into Ec. Therefore, there is only 1 way in which all three letters can be put into the envelopes correctly.

(ii) only two letters into the envelopes correctly?

Is this possible? Is it possible that only 2 letters are put into their envelopes correctly? If La goes into Ea and Lb goes into Eb, where will we put Lc? It has to go into Ec. There is no other option. So it is not possible to put only two letters into their correct envelopes.

(iii) only one letter into the envelope correctly?

First of all, we will need to select the letter which has to be put in correctly. We can select one of the three letters in 3 ways (i.e. we can put either La in Ea or Lb in Eb or Lc in Ec) The leftover 2 letters must be put in the wrong envelopes. Say, we put Lb in Eb (one letter which is put correctly). Now we are left with La and Lc and Ea and Ec. La must go into Ec and Lc must go into Ea. There is only 1 way of ensuring that the other two letters go into the wrong envelopes. Hence, total number of ways such that only one letter goes into the correct envelope = 3*1 = 3 ways.

(iv) no letter into the envelope correctly?

Of the total 6 ways, 1+0+3 = 4 ways are already accounted for. Now, in the leftover 2 ways, no letter must be in its correct envelope. Directly calculating this is a little tricky so we prefer to use the method of 'Total – All Other Cases'. If you still want to get an idea of how we get 2 ways, use an example to understand this:

La can be put in either Eb or Ec (i.e. 2 ways). Say, La is put in Ec. Now we have 2 letters leftover: Lb and Lc and 2 envelopes leftover: Ea and Eb. Mind you, Lb cannot go into Eb so Lb must go into Ea and Lc must go into Eb i.e. there is only one way of putting in the other two letters. So, number of ways of putting in all the letters incorrectly = 2*1 = 2 ways.

The same logic can be used for more letters and envelopes too though it keeps getting more and more complicated.

Question 2: Robin wrote 4 different letters to send to 4 different addresses. For each letter, she prepared one envelope with its correct address. If the 4 letters are to be put into the 4 envelopes at random, in how many ways can we put

(i) all four letters into the envelopes correctly?
(ii) only three letters into the envelopes correctly?
(iii) only two letters into the envelopes correctly?
(iv) only one letter into the envelope correctly?
(v) no letter into the envelope correctly?

Solution: Extending the same logic as used above, we can say that we have 4 letters: La, Lb, Lc and Ld and 4 envelopes: Ea, Eb, Ec and Ed

Total number of ways of assigning 4 different letters to 4 different envelopes = 4*3*2*1 = 24 (using our basic counting principle). These total 24 ways of randomly putting letters in envelopes includes all the above given 5 cases. The number of ways in each case should add up to give us 24.

(i) all four letters into the envelopes correctly?

Using the same logic as in (i) above, there is only 1 way in which all four letters can be put into the envelopes correctly.

(ii) only three letters into the envelopes correctly?

Is this possible? Is it possible that only 3 letters are put in their envelopes correctly? Using the same logic as used in (ii) above, it is not possible to put only three letters into their correct envelope.

(iii) only two letters into the envelopes correctly?

First of all, we will need to select the two letters which have to be put in correctly. We can select two of the four letters in 4*3/2 ways i.e. ${}^4C_2$ = 6 ways (i.e. we can put either La in Ea and Lb in Eb OR La in Ea and Lc in Ec etc) The leftover 2 letters must be put in the wrong envelopes. Say, we put La into Ea and Lb into Eb (two letters which are put correctly). Now we are left with Lc and Ld and Ec and Ed. Lc must go into Ed and Ld must go into Ec. There is only 1 way of ensuring that the other two letters go into the wrong envelopes. Hence, total number of ways such that only two letters go into the correct envelopes = 6*1 = 6 ways

(iv) only one letter into the envelope correctly?

Let's first select the one letter out of 4 that must be put in its correct envelope. We can do this in 4 ways. In how many ways can we put the rest of the 3 letters into 3 envelopes such that all 3 letters go into incorrect envelopes? Does this question sound familiar? Sure it does! It is our case (iv) in question 1 above. We found that 3 letters can be put into 3 envelopes such that each letter is put in incorrectly in 2 ways. So total number of ways such that only two letters go into the envelopes with their correct addresses = 4*2 = 8 ways

(v) no letter into the envelope with its correct address?

Of the total 24 ways, 1+0+6+8 = 15 ways are already accounted for. Now, in the leftover 9 ways, no letter must be in its correct envelope. Again, directly calculating this is a little tricky but if you want to get an idea of how we get 9 ways, use an example to understand this:

La can be put in either Eb or Ec or Ed (i.e. 3 ways). Say, La is put in Ec. Now we have 3 letters leftover: Lb, Lc and Ld and 3 envelopes leftover: Ea, Eb and Ed. Lc, the letter corresponding to Ec, can be put in any one of these three envelopes. Hence Lc can be put in 3 ways too. Say, Lc is put in Ed. Now, we have two letters, Lb and Ld leftover and two envelopes, Ea and Eb leftover. Lb cannot go into Eb so Lb must go into Ea and Ld must go into Eb i.e. there is only one way of putting in the other two letters. So, number of ways of putting in all the letters incorrectly = 3*3*1 = 9 ways.

Something to think about: Here, why don't we put one letter incorrectly and say that the leftover – 3 letters and 3 envelopes, each letter has to go in incorrectly – is the same as case iv in question 1 above? After all, we used this method in case iv of this question.

6. Linking Roots & Coefficient of Quadratics Equations

If you have been following my last few posts, I am sure you are a little wary of today's post. They have been a little convoluted lately since we are dealing with permutations and combinations. Next, we will tackle probability but today, I am going to digress (to give you some much needed respite) and take up a simple yet interesting topic. We deal with quadratic equations on a regular basis. Tackling them effectively is pretty much one of the most basic and important skills you need for GMAT Quant. Today we will look at some relationships between the coefficients of quadratic equations and roots.

I am sure you know how to solve a quadratic equation so I will not delve into that. I am also assuming that you studied the Vieta's formulas in high school.

Let me recap it here:

Given a quadratic equation ax^2 + bx + c = 0 with roots p and q,

Sum of the roots = p + q = -b/a

Product of the roots = pq = c/a

It is good to remember these relations since they can be useful sometimes. (You can easily derive them by expanding the following: ax^2 + bx + c = a(x – p)(x – q))

Let us go through a quick example.

Example: Two friends, Ann and Beth started solving a quadratic equation. Ann made a mistake while copying the constant term and got the roots as 5 and 9. Beth made a mistake in the coefficient of x and she got the roots as 12 and 4.What is the equation?

(A) x^2 + 4x +14 =0

(B) 2x^2 +7x -24 =0

(C) x^2 -14x +48 =0

(D) 3x^2 -17x +52 =0

(E) 2x^2 + 4x +14 =0

Solution:

Ann made a mistake while copying the constant term i.e. c but she copied a and b properly. So the sum of the roots she found must be correct.

-b/a = 5 + 9 = 14

Beth made a mistake while copying the coefficient of x i.e. b but she copied a and c properly. So the product of the roots she found must be correct.

c/a = 12*4 = 48

Only option (C) above has –b/a = 14 and c/a = 48. Therefore, answer must be (C).

You see knowing the Vieta's formulas made a potentially tricky question pretty simple.
Now on to the real thing I had in mind for this post. Once, someone asked me the following question: Is there a quick way to know whether a quadratic equation has one or two positive solutions without the need to solve it?
It made me think how we know so much but are still sometimes unable to connect the dots between inter-linked concepts. The following was my answer to the query. I hope you find it useful too.
Answer: Yes, there is. The Vieta's formulas help us figure it out.
Let's say we have a quadratic equation: $ax^2 + bx + c = 0$
We know that sum of the roots = $-b/a$
Product of the roots = c/a
(We are dealing with only real roots.)
Case 1: Sum of the roots is positive; product of the roots is positive.
If product is positive, it means both roots are either positive or both are negative. Since the sum is also positive, both roots must be positive.
For example: $2x^2 – 8x + 1 = 0$
Product of roots = 1/2
Sum of roots = -(-8)/2 = 4
Both roots positive.
Case 2: Sum of the roots is negative; product of the roots is positive.
If product is positive, it means both roots are either positive or both are negative. Since the sum is negative, both roots must be negative.
For example: $x^2 + 5x + 4 = 0$
Product of roots = 4/1 = 4
Sum of the roots = -5/1 = -5
Both roots negative.
Case 3: Product of the roots is negative.
If product is negative, it means one root is positive and the other is negative. If the sum is positive, the absolute value of the positive root is higher than the absolute value of the negative root. If the sum is negative, the absolute value of the negative root is higher than the absolute value of the positive root.
For example: $3x^2 -6x -2 = 0$
Product of roots = -2/3
One root negative, one positive.
These little insights help you solve quirky questions quickly and neatly. Be on the lookout for inferences you can draw from the knowledge you already possess and as always, keep practicing!

7. The Intricacies of Probability

Now that we have laid the groundwork for permutations and combinations, probability will be a piece of cake. We just need to build up on what we have already learned.

The single most important concept in probability is the following:

The probability of an event A is calculated as P(A) = No. of outcomes when A occurs/Total no. of outcomes.

In this post, we will just extend the combinatorics concepts and apply them to probability. Let me explain how we will do it using some examples.

Example 1: Six friends live in the city of Monrovia. There are four natural attractions around Monrovia – a waterfall, a safari, a lake and some caves. The friends decide to take a vacation together at one of these attractions. To select the attraction, each one of them votes for one of the attractions. What is the probability that each one of them votes for the safari?

Solution: Here, A, the event for which we want to find the probability is 'all six friends vote for the safari'

P(A) = No of ways in which all six can vote for the safari/Total no. of ways in which they can vote.

What is the no. of ways in which all six vote for the safari? Only one way. They all vote for the safari!

What is the no. of ways in which the friends can vote? Say, the friends are A, B, C, D, E and F. A can vote in 4 ways. B can vote in 4 ways. C can vote in 4 ways and so on… Total no of ways in which the 6 friends can vote = 4*4*4*4*4*4 = 4^6 (Using our old friend, the basic counting principle). We discussed this concept in our post on Unfair Distributions.

Therefore, P(A) = 1/(4^6)

Finding this probability involved the use of the concepts we have already learned in combinatorics. I hope you see that it is quite simple and straight forward. Let's tweak this example a little to make it slightly complicated.

Example 2: Six friends live in the city of Monrovia. There are four natural attractions around Monrovia – a waterfall, a safari, a lake and some caves. The friends decide to take a vacation together at one of these attractions. To select the attraction, each one of them votes for one of the attractions. What is the probability that each one of them votes for the same attraction?

Solution: Here, A, the event for which we want to find the probability is 'all six friends vote for the same attraction'. We don't have a specific attraction given to us. So the selected attraction could be any one of the given four.

P(A) = No of ways in which all six can vote for the same attraction/Total no. of ways in which they can vote.

What is the no. of ways in which all six vote for the same attraction? They could all vote for the waterfall or for the safari or for the lake or for the caves. All of them can vote for the same attraction in 4 ways.

What is the no. of ways in which the friends can vote? As we saw in question no. 1, total no of ways in which the 6 friends can vote = 4*4*4*4*4*4 = 4^6

Therefore, P(A) = 4/(4^6) = 1/(4^5)

Now, let's make the question even trickier.

Example 3: Six friends live in the city of Monrovia. There are four natural attractions around Monrovia – a waterfall, a safari, a lake and some caves. The friends decide to take a vacation together at one of these attractions. To select the attraction, each one of them votes for one of the attractions. What is the probability that each attraction gets at least one vote?

Solution: Here, A, the event for which we want to find the probability is 'each attraction gets at least one vote'.

P(A) = No of ways in which each attraction gets at least one vote /Total no. of ways in which the friends can vote.

Each attraction should get at least one vote. 6 votes can be divided among 4 attractions in the following ways: (1, 1, 1, 3) and (1, 1, 2, 2)

Case 1: (1, 1, 1, 3)

First, we select the attraction that will get 3 votes in 4 ways (= 4C1)

Now, we can select the 3 people who will vote for this attraction in 6*5*4/3! = 20 ways (= 6C3)

The other 3 votes will be distributed among the other 3 attractions in 3! = 6 ways

The 6 people could vote for the 4 attractions in this case in 4*20*6 = 480 ways

Case 2: (1, 1, 2, 2)

Let's select the two attractions that will get 2 votes each in 4*3/2! = 6 ways (= 4C2). Say we select caves and waterfall.

Now, we can select the 2 people who will vote for one of the selected attractions in 6*5/2! = 15 ways (= 6C2)

We can select the other 2 people who will vote for the other selected attraction in 4*3/2! = 6 ways (= 4C2)

The other 2 votes will be distributed among the other 2 attractions in 2! = 2 ways

The 6 people could vote for the 4 attractions in this case in 6*15*6*2 = 1080 ways

Total number of ways in which 6 votes can be distributed among 4 attractions such that each attraction gets at least one vote = 480 + 1080 = 1560 ways

As we saw in the questions above, the total no. of ways in which the friends can vote = 4^6

Therefore, P(A) = 1560/(4^6)

I hope you see that probability is just an extension of combinatorics. Some important concepts in Probability e.g. Independent events, mutually exclusive events, dependent events etc are discussed in detail in your Combinatorics and Probability book. Go through that theory before next Monday. We will discuss some tricky questions related to those concepts next week.

8. Is it a Hit or a Miss

I hope you have gone through the theory of probability from your book. I will not replicate that theory here but will assume that you already know it. Instead, what we will do now is take some tricky questions on probability and try and find out the various ways in which they can be solved. Hope they give you ideas and takeaways for other questions too!

Question: At the shooting range, the probability that Robert will hit the target in any one shot is 25%. If he takes four shots one after another, what is the probability that he will hit the target?

Solution: The first thing to understand here is the figure of 25%. What does it mean? It means that the probability of Robert hitting the target in a shot is 1/4. So he is expected to miss the target 3 out of 4 times i.e. his probability of not hitting the target in a shot is 3/4.

To hit the target, he needs to hit it at least once. When he takes four shots in succession, if he hits the target in any one shot, the target is hit. After that, whether he hits it or misses, it doesn't matter at all.

Another thing, the question asks you for the probability that he will hit the target in 4 shots? Is this probability 1 then? (since he is expected to hit it once in every four shots) No. Each shot he takes is independent of the shot he took before that and the shot he will take after that. In every shot the probability of hitting the target remains 1/4. Not that after he takes 3 shots and fails to hit the target, his probability of hitting the target in the fourth shot will become 1; it will still remain 1/4.

A similar case that might make this concept clearer is 'tossing a coin'. What is the probability of getting 'heads' in a toss? 1/2, I am sure you agree. So if I toss a coin once and get 'tails', does it mean I will definitely get 'heads' on the next toss? No, because the two tosses are independent of each other. My probability of getting 'heads' is still 1/2 on the second toss.

So we have established that the probability is not 1. Let's try and find out what the probability is. You can do it in various ways. Let's look at some of them.

Method 1: This is the simplest method and should be used. (The other two methods I will discuss are for intellectual purposes. In some questions, these other methods might come in handy.)

To hit the target, you need to hit it at least once in the 4 shots. You would not have hit the target, if you do not hit it in any of the 4 shots. Let's find the probability that he will not hit the target in any of the four shots. We can then subtract it from 1 to get the probability that he will hit the target in at least one shot.

Probability of not hitting the target in a shot = 3/4

Probability of not hitting the target in any of the 4 successive shots = (3/4)*(3/4)*(3/4)*(3/4) = 81/256

(We multiply the '(3/4)'s together because they are independent events.)

Probability of hitting the target at least once = 1 – (81/256) = 175/256

This is the required probability!

Method 2:

You hit the target if you hit it at least once. Say, if you hit the target on the first shot, you are done no matter what you do on the rest of the 3 shots. If you do not hit the target on the first shot, but hit it on the second shot, again, you are done and so on…

Probability of hitting the target on the first shot = 1/4

Probability of not hitting the target on the first shot but hitting it on the second shot = (3/4) * (1/4)

Probability of not hitting the target on the first and second shot but hitting it on the third shot = (3/4) * (3/4) * (1/4)

Probability of not hitting the target on the first, second and third shot but hitting it on the fourth shot = (3/4) * (3/4) * (3/4) * (1/4)
He could have hit the target in any one of the four ways given above.

Therefore, probability of hitting the target = (1/4) + (3/4)*(1/4) + (3/4)*(3/4)*(1/4) + (3/4)*(3/4)*(3/4)*(1/4) = (1*64 + 3*16 + 9*4 + 27)/256 = 175/256

As expected, the answer we get is the same as the one obtained in method 1 above.

Method 3: There is another way of thinking about this.

Say H = Hit and M = Miss

He can hit the target by hitting it only once. He can also hit the target by hitting it on two shots. He can also hit the target by hitting it on three shots and so on…

If he hits the target only once, it means he is successful (probability 1/4) once and unsuccessful (probability 3/4) three times. This can happen in various ways e.g. HMMM or MHMM or MMHM or MMMH. The hits and misses can be arranged in 4!/3! ways. (You arrange 4 things out of which 3 are identical in 4!/3! ways.)

Probability of hitting the target only once = (1/4)*(3/4)*(3/4)*(3/4) * 4!/3! = 27*4/256

Similarly, probability of hitting the target twice = (1/4)*(1/4)*(3/4)*(3/4) * 4!/(2!*2!) = 54/256
(Why do we multiply by 4!/(2!*2!)? Same logic as above. You arrange 4 things out of which 2 pairs are identical in 4!/(2!*2!) ways. HHMM or HMHM or MHHM etc.)

Using the same logic, probability of hitting the target three times = (1/4)*(1/4)*(1/4)*(3/4) * 4!/3! = 12/256 and probability of hitting the target four times = (1/4)*(1/4)*(1/4)*(1/4)*1 = 1/256

To get the probability of hitting the target, all we need to do now is add up all the four cases above.

Therefore, probability of hitting the target = 27*4/256 + 54/256 + 12/256 + 1/256 = 175/256

Of course, the answer still remains 175/256

I am sure different methods work for different people. Similarly, different methods work for different questions too. My suggestion is to be comfortable with all three methods. In weeks to come, I might point back to one of these 3 methods and show you why it works best for a particular question. Till then, keep practicing!

9. Removal/replacement in Mixtures

Today, as requested by Pratap, we are going to take removal/replacement in mixtures. For those of you who were looking forward to some more tricky probability questions, I will make up for your disappointment next week. Meanwhile, rest assured, replacement is a very interesting, not to mention useful, concept in GMAT. So brace yourself to learn some new things today.

First of all, many "replacement" questions are nothing but the plain old mixture questions, the type we discussed in this post, with an extra step. So don't flip out the moment you read the word "replace." Let me show you what I mean:

Example 1: If a portion of a 50% alcohol solution (in water) is replaced with 25% alcohol solution, resulting in a 30% alcohol solution, what percentage of the original alcohol was replaced?

(A) 3%
(B) 20%
(C) 66%
(D) 75%
(E) 80%

Solution: What this question says is that a solution of 25% alcohol (say solution 1) is mixed with another solution of 50% alcohol (say solution 2) to give us a 30% alcohol solution. We don't know in what quantities they were mixed. Can we find out the ratio in which these two solutions were mixed? If you are not sure, check out this post.

w1/w2 = (A2 – Aavg)/(Aavg – A1) = (50 – 30)/(30 – 25) = 4/1

So the two solutions were mixed in the ratio 4:1. Mind you,

(volume of 25% alcohol solution) : (volume of 50% alcohol solution) = 4:1

Out of 5 parts of total solution obtained, 50% solution was 1 part while 4 parts was 25% solution. So what part of the 50% solution was removed and replaced by the 25% solution? Would you agree it is (4/5)th? We can say that 80% of the 50% solution was replaced by the 25% solution.

But the actual question is something else: What percentage of **original alcohol** was replaced?

We need to find the percentage of original alcohol that was replaced, not the percentage of original solution! Now, here is the interesting thing: Since the solution is homogenous, it you replace 80% of it, 80% of the original amount of alcohol in the solution will be replaced. So the answer will still be 80%. I will explain this point in detail since it is extremely important while dealing with the really treacherous replacement questions.

Let's say we have 100 liters of 50% alcohol solution (so alcohol = 50 liters and water = 50 liters). When we remove 80% of the solution, we remove 80 liters of the solution. In the solution we remove, we will still have 50% alcohol i.e. we will have 40 liters alcohol and 40 liters water. In the 20 liters solution that is remaining, we will have 10 liters alcohol and 10 liters water. So amount of alcohol removed is 40/50 = 80%

Quarter Wit_Quarter Wisdom- Part-2

Important Points to Remember:

1. When a fraction of a homogenous solution is removed, the percentage of either part does not change. If milk:water = 1:1 in initial solution, it remains 1:1 in the leftover solution.

2. When you add one component to a solution, the amount of the other component does not change. In milk and water solution, if you add water, amount of milk is still the same (not percentage but amount). If milk:water = 1:1 in 10 liters of solution, it means, milk = 5 liters and water = 5 liters. Now, if you add 2 liters of water, amount of water = 7 liters but amount of milk is still 5 liters. The percentage of milk has changed but the amount of milk is still the same.

3. Amount of A = Concentration of A * Volume of the mixture

Amount = C*V

In a 10 liter mixture of milk and water, if milk is 50%, amount of milk = 50%*10 = 5 liter
When you add water to this solution, the amount of milk does not change (as discussed in point 2 above). The concentration of milk changes of course since the solution is diluted.

Amount of milk before addition = Amount of milk after addition

So Initial Concentration of milk * Initial Volume of solution = Final Concentration of milk * Final Volume of solution

Ci * Vi = Cf * Vf

Or

Cf = Ci * (Vi/Vf)

Remember, this is the relation between the initial and final concentration of milk since the amount of milk remains the same. The amount of water does not remain the same since more water is added. Hence, this relation does not hold for water.

Go through these points repeatedly till you are very comfortable with them!

Example 2: 10% of a 50% alcohol solution is replaced with water. From the resulting solution, again 10% is replaced with water. This step is repeated once more. What is the concentration of alcohol in the final solution obtained?

(A) 3%
(B) 20%
(C) 25%
(D) 36%
(E) 40%

Solution: In each step, we are replacing the solution with water. Every time we remove p% of the solution, the amount of alcohol goes down but the concentration of alcohol in the mixture remains the same (point 1 above). When we add water, the amount of alcohol remains the same.

Let's try and perform the steps to see what happens:

Step 1: 10% of a 50% alcohol solution is removed – In the leftover solution, concentration of alcohol remains the same i.e. 50%. If initial volume of the solution was 10 liters, new volume is 9 liters.

Step 2: Water is added to the solution to replace the 10% shortfall – the concentration of alcohol changes now (but the amount of alcohol is still the same). Also, the volume of the solution is 10 liters again. In this new solution, The concentration of alcohol after this step Cf1 = (50%)*(9/10) (using point 3)

Step 3: 10% of the solution with concentration of alcohol = Cf1 is removed – In the leftover solution, concentration of alcohol is still Cf1. The volume of the solution reduces to 9 liters again.

Step 4: Water is added to the solution to replace the 10% shortfall – The concentration of alcohol changes now. Also, the volume of the solution is 10 liters again. In this new solution,

The concentration of alcohol after this step Cf2 = Cf1*(9/10) = (50%)*(9/10) *(9/10)

Step 5: 10% of the solution with concentration of alcohol = Cf2 is removed – In the leftover solution, concentration of alcohol is still Cf2. The volume of the solution reduces to 9 liters again.

Step 6: Water is added to the solution to replace the 10% shortfall – The concentration of alcohol changes now. Also, the volume of the solution is 10 liters again. In this new solution,

The concentration of alcohol after this final step Cf3 = Cf2*(9/10) = .5*(9/10) *(9/10) *(9/10)

The concentration changes only when water is added. Each time water is added, the concentration becomes (9/10)th of the previous concentration.

Final concentration of alcohol = (50%) *(9/10) *(9/10) *(9/10) = 36.45%

Answer (D)

Now try the following question to see if the theory makes sense to you:

Example 2: 20% of a 40% alcohol solution is removed and replaced with water. From the resulting solution, again 20% is replaced with water. This step is repeated once more. What is the concentration of alcohol in the final solution obtained?

Solution:

Concentration of alcohol in the final solution = (40%) * (8/10) * (8/10) * (8/10) = 20.48%

I will leave you here with a complicated question. See if you can arrive at the answer on your own. If not, let me know!

Question 1: A container has 3 liters of pure lime juice. 1 liter from the container is taken out and 2 liter water is added. The process is repeated several times. After 19 such operations, quantity of lime juice in the mixture is

(A) 2/7 L
(B) 3/7 L
(C) 5/14 L
(D) 5/19 L
(E) 6/19L

This question can be solved in under a minute if you understand the concept of concentration and volume. Take your time and see if you can do it on your own!

10. Separating the couples

Let's take another tricky probability question today and employ two different methods to solve it.

Question: Two couples and one single person occupy a row of five chairs at random. What is the probability that neither couple sits together (the husband and the wife should not occupy adjacent seats)?

(A) 1/5

(B) 1/3
(C) 3/8
(D) 2/5
(E) 1/2

Solution: You can approach this GMAT problem in different ways. One way is a step-by-step case evaluation. Another is to go the reverse way: count all the arrangements in which at least one couple sits together and subtract that from the total arrangements possible. My method of choice is generally the second one. The only catch is that you have to remember to subtract from the total number of arrangements.

What is the total number of arrangements (without any restrictions)? I hope you remember your basic counting principle and will agree that it should be 5! (Five people arranged in five seats). Now, let's find out the number of favorable cases.

We will discuss both the methods in detail.

<u>Method 1</u>: Step – by – Step Case Evaluation

Let's say the two couples are {C1h, C1w} and {C2h, C2w} and the single person is S.

Case 1: S takes the first/last chair

If S takes the first chair, any one of the remaining people can take the chair next to him (4 ways). Say, C1h takes this spot. The next place cannot be occupied by C1w but either of C2h and C2w can occupy it (2 ways). Say, C2h occupies it. The fourth chair can be occupied by only one of the remaining 2 people since C2w cannot take it now (1 way). The last chair has only one person remaining for it.

Total number of acceptable arrangements in which S takes the first chair = 4*2*1 = 8

The case would be exactly the same if S took the fifth seat. Think of it this way: The chairs have seat numbers 1-5. Now the numbers have reversed, 1 switched with 5, 2 switched with 4 and 3 is as it is. Now S is sitting on seat number 5 and we have exactly 8 more arrangements possible.

Total number of acceptable arrangements in which S takes the first or fifth chair = 8*2 = 16

Case 2: S takes the second/fourth chair

If S takes the second seat, any of the remaining four people could sit next to S on either side. However, we need to ensure that both people sitting on either side of S are not a couple i.e. C1h, S, C1w should not occupy the first, second and third seats respectively because then C2h and C2w are left and only 2 adjacent seats are vacant. But they cannot take adjacent seats. This means that there are 4 ways in which the first seat can be occupied — i.e., anyone can take it but there are only 2 ways in which the third seat can be occupied since the person taking the third seat must be from the other couple — i.e., if C1h takes the first seat, only C2h or C2w could take the third one. Now we have 2 people and 2 seats leftover. Fourth seat can be occupied in only one way since if C2h takes the third seat, C2w cannot take it i.e. one of the remaining two people cannot take it. Thereafter, one person and one seat are leftover so the fifth seat can be occupied in one way.

Total number of acceptable arrangements in which S takes the second chair = 4*2*1*1 = 8

The case would be exactly the same if S took the fourth seat.

Total number of acceptable arrangements in which S takes the second or fourth chair = 8*2 = 16

Case 3: S takes the middle seat i.e. the third seat

If S takes the third seat, there are two seats on his left and two on his right. We have to ensure that a couple doesn't sit on one side and the other side would automatically be couple-free. Any one of the four people can occupy the first seat (say C1h takes it). The second seat can be taken by one person from the other couple i.e. by C2h or C2w so it can be occupied in only 2 ways. Now we have two people leftover and two seats. Either one of them could take the fourth seat so it can be occupied in 2 ways. The fifth seat can be occupied in one way.

Total number of acceptable arrangements in which S takes the third chair = 4*2*2*1 = 16

Total number of favorable arrangements = 16 + 16 + 16 = 48

Total number of arrangements = 120

Probability that neither couple sits together = 48/120 = 2/5

Answer (D)

Method 2:

The logic I use here is the one we use to solve SETS questions. It needs a little bit of thought but minimum case evaluations.

There are two couples. We don't want either couple to sit together. Let's go the reverse way – let's make at least one of them sit together. We can then subtract this number from the total arrangements to get the number of arrangements in which neither couple sits together.

Would you agree that it is easy to find the number of arrangements in which both couples sit together? It is. We will work on it in a minute. Let's think ahead now.

How about 'finding the number of ways in which one couple sits together?' Sure we can easily find it but it will include those cases in which both couples are sitting together too. But we would have already found the number of ways in which both couples sit together. We just subtract 'both couples together' number from 'one couple together' number and get the number of arrangements in which ONLY one couple sits together. Think of SETS here.

Let's do this now.

Number of arrangements in which both couples sit together: Let's say the two couples are {C1h, C1w} and {C2h, C2w} and the single person is S. There are three groups/individuals. They can be arranged in 3! ways. But in each couple, husband and wife can be arranged in 2 ways (husband and wife can switch places)

Hence, number of arrangements such that both couples are together = 3!*2*2 = 24

Number of arrangements such that C1h and C1w are together: C1 acts as one group. We can arrange 4 people/groups in 4! ways. C1h and C1w can be arranged in 2 ways (husband and wife can switch places).

Number of arrangements in which C1h and C1w are together = 4! * 2 = 48
But this 48 includes the number of arrangements in which C2h and C2w are also sitting together.
Therefore, number of arrangements such that ONLY C1h and C1w sit together = 48 – 24 = 24
Similarly, number of arrangements such that ONLY C2h and C2w sit together = 24

Number of arrangements in which at least one couple sits together = 24 + 24 + 24 = 72

Number of arrangements in which neither couple sits together = 120 – 72 = 48

Probability that neither couple sits together = 48/120 = 2/5

I believe that the second method is much faster and easier. Nevertheless, it's good to know and understand both.

11. Braving the binomial Probability

I would like to take up a couple of questions on binomial probability today. The concepts of the topic have been covered in detail in the book so I am assuming that you know how to solve questions such as "What is the probability of getting at least 3 heads on 5 tosses of a coin?" etc. Therefore, let's work on a couple of questions which use the binomial probability with a twist.

Question 1: Martin and Joey are playing a coin game in which each player tosses a fair coin alternately. The player who gets a 'Heads' first wins. The maximum number of tosses allowed in a single game for any player is 6. What is the probability that the person who tosses first will win the game?

Solution:

Probability of getting 'Heads' on a single toss = 1/2

Probability of getting 'Tails' on a single toss = 1/2

The person who starts the game can win the game if one of the following scenarios plays out:

1. The first person tosses the coin and gets a 'Heads' right away. The first person wins!
2. The first person tosses the coin and gets a 'Tails'. The second person gets 'Tails' too. The first person tosses again and gets a 'Heads'. The first person wins!
3. The first person tosses the coin and gets a 'Tails'. The second person gets 'Tails' too. The first person tosses again and gets a 'Tails' again. The second person gets 'Tails' again too. Finally, the first person tosses and this time, gets a 'Heads'. The first person wins!

and so on…

In the worst case, the first person will have to toss 6 times to get a 'Heads'. He and the second person would end up getting 'Tails' on five previous tosses.

Probability that the first person tosses the coin and gets a 'Heads' right away = 1/2

Probability that the first person tosses the coin and gets a 'Tails' (1/2), the second person gets 'Tails' (1/2) and then the first person gets a 'Heads'(1/2) = (1/2)*(1/2)*(1/2) = (1/2)^3

Probability that the first person tosses the coin and gets a 'Tails' (1/2), the second person gets 'Tails' (1/2) , the first person tosses again and gets a 'Tails' again (1/2), the second person gets 'Tails' again (1/2) and finally, the first person tosses and this time, gets a 'Heads' (1/2) = (1/2)^5

and so on…

Probability that the first person will have to toss 6 times to get a 'Heads' = (1/2)^11

To get the probability of the first person winning, we just need to add all these probabilities now.

Probability that the first person will win = (1/2) + (1/2)^3 + (1/2)^5 + (1/2)^7 + (1/2)^9 + (1/2)^11

On the same lines, can you find the probability that the person who tosses second wins? I hope you understand that it is very similar to what we have already discussed. The person who tosses second will win if one of the following happens:

1. The person who tosses first gets 'Tails' and then the person who tosses second gets 'Heads'.
2. The person who tosses first gets 'Tails', the person who tosses second gets 'Tails', the person who tosses first gets 'Tails' again and the second person then gets 'Heads'.

and so on…

Probability that the second person will win = (1/2)^2 + (1/2)^4 + (1/2)^6 + (1/2)^8 + (1/2)^10 + (1/2)^12

I hope you see that the question is quite straight forward. Now, let's take a question very similar to one from a GMAT Prep test.

Question 2: For one toss of a certain coin, the probability that the outcome is heads is 0.7. If the coin is tossed 6 times, what is the probability that the outcome will be tails at least 5 times?

Solution: This question is very similar to the questions we saw in the Probability book. The only difference is that we are not tossing a fair coin. The probability of getting heads is 0.7 not 0.5. So the probability of getting tails must be 0.3 since the total probability has to add up to 1.

The only acceptable cases are those in which we get 'tails' on all 6 tosses or we get tails on exactly 5 of the 6 tosses.

P(Tails on all 6 tosses) = (0.3)*(0.3)*(0.3)*(0.3)*(0.3)*(0.3) = (0.3)^6

P(Tails on exactly 5 tosses and Heads on one toss) = (0.3)^5*(0.7)*6

We multiply by 6 because 5 tails and 1 heads can be obtained in 6 different ways: HTTTTT, THTTTT, TTHTTT, TTTHTT, TTTTHT, TTTTTH

Probability that the outcome will be tails at least 5 times = Probability that the outcome will be tails 5 times + Probability that the outcome will be tails 6 times

Probability that the outcome will be tails at least 5 times = (0.3)^6 + (0.3)^5*(0.7)*6

Again, the question is straight forward. It just has a little twist which sometimes throws people off during the test. It is these little things that differentiate a medium level question from a high level question.

12. Probability with Conditions

Let's look at the concept of conditional probability in detail today. (As if the probability questions weren't tricky enough!) But since I like to discuss advanced concepts in this blog (in addition to alternative approaches and very important fundamentals), it would not be fair on my part to end the probability discussion without a quick review of conditional probability. Let me start by tossing a question at you.

Quarter Wit_Quarter Wisdom- Part-2

Question 1: Alex tosses a coin four times. On two of the tosses (we don't know which two), he gets 'Heads'. What is the probability that he gets 'Tails' on other two tosses?

Solution: Wait a minute! Isn't it something like the Binomial Probability questions we saw last week? It is but notice that it is also a conditional probability question. You are given that on at least 2 tosses, he got 'Heads'. Under this condition, you want to find the probability that he got 2 tails i.e. he got 2 heads and 2 tails on his 4 tosses.

Conditional Probability is calculated as given below:

P(A given B) = P(A)/P(B)

Here, we are trying to find the probability that event A happens given that event B happens. To understand this formula, think of it this way:

Say there are a total of 100 cases and event B takes place in 10 cases. Also, event A takes place in 5 of the 10 cases in which event B takes place (A is a more restricted event under event B). Let's say we know that event B has taken place. This means that one of the 10 cases has occurred. The probability that A has taken place is 5/10 = 1/2 and not 5/100. I hope this makes sense to you. Let me take an example to make this clearer.

GMAT score can take one of 61 values (200/210/220 … 780/790/800). So there are a total of 61 cases. What is the probability that I will score above 700 on GMAT? (well, it should be 100% because otherwise I should not be writing blog posts on GMAT but let's assume that all the scores are equally likely)

There are 10 possible scores above 700 (710/720/730 … 800). Probability of a score above 700 = 10/61. That is our simple probability that we have been working on till date.

Now, consider this: You know that I scored above 600. How much exactly, you do not know! What will you say is the probability that I scored above 700? (again assuming that all the scores are equally likely)

I did score above 600. Now, what is the probability that I scored above 700? There are 20 possible scores above 600 (610/620/630 … 800). Any of them could have been my score. What is the probability that I actually scored above 700? It is 10/20. The event that I scored more than 700 is event A. It is more restrictive than event B i.e. the event that I scored more than 600. Given that event B took place i.e. I scored above 600, the probability that event A took place i.e. I scored above 700 is P(Score above 700)/P(Score above 600). This is conditional probability.

I hope you see the difference between probability and conditional probability.

Let's go back to the original question now.

We want to find this probability: P('2 Heads and 2 Tails' given 'At least 2 Heads') = P(2 Heads and 2 Tails)/P(At least 2 Heads)

We can easily find P(2 Heads and 2 Tails) and P(At least 2 Heads) since we are comfortable with the concepts of binomial probability! (right?)

P(2 Heads and 2 Tails) = (1/2)*(1/2)*(1/2)*(1/2) * 4!/(2!*2!) = 3/8
You multiply by 4!/(2!*2!) because out of the four tosses, any 2 could be heads and the other two would be tails. So you have to account for all arrangements: HHTT, HTHT, TTHH etc

P(Atleast 2 Heads) = P(2 Heads and 2 Tails) + P(3 Heads, 1 Tails) + P(4 Heads)

Let me remind you here that we can also find P(Atleast 2 Heads) in the reverse way like this:

P(Atleast 2 Heads) = 1 – [P(4 Tails) + P(3 Tails, 1 Heads)]

Let me show you the calculations involved in both the methods.

P(2 Heads and 2 Tails) = 3/8 (calculated above)

P(3 Heads, 1 Tails) = (1/2)*(1/2)*(1/2)*(1/2) * 4!/3! = 1/4

We multiply by 4!/3! to account for all arrangements e.g. HHHT, HHTH etc

P(4 Heads) = (1/2)*(1/2)*(1/2)*(1/2) = 1/16

P(Atleast 2 Heads) = 3/8 + 1/4 + 1/16 = 11/16

OR

P(4 Tails) = (1/2)*(1/2)*(1/2)*(1/2) = 1/16

P(3 Tails, 1 Heads) = (1/2)*(1/2)*(1/2)*(1/2) * 4!/3! = 1/4

P(Atleast 2 Heads) = 1 – (1/16 + 1/4) = 11/16

As expected, the value of P(Atleast 2 Heads) is the same using either method.

P('2 Heads and 2 Tails' given 'At least 2 Heads') = P(2 Heads and 2 Tails)/P(At least 2 Heads) = (3/8)/(11/16) = 6/11

Notice here that you can ignore all the (1/2)s since in every case, you get (1/2)*(1/2)*(1/2)*(1/2) because Heads and Tails have equal probability. You can simply solve this question using this method:

No of arrangements with 2 Heads and 2 Tails = 4!/(2!*2!) = 6

No of arrangements with 3 Heads and 1 Tails = 4!/3! = 4

No of arrangements with 4 Heads = 4!/4! = 1

No of arrangements with at least 2 Heads = 6 + 4 + 1 = 11

P('2 Heads and 2 Tails' given 'At least 2 Heads') = 6/11

Out of the total number of arrangements of 'At least 2 Heads' (which is 11), only 6 are such that you get 2 Heads and 2 Tails.

Mind you, you cannot do that if the probabilities differ. Look at the question given below:

Question 2: Alex has five children. He has at least two girls (you do not know which two of his five children are girls). What is the probability that he has at least two boys too? (The probability of having a boy is 0.4 while the probability of having a girl is 0.6)

Think about what you are going to do here. We will look at the solution of this question next week.

13. Probability with Conditions-II

Last week I left you with a conditional probability question. Let's look at its solution now. This will be my last post on GMAT Combinatorics and Probability (for a while at least) until and unless you want me to take up a particular concept/question related to this topic. Next week, we will start a new topic.

Back to question at hand:

Question 2: Alex has five children. He has at least two girls (you do not know which two of her five children are girls). What is the probability that he has at least two boys too? (The probability of having a boy is 0.4 while the probability of having a girl is 0.6)

Solution:

We want to find this probability: P(‘At least 2 Boys and at least 2 Girls’ given ‘At least 2 Girls’) = P(At least 2 Boys and at least 2 Girls)/P(At least 2 Girls)

Let’s try and find P(At least 2 Boys and at least 2 Girls) and P(At least 2 Girls)

‘At least 2 Boys and at least 2 Girls’ can be obtained in two ways: ‘3 Boys and 2 Girls’ or ‘2 Boys and 3 Girls’

P(At least 2 Boys and at least 2 Girls) = P(3 Boys and 2 Girls) + P(2 Boys and 3 Girls)

P(2 Boys and 3 Girls) = 0.4*0.4*0.6*0.6*0.6 * 5!/(2!*3!) = (0.4)^2 * (0.6)^3 * 10

You multiply by 5!/(2!*3!) because out of the five children, any 2 could be boys and the other three would be girls. So you have to account for all arrangements: BBGGG, BGBGG, GGBGB etc

P(3 Boys and 2 Girls) = 0.4*0.4*0.4*0.6*0.6 * 5!/(3!*2!) = (0.4)^3 * (0.6)^2 * 10

P(At least 2 Boys and at least 2 Girls) = [(0.4)^2 * (0.6)^3 * 10] + [(0.4)^3 * (0.6)^2 * 10] = (0.4)^2 * (0.6)^2 *10 (0.6 + 0.4) = (1.6)(0.36)

Now that we have P(At least 2 Boys and at least 2 Girls), let’s focus on getting P(At least 2 Girls). Again, as we saw last week, there are 2 ways of arriving at P(At least 2 Girls).

P(At least 2 Girls) = P(2 Girls and 3 Boys) + P(3 Girls and 2 Boys) + P(4 Girls + 1 Boy) + P(5 Girls)

OR

P(At least 2 Girls) = 1 – P(5 Boys) – P(1 Girl and 4 Boys)

Let me show you the calculations involved in both the methods.

Method 1:

P(At least 2 Girls) = P(2 Girls and 3 Boys) + P(3 Girls and 2 Boys) + P(4 Girls + 1 Boy) + P(5 Girls)

P(2 Girls and 3 Boys) = (0.4)^3 * (0.6)^2 * 10 (from above)

P(3 Girls and 2 Boys) = (0.4)^2 * (0.6)^3 * 10 (from above)

P(4 Girls + 1 Boy) = (0.4)*(0.6) *(0.6)*(0.6)*(0.6)*5!/4! = (0.4) * (0.6)^4 * 5

P(5 Girls) = (0.6)*(0.6)*(0.6)*(0.6)*(0.6) = (0.6)^5

P(At least 2 Girls) = [(0.4)^3 * (0.6)^2 * 10] + [(0.4)^2 * (0.6)^3 * 10] + [(0.4) * (0.6)^4 * 5] + [(0.6)^5]

Method 2:

P(At least 2 Girls) = 1 – P(5 Boys) – P(1 Girl and 4 Boys)

P(5 Boys) = (0.4)* (0.4)* (0.4)* (0.4)* (0.4) = (0.4)^5

P(1 Girl and 4 Boys) = (0.6)* (0.4)*(0.4)*(0.4)*(0.4)*5!/4! = (0.6)*(0.4)^4 * 5

P(At least 2 Girls) = 1 – [(0.4)^5] – [(0.6)*(0.4)^4 * 5]

The values in bold are the same even if they don't look same. (Trust me, I checked on my financial calculator!)

P('At least 2 Boys and at least 2 Girls' given 'At least 2 Girls') = P(At least 2 Boys and at least 2 Girls)/P(At least 2 Girls)

P('At least 2 Boys and at least 2 Girls' given 'At least 2 Girls') = (1.6)(0.36)/[1 – (0.4)^5 - (0.6)*(0.4)^4 * 5]

Even though the solution looks complicated, I hope you see that the approach is quite logical and straight forward. Let's bid farewell to combinatorics and probability now. We will take up some other topic next week. Till then, keep practicing!

14. Polarity of Exponents

Some of the trickiest questions in GMAT are based on positive/negative bases and powers. Today, let's look at some of their properties. First thing you must understand is that if the base is positive, it will stay positive no matter what the power . a^n is equal to a*a*a* … (n times). Since only positive numbers are multiplied with each other, the product will always be positive. (We cannot say the same thing about negative bases but let's ignore them in today's post.)

For example, 5^n must be positive no matter what the value of n.
The base 'a', which is positive, can belong to one of the two ranges – 'Greater than 1' or 'Between 0 and 1' (or it can be equal to 1). Let's see what happens to a^n is each case.

Case 1: Base 'a' greater than 1
If n is positive, a^n > 1 (for example, 3^2 = 9 which is greater than 1)
If n = 0, a^n = 1 (for example, 3^0 = 1)
If n is negative, 0 < a^n < 1 (for example, 3^(-2) = 1/9 which is less than 1)

We saw what happens if the base is greater than 1. Let's see what happens if the base is between 0 and 1.

Case 2: Base 'a' lies between 0 and 1
If n is positive, a^n < 1 (for example, (1/2)^2 = 1/4 which is less than 1)
If n = 0, a^n = 1 (for example, (1/2)^0 = 1)
If n is negative, a^n > 1(for example (1/2)^(-2) = 4 which is greater than 1)

If the base is equal to 1, it will stay 1 no matter what the power.

These relations hold the other way round too. If a is between 0 and 1, and a^n > 1, n must be negative etc. Plug in some values to understand them.

Let's look at a question dealing with these concepts now.

Question: In which one of the following choices must m be greater than n?

(A) 0.8^m = 0.8^n
(B) 0.8^m = 0.8^2n
(C) 0.8^m > 0.8^n
(D) 8^m < 8^n
(E) 8^m > 8^n

Solution:
The question asks you for the option where m MUST be greater than n. This means that there must be no values of m and n which satisfy the equation/inequality but where m is equal to or less than n. If we can find a single pair of values such that m = n or m < n which satisfy the equation/inequality, that equation/inequality cannot be the answer.

Options (A) and (B) are quite straight forward.

Option (A): Here, we can see that m must be equal to n. Hence this cannot be the answer.

Option (B): Values m = 0 and n = 0 hold for the equation but m is equal to n. Hence this cannot be the answer.

Let's go on to the inequalities now.

In the following three options, each term is positive (since a positive term to any power will stay positive). Therefore, we can take the term on the right to the left hand side so that we have to deal with a single base.

(C) (0.8)^m > (0.8)^n
(0.8)^(m-n) > 1

When the base is between 0 and 1 and the expression is greater than 1, what can we say about the power? We can say that the power must be negative. (Go and check back the third point of case 2.) Here, m-n must be negative i.e. m – n < 0 or m < n. Hence this cannot be the answer.

(D) 8^m < 8^n
8^(m – n) < 1

When the base is greater than 1 and the expression is less than 1, what can we say about the power? We can say that the power must be negative. (Go and check back the third point of case 1.) Here, m – n must be negative i.e. m – n < 0 or m < n. Hence this cannot be the answer.

(E) 8^m > 8^n
8^(m – n) > 1

When the base is greater than 1 and the expression is greater than 1, what can we say about the power? We can say that the power must be positive. (Go and check back the first point of case 1.) Here, m-n must be positive i.e. m – n > 0 or m > n. That is, m must be greater than n. That is what we were looking for! Option (E) must be the answer.

Answer (E)

This was a direct application of the rules we saw above. Hope you understand them well. They could help you extensively while dealing with equaltions/inequalities involving exponents.

15. Scrutinizing Sequences

Today, let's dig into sequences on the GMAT. Let's first understand what a sequence is (from Wikipedia):

A sequence is an ordered list of objects. The number of terms it contains (possibly infinite) is called the length of the sequence. Unlike a set, order matters in a sequence, and exactly the same elements can appear multiple times at different positions in the sequence. Since order matters, (A, B, C) and (B, C, A) are two different sequences. (A series is the sum of the terms of a sequence but we will not deal with series today.)

There are some special sequences e.g. arithmetic progressions and geometric progressions. We will deal with these in subsequent weeks. Today we will look at some generic sequence questions and will learn how to approach them. I will start with a very basic question. Mind you, most sequence questions will be higher level questions since sequence questions look complicated (even though they are very straight forward, believe me!). Let me show you using some questions from external sources:

A note on notation: The first term of a sequence will be denoted by x(1), second term by x(2) and nth term by x(n). (If I want to show multiplication e.g. multiply x by 2, I will show it by writing x*2)

Quarter Wit_Quarter Wisdom- Part-2

Question 1: In a certain sequence, the term x(n) is given by the formula x(n) = 2*x(n-1) – (1/2)*x(n-2) for all n>= 2. If x(0) = 3 and x(1) = 2, what is the value of x(3)?

(A) 2.5
(B) 3.125
(C) 4
(D) 5
(E) 6.75

Solution: This is a straight forward question event though the formula given is discomforting. Whenever you have a generic formula for the nth term of a sequence, plug in some numbers to see what pattern you get.

x(0) = 3 (given)

x(1) = 2 (given)

If n = 2, x(2) = 2*x(1) – (1/2)*x(0) = 2*2 – (1/2)*3 = 5/2

If n = 3, x(3) = 2*x(2) – (1/2)*x(1) = 2*(5/2) – (1/2)*2 = 4

Answer: C.

I hope you agree that this was very simple. For x(3), you needed x(2). For x(2), you needed x(1) and x(0), both of which you had! So it was a simple matter of quick substitution. Now let's look a teeny bit complicated question

Question 2: The infinite sequence a(1), a(2),… a(n),… is such that a(1) = 4, a(2) = -2, a(3) = 6, a(4) = -1, and a(n) = a(n-4) for n > 4. If T = a(10) + a(11) + a(12) + … a(84) + a(85), what is the value of T?

(A) 119
(B) 120
(C) 121
(D) 126
(E) 133

Solution: We know the first four terms: a(1) = 4, a(2) = -2, a(3) = 6, a(4) = -1

Also it is given that a(n) = a(n-4) i.e. the nth term is equal to the (n-4)th term e.g. 5th term is equal to the 1st term. 6th term is equal to the 2nd term. 7^{th} term is equal to the 3^{rd} term etc.

Hence, the sequence becomes: 4, -2, 6, -1, 4, -2, 6, -1, 4, -2, 6, -1 … (It is always helpful to write down the first few terms of the sequence. It helps you see the pattern.)

The sequence has a cyclicity of 4 i.e. the terms repeat after every 4 terms (go back to the definition of sequence above – it says that the same element can appear multiple times at different positions). Therefore, first to fourth terms will form the first cycle, fifth to eighth terms will form the second cycle, ninth to twelfth terms will form the third cycle and so on…
The sum of each group of 4 terms = 4 – 2 + 6 – 1 = 7

What will be the tenth term, a(10)? A new cycle starts from a(9) so a(9) = 4. Then, a(10) must be -2.

a(10) + a(11) + a(12) is the sum of last three terms of a cycle so this sum must be – 2 + 6 – 1 = 3

a(13) to a(16) is a complete cycle, a(17) to a(20) is another complete cycle and so on… The sum of each of the complete cycles is 7.

How many such complete cycles will there be? The first complete cycle will end at a(16), the second one at a(20), the third one at a(24) etc (i.e. at multiples of 4). The last complete cycle will end at a(84). How many complete cycles do we have here then?

16 = 4*4 and 84 = 4*21 so you start from the fourth multiple to the 21st multiple i.e. you have (21 – 4 + 1) = 18 total cycles. If you are confused about the '+1' here, hang on – I will take it up at the end of this post.

The sum of these 18 cycles will be 7*18 = 126 (I know the multiplication table of 18 as should you!)

We still haven't accounted for a(85), which will be the first term of the next cycle. The first term is 4.

a(10) + a(11) + a(12) + … a(84) + a(85) = 3 + 126 + 4 = 133 = T

or you could just consider this:

You have 18 complete cycles except for the first 3 terms and the last term of the sequence. The last term of the sequence is the first term of a cycle and the first three terms of the sequence are the last three terms of the cycle. So these four terms make one complete cycle. Therefore, instead of 18, you have 19 complete cycles.

T = 7 * 19 = 133

Answer (E)

These were some basic sequences questions. I want to leave you with a sequence question from GMAT prep now. Try and work it out. We will look at its solution next week.

Question 3: For every integer m from 1 to 10 inclusive, the mth term of a certain sequence is given by [(-1)^(m+1)]*[(1/2)^m]. If T is the sum of the first 10 terms in the sequence, then T is:

(A) greater than 2
(B)between 1 and 2
(C) between 0.5 and 1
(D) between 0.25 and 0.5
(E) less than 0.25

Note on the '+1' above: How many numbers are there from 11 to 25, both inclusive? If your answer is 25 – 11 = 14, then you are wrong. When we say 25 – 11, we are saying that we have 25 numbers and we are throwing away 11 of them. But we want to keep the 11 (since we have both inclusive); we want all the numbers starting from 11 and ending at 25. We need to add 1 to the result to ensure that the 11 that we threw out, is retained. Consequently, the number of numbers from 11 to 25, both inclusive is 14 + 1 = 15.

16. Scrutinizing a 700+ level Question on Sequences

Today, I will take the question I gave you in the last post (and that is all we will tackle today!) It is a question from GMAT prep test so it is quite indicative of the tricky questions you might get on actual GMAT. Solving the question takes less than two minutes since the calculations required are negligible. However, if you start calculating the actual value, you could end up spending many painful minutes before giving up. I implore you to always remember that GMAT does not give you calculation intensive questions. Since they do not provide you with an HP12C, there will always be a logical solution — you will just need to think a little harder! Let's get going then…

Question 3: For every integer m from 1 to 10 inclusive, the mth term of a certain sequence is given by [(-1)^(m+1)]*[(1/2)^m]. If T is the sum of the first 10 terms in the sequence, then T is

(A) greater than 2
(B) between 1 & 2
(C) between 0.5 and 1
(D) between 0.25 and 0.5
(E) less than 0.25

Quarter Wit_Quarter Wisdom- Part-2

Solution:

I agree that the expression for the mth term looks daunting. But as we discussed last week, the first step in a sequence question should be to write down the first few terms of the sequence. So let's do that and see what we get.

We get the first term by putting m = 1

First term = [(-1)^(1+1)]*[(1/2)^1] = **1/2** (Not so bad, eh?!)

Second term = [(-1)^(2+1)]*[(1/2)^2] = -1/2^2 = **-1/4**

Third term = [(-1)^(3+1)]*[(1/2)^3] = 1/2^3 = **1/8**

Do we see a pattern?

The tenth term will be [(-1)^(10+1)]*[(1/2)^10] = -1/2^10 = **-1/1024** (You don't need to calculate this of course. You can keep it as 2^10. I do suggest though that you should be good with the first 10 powers of 2, first 6 powers of 3, first 4 powers of 4 and first 3 powers of 5 to 10.)

The sequence looks like this: 1/2, – 1/4, 1/8, – 1/16, …

T = 1/2 – 1/4 + 1/8 – 1/16 +…+ 1/512 – 1/1024

Of course GMAT doesn't expect us to calculate this. One could end up wasting a lot of precious time if one did. The trick is to know that we need to figure out the answer using some shrewdness. Fortunately (or unfortunately), GMAT software rewards cunning and craft!

The problem is that we have positive and negative terms so it is very hard to say what the value of T will be. But the terms are not random. There is a positive term followed by a negative term which is then followed by a positive term and so on. Also, every subsequent term is smaller than the previous term (in fact it is a Geometric Progression but we don't need to know that to solve this question. Nevertheless, we will take up GP too in a couple of weeks).

We need to create some uniformity so that we can deduce something about T. We have 10 terms. If we couple them up, two terms each, we get 5 groups:
T = (1/2 – 1/4) + (1/8 – 1/16) + … + (1/512 – 1/1024)

Tell me, can we say that each group is positive? From a larger number, you are subtracting a smaller number in each bracket. The first number is greater than the second number in each group e.g. 1/2 is greater than 1/4, therefore, (1/2 – 1/4) = 1/4 i.e. a positive number
Similarly, (1/8 – 1/16) = 1/16, again a positive number.

This means T = 1/4 + 1/16 +…. (all positives)
Definitely this sum, T, is greater than 1/4 i.e. 0.25

So we can rule out option (E). But we still have to choose one out of the four remaining options.

Now, let's group the terms in another way.

T = 1/2 + (- 1/4 + 1/8) + (- 1/16 + 1/32) … – 1/1024

You leave out the first term and start grouping two terms at a time. The last term will be left alone too! You will be able to make four groups with the 8 terms in the middle.

Now look closely at each group: The first term is a negative number with a higher absolute value while the second term is a smaller positive number so the sum will give you a negative number, e.g.:

(-1/4 + 1/8) = -1/8

(-1/16 + 1/32) = -1/32 etc

This means T = 1/2 – 1/8 – 1/32 … -1/1024

All 4 of the groups will give you a negative number and the last term is also negative. Since the first term is 1/2 i.e. 0.5, we can say that the sum T will be less than 0.5 since all the other terms are negative.

So the sum, T, must be more than 0.25 but less than 0.5.

Answer has to be option (D).

There are other ways of arriving at the answer here. We will look at it from the Geometric Progression perspective some time later.

I will leave you now with a question I saw somewhere once. Let's see if you can use your craft to arrive at the answer in a minute! (absolutely do-able)

Question:

In the infinite sequence A, the nth term, A(n), is given by x^(n-1) + x^n + x^(n+1) + x^(n+2) + x^(n+3) where x is a positive integer constant. For what value of n is the ratio of A(n) to x(1+x(1+x(1+x(1+x)))) equal to x^5?

(A) 8
(B) 7
(C) 6
(D) 5
(E) 4

17. Progressing to Arithmetic Progressions

Today's topic is Arithmetic Progression (AP). An AP is a sequence of numbers such that the difference between the consecutive terms is constant.

For example:

2, 4, 6, 8…

-1, 10, 21, 32…

4, 3, 2, 1, 0, -1, -2, -3

-1/2, -3/2, -5/2, -7/2…

and so on.

Note that the numbers could be increasing or decreasing. As long as the difference between the consecutive terms is constant, it is an AP. The general form of an AP is given by

a, a+d, a+2d, a+3d…

I hope you see how we get it. 'a' is the first term and 'd' is the common difference. The nth term has the form 'a + (n-1)d'. The first term is given by 'a + (1-1)d' = 'a'

The second term is given by 'a + (2-1)d' = 'a + d' etc

The sum of n terms will be found by doing the following:

a + (a+d) + (a+2d) + … + (a+(n-1)d) = na + (d + 2d + 3d + … + (n-1)d) (summing the n first terms together and clubbing the common differences together in a bracket)

= na + d(1 + 2 + 3 + …(n-1)) (taking d common)

= na + d(n-1)n/2 (this is because the sum of n consecutive integers starting from 1 is given by n(n+1)/2. We will work on this concept in detail next week.)

= n/2 (2a + (n-1)d)

If you notice carefully, 2a + (n-1)d can be re-written as: [a] + [a + (n-1)d] i.e. the sum of first and last terms. So basically the sum of n terms of the AP is [n * (First term + Last term)/2] which is the same as [number of terms * Average of the first and the last terms].

Questions on APs are very simple. Sometimes, people just don't realize that the question is based on an AP. The questions in GMAT may not give you that the sequence is an AP but it is not tough to figure out. Let us look at an example now.

Question 1: The nth term, t(n), of a certain sequence is defined as t(n) = t(n-1) + 4. If t(1) = -11, then t(82) =

(A) 313
(B) 317
(C) 320
(D) 321
(E) 340

Solution:

The given relation says that every nth term is 4 more than the previous term. So basically, it tells us that the sequence is an AP. Whew! (An AP is very easy to work with)

What is the nth term of an AP? It's [a + (n-1)d]

What is the 82nd term of this AP? It's [-11 + 81*4] = 313

Answer (A)

Question 2: If S is the infinite sequence such that t(1) = 4, t(2) = 10, …, t(n) = t(n-1) + 6,…, what is the sum of all the terms from t(10) to t(18)?

(A) 671
(B) 711
(C) 738
(D) 826
(E) 991

Solution: Again, since the nth term is 6 more than the previous term, it is an AP.

We need to find the following sum: t(10) + t(11) + t(12) + … + t(18)

But we only know how to find the sum starting from t(1). Let's manipulate what we have to find a little to make it similar to what we know.

t(10) + t(11) + t(12) + … + t(18) = Sum of first 18 terms – Sum of first 9 terms

Sum of first 18 terms = (18/2)(2*4 + 17*6)

Sum of first 9 terms = (9/2)(2*4 + 8*6)

t(10) + t(11) + t(12) + … + t(18) = (18/2)(2*4 + 17*6) – (9/2)(2*4 + 8*6)

= 990 – 252 = 738

Answer (C)

As you see, AP questions are easy to work with. Next week, we will discuss some properties of a specific type of AP i.e. consecutive integers. Till then, keep practicing!

18.Special Arithmetic Progression

Last week, we looked at some basic formulas related to Arithmetic Progressions. This week, we will look at a particular (and related) type of Arithmetic Progression — Consecutive Integers.

Look at the following three sequences:

S1 = 3, 4, 5, 6, 7

S2 = -1, 0, 1, 2, 3, 4, 5, 6, 7

S3 = 1, 2, 3, 4, 5, 6, 7, 8, 9

All of them are APs of consecutive integers so every formula we looked at last week is applicable here.

Sum of an AP = (n/2)[2a + (n-1)d]

Sum of the terms of S1 = (5/2)[2*3 + (5-1)1] = 25

Sum of the terms of S2 = (9/2)[2*(-1) + (9-1)1] = 27 (or treat it as a sum of 2, 3, 4, 5, 6, 7)

Sum of the terms of S3 = (9/2)[2*1 + (9-1)1] = 45

We will pay special attention to S3 i.e. an AP of consecutive integers starting from 1

Sum of the terms in this case = (n/2)[2*1 + (n-1)*1] (since a = 1 and d = 1)

(n/2)[2*1 + (n-1)*1] = (n/2)(n+1) = n*(n+1)/2

This is a formula we should be very comfortable with: sum of first n terms of a sequence of consecutive integers starting from 1 = n*(n+1)/2. It is very useful in a lot of situations. We can also derive a lot of other relations using this single formula.

For Example:

Example 1. What is the sum of positive consecutive even integers starting from 2?

- It is n(n+1) where n is the number of even integers (remember, n is not the last term; it is the number of total even integers).

- How?

- Say there are n numbers in the sequence and we want to find their sum: 2 + 4 + 6 + 8 + … We take 2 common out of them to get 2(1 + 2 + 3+ 4…). This is twice the sum of n consecutive integers starting from 1. So Sum = 2*n(n+1)/2 = n(n+1).

Example 2. What is the sum of positive consecutive odd integers starting from 1?

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- It is n^2 where n is the number of odd integers (again, n is not the last term; it is the number of total odd integers).

- How?

- Say there are n numbers in the sequence and we want to find their sum: 1 + 3 + 5 + 7 + … Sum of 2n consecutive integers (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8…) will be 2n(2n+1)/2 and sum of n even consecutive integers (2 + 4 + 6 + 8 + …) starting from 2 will be n(n+1) so sum of the leftover n consecutive odd integers will be n(2n+1) – n(n+1) = n^2.

With this background, let's look at a question similar to an OG12 question:

Question 1: What is the sum of all the even integers between 99 and 401 ?

(A) 10,100
(B) 20,200
(C) 37,750
(D) 40,200
(E) 45,150

Solution: We can solve this question in multiple ways.

Required Sum = 100 + 102 + 104 + 106 + … + 400

Method 1:

We know the sum of consecutive even integers but only when they start from 2. So what do we do? We find the sum of even integers starting from 2 till 400 and subtract the sum of even integers starting from 2 till 98 from it! Note that we subtract even numbers till 98 because 100 is a part of our series.

How many even integers are there from 2 to 400? I hope you agree that we will have 200 even integers in this range (both inclusive)

Sum of these 200 integers = 200(201)

How many even integers are there from 2 to 98? Now we have 49 even integers here.

Sum of these 49 even integers = 49(50)

What is the sum of integers from 100 to 400? It will be 200(201) – 49(50) = 40200 – 2450 = 37750

Method 2:

100 + 102 + 104 + 106 + … + 400 = 2(50 + 51 + 52 + 53 + … + 200) (We take out 2 common and find the sum in brackets)

Sum in brackets = 50 + 51 + 52 + 53 + … + 200

We know the sum of consecutive integers but only when they start from 1. So we find the sum of first 200 numbers and subtract the sum of first 49 numbers from it. That will give us the sum of numbers from 50 to 200. Note that we subtract 49 numbers because 50 is a part of our series.

Sum of 1 + 2 + 3 + … + 200 = 200*201/2 = 20100
Sum of 1 + 2 + 3 + … + 49 = 49*50/2 = 1225 (I am doing these calculations here only for clarity. Normally, I would like to carry all these till the last step, then take common, divide by whatever I can etc so that I have very few actual calculations left.)

Therefore, 50 + 51 + 52 + 53 + … + 200 = 20100 – 1225 = 18875

Then 100 + 102 + 104 + 106 + … + 400 = 2*18875 = 37750

Answer (C)

I hope you see that questions on arithmetic progressions are generally quite simple. Next week, we will move on to geometric progressions. Till then, keep practicing!

19. Time to tackle GP

Let's look at geometric progressions (GP) now. Before I start, let me point out that GMAT is unlikely to give you a statement which looks like this: "If S denotes a geometric progression whose first term is… " GMAT will not test your knowledge of GP (i.e. you don't really need to learn the formulas of the sum of n terms of a GP or sum of infinite terms of a GP etc) though it may give you a sequence which is a geometric progression and ask you questions on it. You will be able to solve the question without using the formulas but recognizing a GP can help you deal with such questions in an efficient manner. That is the reason we are discussing GPs today.

For those of you who are wondering what exactly a GP is, let me begin by giving you the definition.

(I will quote Wikipedia here.)

A geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio. For example, the sequence 2, 6, 18, 54, … is a geometric progression with common ratio 3 (each term after the first is obtained by multiplying the previous term by 3). Similarly 10, 5, 5/2, 5/4, … is a geometric sequence with common ratio 1/2.

The first term of a GP is generally denoted by a and the common ratio is denoted by r. So the general form of a GP is: a, ar, ar^2, ar^3, …, ar^(n – 1) (The GP has n terms here.)

Sum of n terms of a GP is given by a*(1 – r^n)/(1 – r).

Sum of all the terms of an infinite GP is given by a/(1 – r).

We generally look at the derivation of the formula to help us understand it better (and hence remember it) but the derivation of this formula is more mathematical and less intuitive so we will not discuss it here. If you are interested in the derivation, check out the Wikipedia link given above. We will directly jump to GMAT relevant questions now and see how they can be solved without the aid of the formulas and how they can be solved with the formulas. First we will look at a simple DS question which deals with GP but doesn't mention it in the question stem. The challenge is to figure out that it is a GP. Once you do, you can solve it in a few moments.

Question 1: In a certain sequence, when you subtract the Mth element from the (M-1)th element, you get twice the Mth element (M is any positive integer). What is the fourth element of this sequence?

1. The first element of the sequence is 1.
2. The third element of the sequence is 1/9.

Solution: The question stem doesn't tell us that the sequence is a geometric progression. It only tells us that t(m-1) – t(m) = 2*t(m)

This tells us that t(m) = t(m-1)/3
Every subsequent term should be a third of the previous term. This means that the sequence is a GP with common ratio = 1/3.

So the GP looks something like this: a, a/3, a/9, a/27 …

Statement 1: The first term of the sequence is 1. Now we know all the terms of the sequence.

1, 1/3, 1/9, 1/27…

The fourth element is 1/27 so this statement alone is sufficient.

Statement 2: The third term of the sequence is 1/9.

The third term of our GP is a/9. If a/9 = 1/9, this implies that a = 1. Hence the fourth term = 1/27. This statement alone is also sufficient to answer the question.

Answer (D).

Now we will look at a trickier question.

Question 2: What is the value of 7 + 6*7 + 6*7^2 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6?

(A) 6^7
(B) 6^9
(C) 7^7
(D) 7^8
(E) 7^9

Solution: First let's solve the question without using the GP formula.

Method 1:

S = 7 + 6*7 + 6*7^2 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6

S = 7*(1 + 6) + 6*7^2 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6 (Take 7 common from the first two terms)

S = 7^2 + 6*7^2 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6

S = 7^2 * (1 + 6) + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6 (Take 7^2 common from the first two terms)

S = 7^3 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6

I hope you see where we are going with this. The last step would be:

S = 7^6 * (1 + 6) = 7^7

Answer (C)

Method 2:

Except for the first term of the sequence, the rest of the sequence is a GP with first term as 6*7 and the common ratio as 7.

S = 7 + 6*7 + 6*7^2 + 6*7^3 + 6*7^4 + 6*7^5 + 6*7^6 = 7 + GP

There are 6 terms in the GP.

Sum of the GP = a*(1 – r^n)/(1 – r) = 6*7*(1 – 7^6)/(1 – 7) = 6*7 * (7^6 – 1)/6 = 7^7 – 7

Substituting this sum back in S, we get

S = 7 + 7^7 – 7 = 7^7

Answer (C)

The first method is not difficult. It is just hard to figure out when you are under time pressure. A GP is easy to notice and you know exactly how to handle it. I will not advice you to use one method over the other – both are equally valid and good so use whatever works for you. The only thing is that knowing how to deal with GPs can help save time. It may not be very apparent in this question but I will leave you with a question in which it will be apparent! It is something similar to the 700+ level GMAT prep test question we saw while working on arithmetic progressions. We will discuss this question in detail next week.

Question 3: For every integer n from 1 to 200, inclusive, the n*th* term of a certain sequence is given by (-1)^n*2^(-n). If N is the sum of the first 200 terms in the sequence, then N is

(A) less than -1
(B) between -1 and -1/2
(C) between -1/2 and 0
(D) between 0 and 1/2
(E) greater than 1/2

See you next week!

20. How to benefit from GP perspective

Last week, at the end of the Geometric Progression (GP) post, I gave you a question to figure out. I hope some of you did try it. Today we will discuss the question in detail and look at two different approaches – one without using GP formula (we discussed this approach in a previous post) and another with the formula. As I said before, you can solve every sequence question on GMAT without using the formulas we are discussing. We are still investing time in these formulas so that we can save some in the actual exam. Let me show you how.

Question 3: For every integer n from 1 to 200, inclusive, the nth term of a certain sequence is given by (-1)^n*2^(-n). If N is the sum of the first 200 terms in the sequence, then N is:

(A) less than -1
(B) between -1 and -1/2
(C) between -1/2 and 0
(D) between 0 and 1/2
(E) greater than 1/2

Solution: The question seems a little intimidating since the options give you ranges. This means that it is probably hard to find the exact value and that's just not good. Anyway, let's begin by doing what we know we should do with every sequence question if possible: we should write out the first few terms of the sequence.

First term: (-1)^1*2^(-1) = -1/2

Second term: (-1)^2*2^(-2) = 1/4

Third term: (-1)^3*2^(-3) = -1/8

and so on…

The sequence looks like this: -1/2, 1/4, -1/8, 1/16, -1/32…

N = -1/2 + 1/4 – 1/8 + 1/16 – 1/32 … (200 terms)

Method 1:

Say, we do not want to work with GPs. Let's sum the pair of consecutive terms. (Guess you are reminded of the 700+ level GMAT prep question we discussed a couple of weeks back.)

N = (-1/2 + 1/4) + (- 1/8 + 1/16) + (- 1/32 + 1/64) + …

-1/2 + 1/4 = -1/4

$-1/8 + 1/16 = -1/16$

$-1/32 + 1/64 = -1/64$

…

$N = -1/4 - 1/16 - 1/64 \ldots$

We can see that the sum of all these terms will be less than -1/4 since all the rest of the terms are negative too.

$N < -1/4$

Let's look at it in another way. Let's leave the first term and sum the pair of consecutive terms thereafter.

$N = -1/2 + (1/4 - 1/8) + (1/16 - 1/32) + \ldots$

$N = -1/2 + 1/8 + 1/32 + \ldots$

Since all the terms after the first one are positive, N must be greater than -1/2.

Therefore, $-1/2 < N < -1/4$.

Hence, of the given ranges, N must lie between -1/2 and 0.

Answer (C)

Let's look at a more straight forward method now.

Method 2:

Let's say we recognize that the given sequence is a GP.

$N = -1/2 + 1/4 - 1/8 + 1/16 - 1/32 \ldots$ (200 terms)

The first term is -1/2 and the common ratio is -1/2.

Sum of 200 terms of a GP = $a(1 - r^n)/(1 - r) = (-1/2)(1 - (-1/2)^{200})/(1 - (-1/2))$

Notice that in the bracket $(1 - (-1/2)^{200})$, $(-1/2)^{200}$ is a very very small number compared to 1 so the value of the bracket is approximately 1.

Sum of the GP = $(-1/2)(1)/(1- (-1/2)) = -1/3$

Since -1/3 lies between -1/2 and 0, answer is (C).

The second method was much faster and much more mechanical than the first one. I don't particularly encourage mechanical thought process but during the exam thinking up innovative methods is a little hard if you haven't trained your mind to do so. Hence, knowing how to deal with APs and GPs is a good idea.

21. Blundering through Calculations

I have my CFA level II exam in June and I am certainly looking at disaster (but that's not what I am going to discuss today). While studying for it yesterday, I did something really stupid and that gave me an insight on 'mind matters'. That is what I want to talk about today but I will have to give you some background to make my point clearer.

Obviously, the two exams – GMAT and CFA are very different and one of the differences is that on the CFA exam, we are allowed to use a financial calculator. Now, if you have read some of my posts before, I guess you will agree that I am a devoted 'logical solution' fan. To increase speed, I advice my students to try to solve all quant questions orally – to sit without a pen and paper and see if you can force yourself to figure out the answer without writing a word. I solve most of the GMAT relevant questions orally so it is certainly possible. If a question makes me get up and get my pen and paper and then, if I can solve the question within two minutes, I consider it a very tricky but good GMAT question.

But yesterday, while studying for my CFA exam with my calculator and pen right next to me, I found myself punching in 138 and then the division sign and then, 10. I am sure you can guess the answer I got (which was 13.8 of course) and I found myself staring at my calculator for the next few seconds. I couldn't believe that I worked out that calculation on the calculator. I was highly intrigued by the fact that I missed the 10 in the denominator when I suggest people to develop oral problem solving skills!

The point is that it is all about perception. When I see a GMAT question, my mind automatically goes to the 'solve yourself' mode. No matter how insane the numbers LOOK, I KNOW they will fall in place. CFA exam affords no such luxuries. The numbers are crazy and you need to use a calculator so even if you have easy numbers in front of your eyes, you just don't register it.

Can you change your perception? Sure! Give up your pen right away and put it far away from yourself before you start doing some questions. If your writing material is far away and you are half as lazy as I am, you will try to work the questions out in your mind instead of getting up to go and get your pen. You will struggle initially but you can slowly train your mind to figure out how the numbers fit in the puzzle to show you the whole picture. But before you start this exercise, there is some preparation you need to do. You need to memorize the following things:

1. Multiplication tables till 20 (e.g. 13*2 = 26, 13*3 = 39 till 13*10 = 130 etc)
2. Squares of all positive integers till 20 (e.g. 12^2 = 144, 14^2 = 196 etc)
3. Cubes of all positive integers till 10 (e.g. 6^3 = 216, 7^3 = 343 etc)
4. Factorials till 7 (e.g. 3! = 6, 6! = 720 etc)
5. All powers of 2 upto 2^10
6. All powers of 3 upto 3^6
7. All powers of 4 and 5 upto 4^5 and 5^5

I can't stress the importance of these points enough. Knowing these things help you see patterns, figure out tricks in number properties questions, find prime numbers and of course, solve questions quickly.

Try to train your mind. You could end up with spare 15 mins in the Quant section of the exam!

22. The meaning of Arithmetic Mean

Let's start today with statistics – mean, median, mode, range and standard deviation. The topics are simple but the fun lies in the questions. Some questions on these topics can be extremely tricky especially those dealing with median, range and standard deviation. Anyway, we will tackle mean today.

So what do you mean by the arithmetic mean of some observations? I guess most of you will reply that it is the 'Sum of Observations/Total number of observations'. But that is how you *calculate* mean. My question is 'what *is*mean?' Loosely, arithmetic mean is the number that represents all the observations. Say, if I know that the mean age of a group is 10, I would guess that the age of Robbie, who is a part of that group, is 10. Of course Robbie's actual age could be anything but the best guess would be 10.

Say, I tell you that the average age of a group of 10 people is 15 yrs. Can you tell me the sum of the ages of all 10 people? I am sure you will say that it is 10*15 = 150. You can think of it in two ways:

Mean = Sum of all ages/No of people

So Sum of all ages = Mean * (No of people) = 15*10

Or

Since there are 10 people and each person's age is represented by 15, the sum of their ages = 10*15. Basically, the total sum was distributed evenly among the 10 people and each person got 15 yrs.

Now, let's say you made a mistake. A boy whose age you thought was 20 was actually 30. What is the correct mean? Again, you can think of it in two ways:

New sum = 150 + 10 = 160

New average = 160/10 = 16

Or

You can say that there is an extra 10 that has to be distributed evenly among the 10 people, so each person gets 1 extra. Hence, the average becomes 15 + 1 = 16.

As you might have guessed, we will work on the second interpretation. Let's look at an example now.

Example 1: The average age of a group of n people is 15 yrs. One more person aged 39 joins the group and the new average is 17 yrs. What is the value of n?

(A) 9
(B) 10
(C) 11
(D) 12
(E) 13

Solution: First tell me, if the age of the additional person were 15 yrs, what would have happened to the average? The average would have remained the same since this new person's age would have been the same as the age that represents the group. But his age is 39 – 15 = 24 more than the average. We know that we need to evenly split the extra among all the people to get the new average. When 24 is split evenly among all the people (including the new guy), everyone gets 2 extra (since average age increased from 15 to 17). There must be 24/2 = 12 people now (including the new guy) i.e. n must be 11 (without including the new guy).

Let's look at another similar example though a little trickier. Try solving it on your own first. If not logically, try using the formula approach. Then see how elegant the solution becomes once you start 'thinking' instead of just 'calculating'.

Example 2: When a person aged 39 is added to a group of n people, the average age increases by 2. When a person aged 15 is added instead, the average age decreases by 1. What is the value of n?

(A) 7
(B) 8
(C) 9
(D) 10
(E) 11

Solution: What is the first thing you can say about the initial average? It must have been between 39 and 15. When a person aged 39 is added to the group, the average increases and when a person aged 15 is added, the average decreases.

Let's look at the second case first. When the person aged 15 is added to the group, the average becomes (initial average – 1). If instead, the person aged 39 were added to the group, there would be 39 – 15 = 24 extra which would make the average = (initial average + 2). This difference of 24 creates a difference of 3 in the average. This means there must have been 24/3 = 8 people (after adding the extra person). The value of n must be 8 – 1 = 7.

If you use the formula instead, it would take you quite a while to manipulate the two variables to get the value of n. I hope you see the beauty of this method. Next week, we will discuss some GMAT questions based on Arithmetic mean!

23.Some Mean Questions

I hope the theory of arithmetic mean we discussed last week is clear to you. Let's see the theory in action today. I will pick some mean questions from various sources (Official Guide, GMAT prep tests, etc.) and we will try to use the concepts we learned last week to solve them.

Let's start with a simple question.

Question 1: For the past *n* days the average daily production at a company was 60 units. If today's production of 100 units raises the average to 65 units per day, what is the value of *n*?

(A) 30
(B) 18
(C) 10
(D) 9
(E) 7

Solution: If today's production were also 60 units, what would have happened to the average? Obviously, it would have stayed the same! But today's production is 40 units extra and hence it raised the average. It raised the average by 5 units which means that each one of the *n* observations and today's observation got an extra 5. Since 40 got distributed and each was given 5, there must have been a total of 40/5 = 8 observations including today's. Therefore, the value of *n* must have been 8 – 1 = 7.

Answer (E)

I know you can solve the question using the formula of averages. In fact, you can solve every question using the formula and working out the values. But the point is that the logical method helps you solve the question very quickly and you are less likely to make calculation errors since there aren't too many calculations to perform! Let's go on now.

Question 2: When Anna makes a contribution to a charity fund at school, the average contribution size increases by 50%, reaching $75 per person. If there were 5 other contributions made before Anna's, what is the size of her donation?

(A) $100
(B) $150
(C) $200
(D) $250
(E) $450

Solution: After Anna's contribution, the average size increases by 50% and reaches $75. What must have been the average size of contribution before Anna's donation? It must have been $50 since a 50% increase would lead us to $75. So, $50 was the average size of 5 donations before Anna made her donation. Had Anna donated $50 as well, the average would have stayed the same i.e. $50. But the average increased to $75 which means that Anna donated an extra $25 for each of the 6 observations (including her) in addition to the $50 she would have donated to keep the average same.

Hence, the amount Anna donated = 50 + 6*25 = $200

Answer (C)

Again, this was a relatively straight forward question. Let's look at a tricky one now.

Question 3: A set of numbers has an average of 50. If the largest element is 4 greater than 3 times the smallest element, which of the following values cannot be in the set?

(A) 85
(B) 90
(C) 123
(D) 150
(E) 155

Solution: This question might look a little ominous but it isn't very tough, really! The set has an average of 50 so that already tells us that we can represent each element of the set by 50. If there is an element which is a little less than 50, there will be another element which is a little more than 50.

The largest element is 4 greater than 3 times the smallest element so L = 4 + 3S.

The smallest element must be less than 50 and the largest must be greater than 50. Say, if the smallest element is 20, the largest will be 4 + 3*20 = 64.

Is there any limit imposed on the largest value of the largest element? Yes, because there is a limit on the largest value of the smallest element. The smallest element must be less than 50. The smallest member of the set can be 49.9999… The limiting value of the smallest number is 50. As long as the smallest number is a tiny bit less than 50, you can have the greatest number a tiny bit less than 4 + 3*50 = 154. The number 154 and all numbers greater than 154 cannot be a part of the set. Say if the smallest element is 49, the largest element will be 4 + 3*49 = 151. So the set could look something like this:

S = {49, 49, 49, 49, … (101 times to balance out the extra 101 in 151), 50, 50, 151}

Only option (E) cannot be a part of the set.

These were some of the basic (and not so basic) questions of mean that we could come across in GMAT. Let's call it a day now. We will look at some more stats concepts next week. Till then, keep practicing!

24. Finding Arithmetic Mean using deviations

Today's post is again focused on arithmetic mean. Let's start our discussion by considering the case of arithmetic mean of an arithmetic progression.

We will start with an example. What is the mean of 43, 44, 45, 46, 47? (Hint: If you are thinking about adding the numbers, that's not the way I want you to go.)

As we discussed in our previous posts, arithmetic mean is the number that can represent/replace all the numbers of the sequence. Notice in this sequence, 44 is one less than 45 and 46 is one more than 45. So essentially, two 45s can replace both 44 and 46. Similarly, 43 is 2 less than 45 and 47 is 2 more than 45 so two 45s can replace both these numbers too.

The sequence is essentially 45, 45, 45, 45, 45.

Hence, the arithmetic mean of this sequence must be 45! (If you have doubts, you can calculate and find out.)

It makes sense, doesn't it? The middle number in the sequence of consecutive positive integers will be the mean. The deviations of all numbers to the left of the middle number will balance out the deviations of all the numbers to the right of the middle number.

Quarter Wit_Quarter Wisdom- Part-2

(In this post, we will assume that the given numbers are in increasing/decreasing order. If that is not the case, you can always put them in increasing order and use these concepts.)

Once again, what is the mean of 192, 193, 194, 195, 196, 197, 198?

It is 195 since it is the middle number!

Ok, what about 192, 193, 194, 195, 196, 197? What is the mean in this case? There is no middle number here since there are 6 numbers. The mean here will be the middle of the two middle numbers which is 194.5 (the middle of the third and the fourth number). It doesn't matter that 194.5 is not a part of this list. If you think about it, arithmetic mean of some numbers needn't be one of the numbers.

What about 71, 73, 75, 77, 79? What will be the mean in this case? Even though these numbers are not consecutive integers, the difference between two adjacent numbers in the list is the same (it is an arithmetic progression). So the deviations of the numbers on the left of the middle number will cancel out the deviations of the numbers on the right of the middle number (71 is 4 less than 75 and 79 is 4 more than 75. 73 is 2 less than 75 and 77 is 2 more than 75). Hence, the mean here will be 75 (just like our first example).

Just to re-inforce:

102, 106, 110 –> Mean = 106

102, 106, 110, 114 -> Mean = 108 (Middle of the second and third numbers)

Let's twist this concept a little now. What is the mean of 36, 40, 42, 43, 44, 47?

This is not an arithmetic progression. So do we need to sum and then divide by 6 to get the mean? Not so fast! Let's try and use the deviations concept we have just learned.

Given sequence: 36, 40, 42, 43, 44, 47

It seems that the mean would be around 42, right? Some numbers are less than 42 and others are more than 42.

36 is 6 less than 42.

40 is 2 less than 42.

Overall, the numbers less than 42 are 6+2 = 8 less than 42.

43 is 1 more than 42.

44 is 2 more than 42.

47 is 5 more than 42

Overall, the numbers more than 42 are 1+2+5 = 8 more than 42.

The deviations of the numbers less than 42 get balanced out by deviations of the numbers greater than 42! Hence, the average must be 42.

This method is especially useful in cases involving big numbers which are close to each other.

Example 1: What is the average of 452, 453, 463, 467, 480, 499, 504?

What would you say the average is here? Perhaps, around 470?

Let's see:

452 is 18 less than 470.

453 is 17 less than 470.

463 is 7 less than 470.

467 is 3 less than 470.

Overall, the numbers less than 470 are 18 + 17 + 7 + 3 = 45 less.

480 is 10 more than 470.

499 is 29 more than 470.

504 is 34 more than 470.

Overall, the numbers more than 470 are 10 + 29 + 34 = 73 more than 470.

The shortfall is not balanced by the excess. There is an excess of 73 – 45 = 28.

So what is the average? If we assume the average of these 7 numbers to be 470, there is an excess of 28. We need to distribute the excess evenly among all the numbers and hence the average will increase by 28/7 = 4. (Go back to the first post on arithmetic mean if this is not clear.)

Hence, the required mean is 470 + 4 = 474.

(If we had assumed the mean to be 474, the shortfall would have balanced the excess.)

Let's go through one more example using this concept:

Example 2: What is the mean of 99, 103, 104, 109, 120, 123, 128, 130?

Let's start by guessing a mean for this sequence. Say, around 115?

Let's see if the shortfall is balanced by the excess.

99 is 16 less, 103 is 12 less, 104 is 11 less and 109 is 6 less than 115.

Overall shortfall = 16 + 12 + 11 + 6 = 45

120 is 5 more, 123 is 8 more, 128 is 13 more and 130 is 15 more than 115.

Overall excess = 5 + 8 + 13 + 15 = 41

We are close, but not quite there yet! There is a shortfall of 4. Since there are a total of 8 numbers, the average must be 4/8 = 0.5 less than 115. Hence, the average here is 114.5

Once you get a hang of this method and understand what you are doing, it is much faster than adding all the big numbers and then dividing the sum since you only deal with small numbers in this method.

Let's wrap up today's post here. Next week, we will see these concepts in action!

25. Applications of Arithmetic Mean

Last week we discussed arithmetic means of arithmetic progressions in GMAT math problems. Today, let's see those concepts in action.

Question 1: If x is the sum of the even integers from 200 to 600 inclusive, and y is the number of even integers from 200 to 600 inclusive, what is the value of x + y?

(A) 200*400
(B) 201*400
(C) 200*402
(D) 201*401
(E) 400*401

Solution:

There are various ways of getting the answer here. We will use the concepts we learned last week.

The given sequence is 200, 202, 204, … 600

It is an arithmetic progression. What is the total number of terms here?

You can use one of two methods to get the number of terms here:

Method 1: Using Logic

In every 100 consecutive integers, there are 50 odd integers and 50 even integers. So we will get 50 even integers from each of 200 – 299, 300 – 399, 400 – 499 and 500 – 599 i.e. a total of 50*4 = 200 even integers. Also, since the sequence includes 600, number of even integers = 200 + 1 = 201

Method 2:

Recall that in our arithmetic progressions post, we saw that the last term of a sequence which has n terms will be first term + (n – 1)* common difference.

600 = 200 + (n – 1)*2

n = 201

Hence y = 201 (because y is the number of even integers from 200 to 600)

Let's go on now. What is the average of the sequence? Since it is an arithmetic progression with odd number of integers, the average must be the middle number i.e. 400.

Notice that since this arithmetic progressions looks like this:

(n – m), … (n – 6), (n – 4), (n – 2), n, (n + 2), (n + 4), (n + 6), … (n + m)

We can find the middle number i.e. the average by just averaging the first and the last terms.

[(n – m) + (n + m)]/2 = 2n/2 = n

Average = (200 + 600)/2 = 400

Sum of all terms in the sequence = x = Arithmetic Mean * Number of terms = 400*201 x + y = 400*201 + 201 = 401*201

Answer (D)

This question was simple. You could have found the sum using the formula n/2(2a + (n-1)d) that we saw in the AP post. But this method is more intuitive since if you don't want to, you don't have to use any formula here. Anyway, let's go on to our second question for today.

Question 2: The sum of n consecutive positive integers is 45. What is the value of n?

Statement I: n is even

Statement II: n < 9

Solution: First I will give the solution of this question and then discuss the logic used to solve it.

In how many ways can you write n consecutive integers such that their sum is 45? Let's see whether we can get such numbers for some values of n.

n = 1 -> Numbers: 45
n = 2 -> Numbers: 22 + 23 = 45
n = 3 -> Numbers: 14 + 15 + 16 = 45
n = 4 -> No such numbers
n = 5 -> Numbers: 7 + 8 + 9 + 10 + 11 = 45
n = 6 -> Numbers: 5 + 6 + 7 + 8 + 9 + 10 = 45

Let's stop right here.

Statement I: n must be even.

n could be 2 or 6. Statement I alone is not sufficient.

Statement II: n < 9
n can take many values less than 9 hence statement 2 alone is not sufficient.

Both statements together: Since n can take values 2 or 6 which are even and less than 9, both statements together are not sufficient.

Answer (E)

Now, the interesting thing is how do we get these numbers for different values of n. How do we know the values that n can take? It's pretty easy really. Follow my thought here.

Of course, n can be 1. In that case we have only one number i.e. 45.

n can be 2. Why? When we divide 45 by 2, we get 22.5. Since 2*22.5 is 45, we have to find 2 consecutive integers such that their arithmetic mean is 22.5. The integers are obviously 22 and 23.

n can be 3. When we divide 45 by 3, we get 15. So we need 3 consecutive integers such that their mean is 15. They are 14, 15, 16.

When we divide 45 by 4, we get 11.25. Do we have 4 consecutive integers such that their mean is 11.25? No, because mean of even number of consecutive integers is always of the form x.5.

n can be 5. When we divide 45 by 5, we get 9 so we need 5 consecutive integers such that their mean is 9. They must be 7, 8, 9, 10, 11.

n can be 6. When we divide 45 by 6, we get 7.5. We need 6 consecutive integers such that their mean is 7.5. The integers are 5, 6, 7, 8, 9, 10

Obviously, we just need to focus on getting 2 even values of n which are less than 9. So we check for 2, 4 and 6 and we immediately know that the answer is (E). We don't have to do this process for all numbers less than 9 and we don't have to do it for odd values of n.

We will move on to median next week. Till then, keep practicing!

26.Mean Question on Median

As promised, we discuss medians today! Conceptually, the median is very simple. It is just the middle number. Arrange all the numbers in increasing/decreasing order and the number you get right in the middle, is the median. So it is quite straight forward when you have odd number of numbers since you have a "middle" number. What about the case when you have even number of numbers? In that case, it is just the average of the two middle numbers.

Median of [2, 5, 10] is 5

Median of [3, 78, 102, 500] is (78 + 102)/2 = 90

If it's that simple, why are we discussing it? – because it isn't "that simple"! Conceptually it is, but when the test writers make questions using median and arithmetic mean together, they make some very mean questions! I will show you with an example, but first, we will look at a simpler question.

Question 1: A, B and C have received their Math midterm scores today. They find that the arithmetic mean of the three scores is 78. What is the median of the three scores?

(1) A scored a 73 on her exam.

(2) C scored a 78 on her exam.

Solution: Recall from the arithmetic mean post that the sum of deviations of all scores from the mean is 0.
i.e. if one score is less than mean, there has to be one score that is more than the mean.
e.g. If mean is 78, one of the following must be true:

1. All scores are equal to 78.
2. At least one score is less than 78 and at least one is greater than 78.

For example, if one score is 70 i.e. 8 less than 78, another score has to make up this deficit of 8. Therefore, there could be a score that is 86 (8 more than 78) or there could be two scores of 82 each etc.

Statement 1: A scored 73 on her exam.

For the mean to be 78, there must be at least one score higher than 78. But what exactly are the other two scores? We have no idea! Various cases are possible:

73, 78, 83 or

73, 74, 87 or

70, 73, 91 etc.

In each case, the median will be different. Hence this statement alone is not sufficient.

Statement 2: C scored 78 on her exam.

Now we know that one score is 78. Either the other two will also be 78 or one will be less than 78 and the other will be greater than 78. In either case, 78 will be the middle number and hence will be the median. This statement alone is sufficient.

Answer (B)

Were you tempted to say (C) is the answer? I hope this question shows you that median can be a little tricky. Let's go on to the tougher question now.

Question 2: Five logs of wood have an average length of 100 cm and a median length of 116 cm. What is the maximum possible length, in cm, of the shortest piece of wood?

(A) 50
(B) 76
(C) 84
(D) 96
(E) 100

Solution:

First thing that comes to mind – median is the 3rd term out of 5 so the lengths arranged in increasing order must look like this:

___ ___ 116 ___ ___

The mean is given and we need to maximize the smallest number. Basically, the smallest number should be as close to the mean as possible. This means the greatest number should be as close to the mean as possible too (if the shortfall deviation is small, the excess deviation should by equally small).

If this doesn't make sense, think of a set with mean 20:

19, 20, 21 (smallest number is very close to mean; greatest number is very close to the mean too)
1, 20, 39 (smallest number is far away from the mean, greatest number is far away too)

Using the same logic, let's make the greater numbers as small as possible (so the smallest number can be as large as possible). The two greatest numbers should both be at least 116 (since 116 is the median). Now the lengths arranged look like this:

___ ___ 116 116 116

Since the mean is 100 and each of the 3 large numbers are already 16 more than 100 i.e. total 16*3 = 48 more than the mean (excess deviation is 48), the deviations of the two small numbers should be a total of 48 less than the mean. To make the smallest number as great as possible, each of the small numbers should be 48/2 = 24 less than the mean i.e. they both should be 76.

Answer (B).

Hopefully, it made sense to you. See you again next week for a discussion on another Statistics concept!\

27. A Range of Questions

Let's discuss the idea of "range" today. It is simply the difference between the smallest and the greatest number in a set. Consider the following examples:

Range of {2, 6, 10, 25, 50} is 50 – 2 = 48

Range of {-20, 100, 80, 30, 600} is 600 – (-20) = 620

and so on…

That's all the theory we have on the concept of range! So let's jump on to some questions now (therein lies the challenge)!

Question 1: Which of the following cannot be the range of a set consisting of 5 odd multiples of 9?

(A) 72
(B) 144
(C) 288
(D) 324
(E) 436

Solution:

There are infinite possibilities regarding the multiples of 9 that can be included in the set. The set could be any one of the following (or any one of the other infinite possibilities):

S = {9, 27, 45, 63, 81} or

S = {9, 63, 81, 99, 153} or

S = {99, 135, 153, 243, 1071}

The range in each case will be different. The question asks us for the option that 'cannot' be the range. Let's figure out the constraints on the range.

A set consisting of only odd multiples of 9 will have a range that is an even number (Odd Number – Odd Number = Even number)
Also, the range will be a multiple of 9 since both, the smallest and the greatest numbers, will be multiples of 9. So their difference will also be a multiple of 9.

Only one option will not satisfy these constraints. Do you remember the divisibility rule of 9? The sum of the digits of the number should be divisible by 9 for the number to be divisible by 9. The sum of the digits of 436 is 4 + 3 + 6 = 13 which is not divisible by 9. Hence 436 cannot be divisible by 9 and therefore, cannot be the range of the set.

Answer (E).

On to another one now:

Question 2: If the arithmetic mean of n consecutive odd integers is 20, what is the greatest of the integers?

(1) The range of the n integers is 18.

(2) The least of the n integers is 11.

Solution: We have discussed mean in case of arithmetic progressions in the previous posts. If mean of consecutive odd integers is 20, what do you think the integers will look like?

19, 21 or
17, 19, 21, 23 or
15, 17, 19, 21, 23, 25 or
13, 15, 17, 19, 21, 23, 25, 27 or
11, 13, 15, 17, 19, 21, 23, 25, 27, 29
etc.

Does it make sense that the required numbers will represent one such sequence? The numbers in the sequence will be equally distributed around 20. Every time you add a number to the left, you need to add one to the right to keep the mean 20. The smallest sequence will have 2 numbers 19 and 21, the largest will have infinite numbers. Did you notice that each one of these sequences has a unique "range," a unique "least number" and a unique "greatest number?" So if you are given any one statistic of the sequence, you will know the entire desired sequence.

Statement 1: Only one possible sequence: 11, 13, 15, 17, 19, 21, 23, 25, 27, 29 will have the range 18. The greatest number here is 29. This statement alone is sufficient.

Statement 2: Only one possible sequence: 11, 13, 15, 17, 19, 21, 23, 25, 27, 29 will have 11 as the least number. The greatest number here is 29. This statement alone is sufficient too.

Answer (D).

Note that you don't actually have to find the exact sequence. All you need to understand is that each sequence will have a unique "range" and a unique "least number."

That's all for this week. It wasn't very dense but we will more than make up for it next week (that's when we start with standard deviation)! Keep practicing!

28. Dealing with Standard Deviation

We will work our way through the concepts of Standard Deviation (SD) today. Let's take a look at how you calculate standard deviation first:

$$SD = \sqrt{\frac{\sum(Ai - Aavg)^2}{n}}$$

Ai – The numbers in the list

Aavg – Arithmetic mean of the list

n – Number of numbers in the list

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Say you have 3 numbers : 11, 13 and 15. Their standard deviation is the "square root of the average of their squared deviations from the arithmetic mean." Let's see what we mean by this.

Mean of 11, 13 and 15 is 13.

$$\sum(Ai - Aavg)^2 = (11-13)^2 + (13-13)^2 + (15-13)^2 = 8$$

$$SD = \sqrt{\frac{8}{3}}$$

Focus on these words: "deviations from mean"

The important point to note is that SD is a measure of dispersion or deviation from the mean (the mean is approximately the middle of the list if there are no outliers). In other words, SD is a measure of whether the numbers are very far away from the mean or close together. Since GMAT isn't calculation intensive, you probably won't need to calculate the actual SD in the test. The calculations are shown here only to illustrate the concept. But you must have a feel for how the numbers are distributed around the mean and what that implies for the SD.

Your statistics book explains how to visualize SD using the number line in detail, therefore, I am not going to delve deep into it but will quickly recap so that we can move ahead. Recall that if you plot the numbers on the number line, it gives you a sense of how far the numbers are from the mean. The farther the numbers, higher is the SD.

Let's check out a few different cases to internalize the SD concept. Do not calculate anything in these questions. Just look at the number line for each case and figure out whether it makes sense to you.

Question: Which set, S or T, has higher SD?

Case 1: S = {3, 3, 3} or T = {0, 10, 20}

Case 2: S = {3, 4, 5} or T = {5, 6, 7}

Case 3: S = {3, 4, 5, 6} or T = {2, 3, 4, 5, 6, 7}

Case 4: S = {1, 3, 5} or T = {1, 1, 3, 5, 5}

Case 5: S = {1, 3, 5} or T = {1, 3, 3, 5}

Case 6: S = {6, 8, 10} or T = {12, 16, 20}

Case 7: S = {6, 8, 10} or T = {3, 4, 5}

Let me represent the first four cases on the number line. Check them out and then think which set should have the higher SD.

Quarter Wit_Quarter Wisdom- Part-2

Case 1: S = {3, 3, 3} or T = {0, 10, 20}

Case 2: S = {3, 4, 5} or T = {5, 6, 7}

Case 3: S = {3, 4, 5, 6} or T = {2, 3, 4, 5, 6, 7}

Case 4: S = {1, 3, 5} or T = {1, 1, 3, 5, 5}

Let's discuss each of these four cases now.

Case 1: S = {3, 3, 3} or T = {0, 10, 20}

T has higher SD. We will obtain the SD of T by calculating as shown in the example above. But we don't really need to calculate it because we see that for set S, SD = 0. Each number is at the mean and hence has 0 deviation from the mean. Since SD cannot be negative, whatever the SD of T, it will be higher than the SD of S which is 0.

Case 2: S = {3, 4, 5} or T = {5, 6, 7}

Both sets have the same SD. We can see from the number line that they are equally dispersed around their respective means.

Case 3: S = {3, 4, 5, 6} or T = {2, 3, 4, 5, 6, 7}

Set T has higher SD. T has two extra numbers which are farther from the mean. Hence these 2 numbers will add to the total deviation. (There is a caveat here which we will discuss next week.)

Case 4: S = {1, 3, 5} or T = {1, 1, 3, 5, 5}

T has higher SD. It has two extra numbers far from the mean. (There is a caveat here too!)

What do you think about cases 5, 6, and 7? I will give you the answers to these three cases next week!

29. Dealing with Standard Deviation-II

This week, we pick from where we left last week. Let's discuss the last 3 cases first.

Question: Which set, S or T, has higher SD?

Case 5: S = {1, 3, 5} or T = {1, 3, 3, 5}

The standard deviation (SD) of T will be less than the SD of S. Why? The mean of 1, 3 and 5 is 3. If you add another 3 to the list, the mean stays the same and the sum of the squared deviations is also the same but the number of elements increases. Hence, the SD decreases.

Case 6: S = {6, 8, 10} or T = {12, 16, 20}

Put the numbers on the number line. You will see that the SD of T is greater than the SD of S. When you multiply each element of a set by the same number (T is obtained by multiplying each element of S by 2), the SD increases.

Case 7: S = {6, 8, 10} or T = {3, 4, 5}

Put the numbers on the number line. You will see that the SD of T is less than the SD of S. When you divide each element of a set by the same number (T is obtained by dividing each element of S by 2 OR you can say that S is obtained by multiplying each element of T by 2), the SD decreases.

Now that we have an understanding of how SD behaves, let's look at a question.

Question 1: A certain list of 300 test scores has an arithmetic mean of 75 and a standard deviation of d, where d is positive. Which of the following two test scores, when added to the list, must result in a list of 302 test scores with a standard deviation less than d?

(A) 75 and 80
(B) 80 and 85
(C) 70 and 75
(D) 75 and 75
(E) 70 and 80

Solution: As discussed last week, the standard deviation of a set measures the deviation from the mean. A low standard deviation indicates that the data points are very close to the mean whereas a high standard deviation indicates that the data points are spread far apart from the mean.

When we add numbers that are far from the mean, we are stretching the set and hence, increasing the SD. When we add numbers which are close to the mean, we are shrinking the set and hence, decreasing the SD.

Therefore, **adding two numbers which are closest to the mean will shrink the set the most**, thus decreasing SD by the greatest amount.

Numbers closest to the mean are 75 and 75 (they are equal to the mean) and thus adding them will decrease SD the most.

Answer: D.

Now that we have seen that difficult looking questions on SD can be quite simple, I want you to think about something – when you add some new numbers to a set, how do you decide whether SD increases or decreases? If you notice, we have seen two different cases (case 4 and case 5) – in one of them SD increases when you add two numbers to the set and in the other, SD decreases. So how do you decide whether SD will increase or decrease? Say, what happens in case S = {3, 4, 5, 6, 7} and T = {3, 4, 4, 5, 6, 6, 7}? Will SD increase or decrease in this case? How do you decide the point at which the increase in the numerator offsets the increase in the denominator?

Meanwhile, let's look at one more question.

Question 2: If 100 is included in each of sets A, B and C (given A= {30, 50, 70, 90, 110}, B = {-20, -10, 0, 10, 20} and C= {30, 35, 40, 45, 50}), which of the following represents the correct ordering (largest to smallest) of the sets in terms of the absolute increase in their standard deviation?

(A) A, C, B
(B) A, B, C
(C) C, A, B
(D) B, A, C
(E) B, C, A

Solution: The question looks a little convoluted but actually you don't have to calculate anything. SD measures the deviation of the elements from the mean. If a new element is added which is far away from the mean, it will add much more to the deviations than if it were added close to the mean.
The means of A, B and C are 70, 0 and 40, respectively.
100 is farthest from 0 so it will change the SD of set B the most (in terms of absolute increase). It is closest to 70 so it will change the SD of set A the least. Hence the correct ordering is B, C, A.

Answer (E)

Simple enough, right? SD questions are generally straight forward once you understand the basics well. See you next week with a tricky SD question!

30. Some Tricky Standard Deviation questions

Last week we promised you a couple of tricky standard deviation (SD) GMAT questions. We start with a 600-700 level question and then look at a 700 – 800 level one.

Question 1: During an experiment, some water was removed from each of the 8 water tanks. If
the standard deviation of the volumes of water in the tanks at the beginning of the experiment was 20 gallons, what was the standard deviation of the volumes of water in the tanks at the end of the experiment?

Statement 1: For each tank, 40% of the volume of water that was in the tank at the beginning of the experiment was removed during the experiment.

Statement 2: The average volume of water in the tanks at the end of the experiment was 80 gallons.

Solution:

We have 8 water tanks. This implies that we have 8 elements in the set (volume of water in each of the 8 tanks). SD of the volume of water in the tanks is 20 gallons. We need to find the new SD i.e. the SD after water was removed from the tanks.

Statement 1: For each tank, 40% of the volume of water that was in the tank at the beginning of the experiment was removed during the experiment.

Initial SD is 20. When 40% of the water is removed from each tank, the leftover water is 60% of the initial volume of water i.e. 0.6*initial volume of water. This means that each element of the initial set was multiplied by 0.6 to obtain the new set. The SD will change. It will become 0.6*previous SD i.e. 0.6*20 = 12 (think of the formula of SD we discussed in the first SD post). This statement alone is sufficient.

Statement 2: The average volume of water in the tanks at the end of the experiment was 80 gallons.

The average volume doesn't give us the SD of the new set. Hence, this statement alone is not sufficient.

Answer (A)

Now that we are done with the easier one, let's go on to the tougher one.

Question 2: M is a collection of four odd integers. The range of set M is 4. How many distinct values can standard deviation of M take?

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

Solution:

Since the range of M is 4, it means the greatest difference between any two elements is 4. One way of doing this will be M = {1, x, y, 5} (obviously, there are innumerable ways of writing M)
Here, x and y can take one of 3 different values: 1, 3 and 5 (x and y cannot be less than 1 or greater than 5 because the range of the set is 4).

Both x and y could be same. This can be done in 3 ways. Or x and y could be different. This can be done in 3C2 = 3 ways. Total x and y can take values in 3 + 3 = 6 ways.

(Note here that the number of ways in which you can select x and y is not 3*3 = 9. Why?)

For clarification, let me enumerate the 6 ways in which you can get the desired set:
{1, 1, 1, 5}, {1, 3, 3, 5}, {1, 5, 5, 5}, {1, 1, 3, 5}, {1, 1, 5, 5}, {1, 3, 5, 5}

Note here that standard deviations of {1, 1, 1, 5} and {1, 5, 5, 5} are same. Why? Because SD measures deviation from mean. It has nothing to do with the actual value of mean and actual value of numbers.

Mean of {1, 1, 1, 5} is 2. Three of the numbers are distance 1 away from mean and one number is distance 3 away from mean. Mean of {1, 5, 5, 5} is 4. Three of the numbers are distance 1 away from mean and one number is distance 3 away from mean. Sum of the squared deviations will be the same in both the cases and the number of elements is also the same in both the cases. Therefore, both these sets will have the same SD.

Similarly, {1, 1, 3, 5} and {1, 3, 5, 5} will have the same SD.

From the leftover sets, {1, 3, 3, 5} will have a distinct SD and {1, 1, 5, 5} will have a distinct SD.

In all, there are 4 different values that SD can take in such a case.

Note: It doesn't matter what the actual numbers are. Since we have found 4 distinct values for SD, we will always have 4 distinct values of SD for a set under the given constraints.

Answer (B)

Hope the question was fun for you too!

31.Inequalities with Multiple Factors

Students often wonder why 'x(x-3) < 0' doesn't imply 'x < 0 or (x – 3) < 0'. In this post, we will discuss why and we will see what it actually implies. Also, we will look at how we can handle such questions quickly.

When you see '< 0' or '> 0', read it as 'negative' or 'positive' respectively. It will help you think clearly.

So the question we are considering today is:

Question: For what values of x will x(x-3) be negative?

Solution: Before we try to answer this question, think – when will the product of 2 numbers be negative? When one and only one of the factors is negative. Therefore, either x should be negative or (x – 3) should be negative, but not both. Let's consider each case.

Case 1: x is negative and (x – 3) is positive

x < 0

and (x – 3) > 0 which implies x > 3

This is not possible. x cannot be less than 0 and greater than 3 at the same time. Hence this case gives us no appropriate values for x.

Case 2: x is positive and (x – 3) is negative

x > 0

and (x – 3) < 0 which implies x < 3

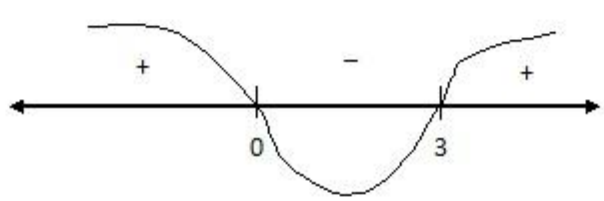
x must be greater than 0 but less than 3.

Therefore, the range of values for which this inequality will be satisfied is 0 < x < 3.

Will we have to do this every time we have multiple factors in an inequality? What will happen in case there are lots of factors? There is an easier way of handling such situations. I will first discuss the method and later explain the logic behind it.

<u>Method</u>: Say we have an inequality of the form (x – a)(x – b)(x – c) < 0. (For clarity, we will work with the example x(x – 3) < 0 discussed above.) This is how we solve for x:

Step 1: Make a number line and plot the points a, b and c on it. In our example, a = 0 and b = 3. The number line is divided into sections by these points. In our example, it is divided into 3 sections – greater than 3, between 0 and 3 and less than 0.



Step 2: Starting from the rightmost section, mark the sections with alternate positive and negative signs. The inequality will be positive in the sections where you have the positive signs and it will be negative in the sections

where you have the negative signs.

Therefore, x(x – 3) will be negative in the section 0 < x < 3 and positive in the other two sections.

Hence, the values of x for which x(x – 3) < 0 is satisfied is 0 < x < 3.

Explanation: When we plot the points on the line, the number line is divided into various sections. Values of x in the right most section will always give you positive value of the expression. The reason for this is that if x > 3, all factors will be positive i.e. x and (x – 3), both will be positive.

When you jump to the next region i.e. between x = 0 and x = 3, the values of x will give you negative values for the entire expression because now, only one factor, (x – 3), will be negative. All other factors will be positive.

When you jump to the next region on the left where x < 0, expression will be positive again because now both factors x and (x – 3) are negative. The product of two negatives is positive so the expression will be positive again and so on…

Similarly, you can solve a question with any number of factors. Next week, we will look at how to easily handle numerous complications that can arise.

32. Inequalities with Complications-I

Last week we learned how to handle inequalities with many factors i.e. inequalities of the form (x – a)(x – b)(x – c)(x – d) > 0. This week, let's see what happens in cases where the inequality is not of this form but can be manipulated and converted to this form. We will look at how to handle various complications.

Complication No. 1: (a – x)(x – b)(x – c)(x – d) > 0

We want our inequality to be of the form (x – a), not (a – x) because according to the logic we discussed last week, when x is greater than a, we want this factor to be positive. The manipulation involved is pretty simple: (a – x) = -(x – a)

So we get: – (x – a)(x – b)(x – c)(x – d) > 0

But how do we handle the negative sign in the beginning of the expression? We want the values of x for which the negative of this expression should be positive. Therefore, we basically want the value of x for which this expression itself (without the negative sign in the beginning) is negative.

We can manipulate the inequality to (x – a)(x – b)(x – c)(x – d) < 0

Or simply, multiply – (x – a)(x – b)(x – c)(x – d) > 0 by -1 on both sides. The inequality sign flips and you get (x – a)(x – b)(x – c)(x – d) < 0

e.g. Given: (4 – x)(2 – x)(-9 – x) < 0

We can re-write this as –(x – 4)(2 – x)(-9 – x) < 0

(x – 4)(x – 2)(-9 – x) < 0

-(x – 4)(x – 2)(x + 9) < 0

(x – 4)(x – 2)(x – (-9)) > 0 (multiplying both sides by -1)

Now the inequality is in the desired form.

Complication No 2: (mx – a)(x – b)(x – c)(x – d) > 0 (where m is a positive constant)

How do we bring (mx – a) to the form (x – k)? By taking m common!

(mx – a) = m(x – a/m)

The constant does not affect the sign of the expression so we don't have to worry about it.

e.g. Given: (2x – 3)(x – 4) < 0

We can re-write this as 2(x – 3/2)(x – 4) < 0

When considering the values of x for which the expression is negative, 2 has no role to play since it is just a positive constant.

Now let's look at a question involving both these complications.

Question 1: Find the range of x for which the given inequality holds.

$-2x^3 + 17x^2 – 30x > 0$

Solution:

Given: $-2x^3 + 17x^2 – 30x > 0$

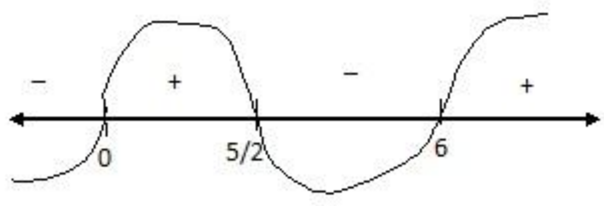
$x(-2x^2 + 17x – 30) > 0$ (taking x common)

$x(2x – 5)(6 – x) > 0$ (factoring the quadratic)

$2x(x – 5/2)(-1)(x – 6) > 0$ (take 2 common)

$2(x – 0)(x – 5/2)(x – 6) < 0$ (multiply both sides by -1)

This inequality is in the required form. Let's draw it on the number line.



We are looking for negative value of the expression. Look at the ranges where we have the negative sign.

The ranges where the expression gives us negative values are $5/2 < x < 6$ and $x < 0$.

Hence, the inequality is satisfied if x lies in the range $5/2 < x < 6$ or in the range $x < 0$.

Plug in some values lying in these ranges to confirm.

Next week, we will look at some more variations which can be brought into this form.

33.<u>Questions on Inequalities</u>

Now that we have covered some variations that arise in inequalities in GMAT problems, let's look at some questions to consolidate the learning.

We will first take up a relatively easy OG question and then a relatively tougher question which looks harder than it is because of the use of mods in the options (even though, we don't really need to deal with the mods at all).

Question 1: Is n between 0 and 1?

Statement 1: n^2 is less than n

Statement 2: n^3 is greater than 0

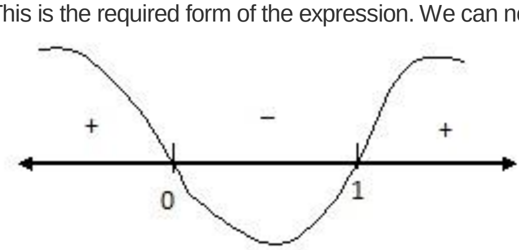
Solution: Let's take each statement at a time and see what it implies.

Statement 1: $n^2 < n$

$n^2 – n < 0$

$n(n – 1) < 0$

This is the required form of the expression. We can now put it on the number line.



For the expression to be negative, n should be between 0 and 1. So we can answer the question with a 'yes'. Statement 1 alone is sufficient.

Statement 2: n^3 > 0

This only implies that n > 0 and we do not know whether it is less than 1 or not. Hence this statement alone is not sufficient.

Answer: (A)

This question could have been easily solved in a minute if you understand the theory we have been discussing for the past few weeks. Let's go on to the trickier question now.

Question 2: Which of the following represents the complete range of x over which x^5 – 4x^7 < 0?

(A) 0 < |x| < ½

(B) |x| > ½

(C) –½ < x < 0 or ½ < x

(D) x < –½ or 0 < x < ½

(E) x < –½ or x > 0

Solution: As I said, it looks harder than it is. We can easily do this in a minute too. First, let's look at the given inequality closely: x^5 – 4x^7 < 0

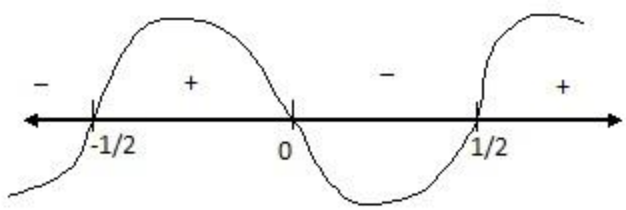
x^5 (1 – 4x^2) < 0 (taking x^5 common)

Just to make things easier right away, take out 4 common and multiply both sides by -1 to get

4(x^5)(x^2 – 1/4) > 0 (notice that the sign has flipped since we multiplied both sides by -1)

4(x^5)(x – ½)(x + ½) > 0

Think of the points you are going to plot: 0, 1/2 and -1/2. Recall that any positive odd power can be treated as a power of 1.



In which region is x positive? -1/2 < x < 0 or x > 1/2.

This is our option (C).

A quick word on the other options: What does 0 < |x| < ½ imply? It implies that distance of x from 0 is less than ½. So x lies between -1/2 and 1/2 (but x cannot be 0).

What does |x| > ½ imply? It implies that distance of x from 0 is more than ½. So x is either greater than ½ or less than -1/2.

If you are wondering what I am talking about, check out an old QWQW post:http://www.veritasprep.com/blog/2011/01/quarter-wit-quarter-wisdom-do-what-dumbledore-did/

We have discussed how to deal with modulus here. We hope this discussion has made such questions easier for you!

34. Some Inequalities, MODS, & Sets

Today, let's look at a question that involves inequalities and modulus and is best understood using the concept of sets. It is not a difficult question but it is still very tricky. You could easily get it right the first time around but if you get it wrong, it could take someone many trials before he/she is able to convince you of the right answer. Even after I write a whole post on it, I wouldn't be surprised if I see "but I still don't get it" in the comments below!

Anyway, enough of introduction! Let's get to the question now.

Question: If x/|x| < x, which of the following must be true about x?

(A) x > 1

(B) $x > -1$

(C) $|x| < 1$

(D) $|x| = 1$

(E) $|x|^2 > 1$

Solution:

First thing we do is tackle the mod. We know that $|x|$ is just the absolute value of x.

So, $x/|x|$ can take only 2 values: 1 or -1

If x is positive, $x/|x| = 1$ e.g. if $x = 4$, then $4/|4| = 1$

If x is negative, $x/|x| = -1$ e.g. if $x = -4$, then $-4/|-4| = -4/4 = -1$

x cannot be 0 because we cannot have 0 in the denominator of an expression.

Now let's work on the inequality.

$x > x/|x|$ implies $x > 1$ if x is positive or $x > -1$ if x is negative.

Hence, for this inequality to hold, either $x > 1$ (when x is positive) or $-1 < x < 0$ (when x is negative)

x can take many values e.g. -1/3, -4/5, 2, 5, 10 etc.

Now think – which of the following MUST BE TRUE about every value that x can take?

(A) $x > 1$

or

(B) $x > -1$

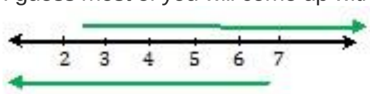
I hope that you agree that $x > 1$ doesn't hold for every possible value of x whereas $x > -1$ holds for every possible value of x. Mind you, every value greater than -1 need not be a possible value of x.

This concept might need some more work. Let me explain with another example.

Forget this question for a minute. Say instead you have this question:

Example 1: $x > 2$ and $x < 7$. What integral values can x take?

I guess most of you will come up with 3, 4, 5, 6. That's correct. I can represent this on the number line.

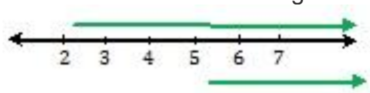


The top arrow shows $x > 2$ and the bottom arrow shows $x < 7$. You see that the overlapping area includes 3, 4, 5 and 6. That is the region that satisfies both the inequalities.

Now consider this:

Example 2: $x > 2$ or $x > 5$. What integral values can x take?

Let's draw that number line again.



So is the solution again the overlapping numbers i.e. all integers greater than 5? No. This question is different. x is greater than 2 OR greater than 5. This means that if x satisfies at least one of these conditions, it is included in your answer. Think of sets. AND means it should be in both the sets (i.e. the overlapping part) as was the case in example 1. OR means it should be in at least one of the sets. Hence, which values can x take? All integral values starting from 3 onwards i.e. 3, 4, 5, 6, 7, 8, 9 …

Now go back to this question. The solution is a one liner.

x/|x| is either 1 or -1.
So x > 1 or x > -1
So which values can x take? All values included in at least one of the sets. Therefore, x > -1.
Note that the confusion lies only between the first two options. The other three options are rejected outright.
(C) |x|< 1 implies -1 < x < 1. Definitely doesn't hold.
(D) |x| = 1 implies x = 1 or -1. Definitely doesn't hold.
(E) |x|^2 > 1 implies either x < -1 or x > 1. Definitely doesn't hold.
So what do you say? Are you convinced that the answer is (B)?

35. Speeding, Relatively

Let's look at the concept of relative speed today. A good understanding of relative speed can be very useful in some questions. If you don't use relative speed in these GMAT questions, you can still solve them but they would be rather painful to work through (they might need multiple variables and you know my policy on variables – use one, if you must)
So the first question is – what is relative speed? To understand this, let's first consider 'speed'. When your car's speedometer shows 60 mph, what does it mean? It means that you are traveling at a speed of 60 mph relative to Earth. For an observer in space, your speed could be different. It would be the resultant of your speed and Earth's speed. If this is hard to imagine, think of a plane flying in the sky. Say, it is flying at a speed of 500 mph. What happens if a strong wind starts blowing in the same direction as the plane? The wind's speed gets added to the plane's speed and the plane travels even faster. So anyway, the speed we usually talk about is relative to Earth.
It is sometimes useful to consider the speed of an object relative to another object. Say, two people are sitting next to each other in the same train which is moving at a speed of 100 mph. Are they moving relative to each other? No. Their speed relative to each other is 0. Ofcourse, relative to the Earth, they are moving at the same speed as the train i.e. 100 mph. When working on relative speed questions, just focus on the two objects we are talking about and how fast they are moving relative to each other. Their actual speed may not be of much relevance to us.
Let's consider a few cases related to relative speed. Say, there are two people, A and B.
Case 1: A is standing still and B is moving at a speed of 5 mph.
What is B's speed relative to A? B is moving away from A at a rate of 5 miles every hour. So B's speed relative to A is 5 mph. What is A's speed relative to B? Is it 0? No! A's speed relative to Earth is 0. A's speed relative to B is 5 mph since distance between A and B is increasing at a rate of 5 mph. Confused? When we say 'relative to B', we assume that B is stationary. Since distance between the two is increasing at a rate of 5 mph, we say A's speed relative to B is 5 mph.
Case 2: A is walking due east at a speed of 5 mph and B is walking due east at a speed of 2 mph.
What is A's speed relative to B? Distance between A and B is increasing by 3 miles every hour. So A's speed relative to B is only 3 mph, not 5 mph. Think of it this way – A is moving fast but not as fast when compared with B since B is also moving. Similarly, B's speed relative to A is also 3 mph. Notice that when they move in the same direction, their relative speed is the difference of their speeds.
Case 3: A is walking due east at a speed of 5 mph and B is walking due west, away from A, at a speed of 2 mph.
What is A's speed relative to B? Distance between A and B is increasing by 7 miles every hour. So A's speed relative to B is 7 mph, not 5 mph. Think of it this way – A is moving fast and even faster as compared with B since B is moving in the opposite direction. Similarly, B's speed relative to A is also 7 mph. Notice that when they move in the opposite directions, their relative speed is the sum of their speeds. It doesn't matter whether they are moving toward each other or away from each other.
Let's look at some easy questions on relative speed to cement the concepts. We will look at some tough nuts next week.

Question 1: Train A starts from station A traveling at 30 miles per hour toward station B. At the same time, on a parallel track, train ? leaves station B at 40 miles per hour toward station A. When the two trains meet, how far is train A from station B if the distance between stations A and B is 700 miles?

(A) 140

(B) 240

(C) 300

(D) 340

(E) 400

Solution: Both the trains start at the same time from stations that are 700 miles away from each other. This means that in the beginning, distance between them is 700 miles. When they meet, they have together covered the entire 700 miles.

Relative to each other, their speed is 30 + 40 = 70 mph.

Time for which they travel till they meet = 700/70 = 10 hrs

Train A covered 30*10 = 300 miles. This means, it is 400 miles away from station B. Answer (E)

Meanwhile, train B covered 40*10 = 400 miles.

Let's look at a variation of this question now.

Question 2: Train A leaves the station and travels at 30 miles per hour. Three hours later, train ? leaves the same station traveling in the same direction at 40 miles per hour on a parallel track. How far from the station was train A overtaken by train B?

(A) 90

(B) 180

(C) 200

(D) 300

(E) 360

Solution: Train A travels at 30 mph for 3 hrs and covers 30*3 = 90 miles in this time. This is when train B leaves the station. Now both the trains are running in the same direction and their relative speed is 40 – 30 = 10 mph. This means that train B covers an extra 10 miles every hour. Since the initial distance between the two trains is 90 miles, it takes train B (90/10 =) 9 hrs to catch up with train A. In 9 hrs, train B must have traveled 40*9 = 360 miles. Hence, train A must also be 360 miles away from the station.

Answer (E)

You can easily do these questions using variables. The relative speed concept is not a must. But, you definitely end up saving some time if you use relative speed. You don't take any variables and don't make any equations. Next week, we will look at some tougher relative speed questions.

36.Questions on Speeding

We discussed the concepts of relative speed in GMAT questions last week. This week, we will work on using those concepts to solve questions. The questions we take today will be 600-700 level. I intend to take the 700+ level questions next week (don't want to scare you away just yet!). Let's get going now.

Question 1: A man walking at a constant rate of 4 miles per hour is passed by a woman traveling in the same direction along the same path at a constant rate of 20 miles per hour. The woman stops to wait for the man 5 minutes after passing him, while the man continues to walk at his constant rate. How many minutes must the woman wait until the man catches up?

(A) 16 mins

(B) 20 mins

(C) 24 mins
(D) 25 mins
(E) 28 mins

Solution: We can solve this question in different ways. I will discuss two methods here – one involving the concept of relative speed and the other using the concept of ratios. The method I will not discuss is the pure algebraic one since I am sure you know how to do it using algebra.

First notice that after the woman crosses the man, she keeps going for 5 mins and in that time, the man keeps going too. So after 5 mins, the distance between them will not be the distance covered by the woman in 5 mins. It will be the difference between the distance covered by the woman in 5 mins and the distance covered by the man in 5 mins.

Method 1:

Man's speed = 4 mph and the woman's speed = 20 mph so their relative speed = 20 – 4 = 16 mph (since they travel in the same direction)

In 5 mins, the woman travels 16*(5/60) = 4/3 miles more than the man.

Hence, the distance between the man and the woman 5 mins after the woman crosses the man is 4/3 miles.

To cover 4/3 miles, the man will take (4/3)/4 = 1/3 hr = 20 mins.

Answer (B)

Method 2:

Man's speed = 4 mph and the woman's speed = 20 mph so the ratio of their speeds = 1:5.

Ratio of the time taken to travel equal distances = 5:1 (inverse of ratio of speeds)

Notice that the woman and the man cover equal distances. The woman covers a certain distance in 5 mins and the man covers the same distance in some time which is more than 5 mins.

Time taken by man: time taken by woman = 5:1

Since the woman takes 5 mins to cover that distance, the man takes 25 mins to cover it. Hence, the woman must wait for 25 – 5 = 20 mins. (we subtract 5 mins because she was traveling during that time, not waiting).

Answer (B)

Another method involves assuming variables and then solving equations but I am not going to get into that. Let's go on to the next question.

Question 2: Two trains of length 100 m and 250 m run on parallel tracks. When they run in the same direction, they take 70 sec to cross each other and when they run in opposite directions, they take 10 sec to cross each other. The speed of the faster train is

(A) 5 m/s
(B) 15 m/s
(C) 20 m/s
(D) 25 m/s
(E) 35 m/s

Solution:

To cross each other (either in same or opposite direction), the trains have to cover a distance of 250 + 100 = 350 m (the faster train should cover the entire slower train and then its own length so that they completely cross each other).

When they run in the same direction, they cover 350 m in 70 sec. This means that their relative speed in this case (which is the difference in their speeds) is 350/70 = 5 m/s

When they run in opposite directions, they cover this distance in 10 sec. So their relative speed in this case (which is the sum of their speeds) is 350/10 = 35 m/s

If the sum of two numbers is 35 and their difference is 5, you should quickly jump to 20 and 15.

The speed of the faster train must be 20 m/s. Answer (C)

If you are not sure how to get the numbers 20 and 15 given their sum and difference, you can go through the following calculations:

x + y = 35

x – y = 5

Adding these two equations, we get 2x = 40 i.e. x = 20. Put x = 20 in any one of the two equations to get y = 15.

But with such easy numbers, I would suggest you to try to figure out these values on your own. Start by thinking this way: since the difference between the numbers is small, try to split 35 in two kind-of-equal numbers. You will get 20 and 15 in no time.

37.Some Tricky relative speed Concepts

As promised, we tackle some 700+ level questions today. Mind you, these questions are not your typical GMAT type questions. The reason we are discussing them is that they look mind boggling but are easily workable when the concepts of relative speed are used. They give insights that help you understand relative speed. Once you are good with the concepts, you can solve most of the relative speed based questions easily.

Question 1: Cities A and B are 20 miles apart. From both of these cities, simultaneously, two people start walking toward each other at a constant speed of 2 miles/hr. At the same time, a dog leaves city B and runs at a constant speed of 5 miles/hr toward city A. When it reaches the person from city A, it immediately turns around and runs back to the person from city B. When it reaches the person from city B, it turns around and runs back to the person from city A. It keeps doing so until the two people meet. How many miles did the dog run?

(A) 15 miles

(B) 20 miles

(C) 25 miles

(D) 30 miles

(E) 35 miles

Solution: Looks really tricky, doesn't it? Actually it is really easy once you look at it from the right perspective. It would take you less than a minute to solve almost any 700+ level GMAT question if you can figure out the most optimum method to solve it. The point is – how long will it take you to figure out the most optimum method? Take a minute to think what you would do in this question.

We know the dog's speed. We need to know the distance it has run. Directly calculating distance is a little complicated since we need to consider the distance covered by the two men too. Instead, if we can figure out the time for which the dog ran, we can easily calculate the distance. For how long did the dog run? He started when the two men started from their respective cities and stopped when the two men met. Therefore, if we can find the time taken by the men to cover the 20 miles, we will get the time for which the dog ran.

This is where the relative speed concept comes in handy.

Total distance between the two men = 20 miles

Relative speed of the men with respect to each other = 2 + 2 mph (since they are travelling in opposite directions)

Time taken by the men to meet = 20/4 = 5 hrs

Therefore, the dog also ran for 5 hrs.

In 5 hrs, the dog must have run 5*5 = 25 miles

Answer (C)

I hope you see that the right perspective simplifies the question immensely. Let's look at another 700+ level question.

Question 2: A man cycling along the road at a constant speed noticed that every 12 minutes a bus overtakes him and every 6 minutes he meets an oncoming bus. If all buses move at the same constant speed and leave the bus station at fixed time intervals, what is the time interval between consecutive buses?

(A) 5 minutes
(B) 6 minutes
(C) 8 minutes
(D) 9 minutes
(E) 10 minutes
Solution: Again, the question is a little tricky but once you understand how to tackle it, it takes less than a minute.
Buses are coming from opposite directions. A bus overtakes the man every 12 minutes i.e. a bus moving in the same direction as the man overtakes him every 12 mins. To clarify: say, at constant intervals, buses leave a bus station located from where the man left and travel on the same road as the man. Since they are faster, they overtake the man. The man noticed that a bus overtakes him every 12 mins. Obviously then, they must be leaving at constant intervals. Also, he meets a bus coming from the opposite direction every 6 mins. So buses must be leaving from a bus station located at the opposite end of the road at constant intervals.
I hope the problem is clear to you. Now let's try to work out the solution.
Say the cyclist is stationary at a point. Buses are coming from opposite directions (same speed, same time interval). A bus will meet the cyclist every t minutes from either direction. This 't minutes' must be the time interval between consecutive buses. Let's say, a bus from each direction just met him. After t minutes, 2 more buses from opposite directions will meet him and so on… We need to find 't'.
Now the cyclist starts moving at speed c. His speed relative to the bus going in the same direction becomes b – c. His speed relative to the bus from the opposite direction becomes b + c. This is the reason that the time interval between two buses is different for the opposite directions. Time interval is in the ratio 12:6. Then, the ratio of the relative speed in the two cases must be inverse i.e. 6:12
(b-c):(b+c) = 6:12 which gives you c = (1/3)b
This means that the bus travelling at a relative speed which is 2/3rd of its usual speed (b-c = 2b/3) takes 12 minutes to meet the man. If it were travelling at its usual speed, it would have taken 12*(2/3) = 8 mins to meet the man. This 8 mins is the value of 't' i.e. the time interval between buses.
Answer (C)
You might need to go through the question a few times before you fully understand it. It will also be helpful to draw a diagram and see what the situation looks like.

38.Moving in Circles

Hopefully, you are a little comfortable with the relative speed concept now. The concept can come in handy in some circular motion questions too. Today, we will show you how you can use the fundamentals we learned in the last few weeks to solve questions involving moving in a circle. Let's try a couple of GMAT questions to put what we learned to use:

Question 1: Two cars run in opposite directions on a circular path. Car A travels at a rate of 4? miles per hour and car B runs at a rate of 6? miles per hour. If the path has a radius of 8 miles and both the cars start from point S at the same time, how long, in hours, after the cars depart will they meet again for the first time after leaving?

(A) 1 hr
(B) 1.2 hrs
(C) 1.5 hrs
(D) 1.6 hrs
(E) 1.8 hrs

Solution:

Quarter Wit_Quarter Wisdom- Part-2

When two objects travel in opposite directions, their relative speed is the sum of their speeds. So the relative speed of the two cars is (4? + 6?) mph = 10? mph

The length of the path is given by the circumference of the circle i.e. 2?*8 = 16?

The two cars leave together from S. When they meet again, together they have covered the length of the path i.e. 16? miles.

Time taken to cover a full circle by both together = Distance Covered/Relative Speed = 16?/10? = 1.6 hrs

The concept is exactly the same. The only thing different is that you need to calculate the length of the path here. Let's look at a trickier question now.

Question 2: Bob starts at point X and runs clockwise around a circular track at a constant rate of 2 mph. Ten hours later, Alan leaves from point X and travels counter-clockwise around the same circular track at a constant rate of 3 mph. If the radius of the track is 10 miles, for how many hours did Bob run from the time he started to the time he passed Alan for the first time and put another 12 miles between them (measured around the curve of the track)?

(A) 4? – 1.6
(B) 4? + 8.4
(C) 4? + 10.4
(D) 2? – 1.6
(E) 2? – 0.8

Solution: The question stem is rather long. You should process one statement at a time. Let me break the question stem into multiple statements. Let's learn how to handle a question one step at a time.

"Bob starts at point X and runs clockwise around a circular track at a constant rate of 2 mph."

- Ok, so Bob covers 2 miles every hour moving clockwise. We don't know the track length yet.

"Ten hours later, Alan leaves from point X and travels counter-clockwise around the same circular track at a constant rate of 3 mph."

- Alan is faster and covers 3 miles every hour counter clockwise. Since their directions are opposite, their relative speed = 2+3 = 5 mph. We still don't know the track length so we cannot say where Bob was when Alan started. All we know is that in 10 hrs, Bob traveled 20 miles.

"If the radius of the track is 10 miles,"

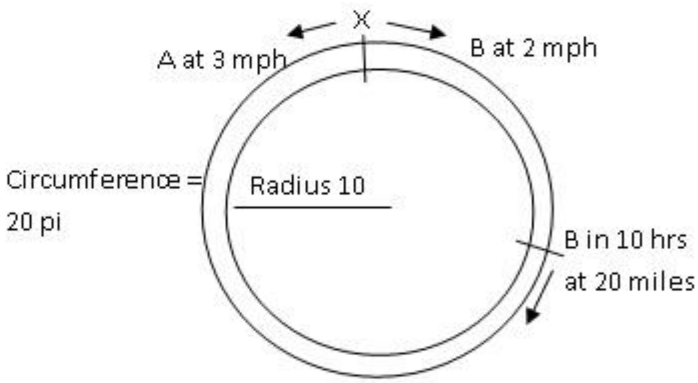
- Now we know the track length. It is 2?r = 2?*10 = 20?. It is greater than the 20 miles that Bob covered in 10 hrs so Bob has not finished one round. Of the 20?, he has covered only 20 miles.

"for how many hours did Bob run from the time he started to the time he passed Alan for the first time and put another 12 miles between them (measured around the curve of the track)?"

- To pass each other for the first time, Alan and Bob together need to cover the remaining distance on the circle i.e. 20? – 20. To create a distance of another 12 miles, they together need to travel 12 miles more away from each other.

Time taken = Total distance to be traveled/ Relative Speed
Time taken = (20? – 20 + 12)/5 = 4? – 1.6 hrs
Bob must have been traveling for 4? - 1.6 + 10 = (4? + 8.4) hrs

Another way to approach this question is using a diagram:



Radius of the track is 10 miles so circumference is 20? i.e. the total length of the track.
Bob starts from X and travels for 10 hrs clockwise at 2 mph i.e. he travels 20 miles. Now Alan starts from X counter clockwise. Distance between Alan and Bob is 20? – 20.

Now, to meet, they have to together cover this distance plus 12 miles more which they have to put between them.
Time taken to cover this distance by them = (20? – 20 + 12)/(3 + 2) = 4? – 1.6 hrs
Bob has been traveling for 10 + 4? – 1.6 = 4? + 8.4 hrs

I hope you see that the questions become very manageable if you use relative speed.

39. Working like Clockwork

In the last few weeks, we have gone through the concepts of relative speed. You might be surprised to know that we can use the same concept to solve some clock problems too. The reason some clock problem can be tricky is that the hour hand and the minute hand move simultaneously so handling them separately is not easy. In such questions you can easily use relative speed i.e. speed of the minute hand relative to the hour hand. Let's try to understand this with the help of an example.

Example: In a circular clock, the minute hand is the radius of the circle. At what time is the smaller angle between the minute hand and the hour hand of the clock not divisible by 10?

I. 7:20

II. 4:30

III. 9:00

(A) Only I
(B) Only II
(C) I + II
(D) II+III
(E) I+II+III

Solution: First let me explain why the question is a little complicated. We need to find the smaller angles between the two hands of the clock at given times. We know that at 9'o clock the hour hand is at 9 and the minute hand is at 12. So the angle between the hour hand and the minute hand is 90 degrees. But what about 7:20? We know that at 7:20, the minute hand is at 4 but where is the hour hand? Is it exactly at 7? No, it is somewhere between 7 and 8. At 4:30, is the hour hand at 4 or mid way between 4 and 5? So you see there is a complication. You have to account for the little bit of distance covered by the hour hand too to get the angle between the hour hand and the minute hand.

Let's see the relevance of relative speed here: Minute hand covers 360 degrees in an hour i.e. it makes one full

rotation. Or we can say that it covers 30 degrees in every 5 mins. On the other hand, the hour hand completes one rotation of 360 degrees in 12 hrs. Or we can say that it covers 360/12 = 30 degrees in an hour.
Speed of minute hand relative to hour hand is 360 – 30 = 330 degrees per hour (since they move in the same direction so the relative speed is the difference between their speeds).

Let's now try to find the angle between the hour hand and the minute hand at the given times. We will start with the easiest one.

III. 9:00

At 9 o clock, the hour hand is at 9 and the minute hand is at 12. The angle between the two hands is 90 degrees. 90 is divisible by 10. This was simple enough.

II. 4:30

At 4 o clock, the minute hand is at 12 and the hour hand is at 4 i.e. the minute hand is 120 degrees behind the hour hand. In half an hour, it covers 330/2 = 165 degrees. This means it covers the 120 degrees between them and further creates a gap of 45 degrees i.e. the minute hand is 45 degrees ahead of the hour hand now. The smaller angle between them is 45 degrees. 45 is not divisible by 10.

I. 7:20

At 7'o clock, the minute hand is at 12 and the hour hand is at 7 i.e. the minute hand is 210 degrees behind the hour hand (going clockwise). In 20 minutes (i.e. at 7:20), it makes up 330*20/60 = 110 degrees. Now the minute hand will be 210 – 110 = 100 degrees behind the hour hand. The smaller angle between them now is 100 degrees. 100 is divisible by 10.

So the smaller angle between the hands is not divisible by 10 at 4:30.

Answer (B)

I hope you see that the question has become quite simple. This is just one of the many applications of relative speed. You can obviously do it without using relative speed concepts too. The method you choose to use during the exam should be the one you are most comfortable with.

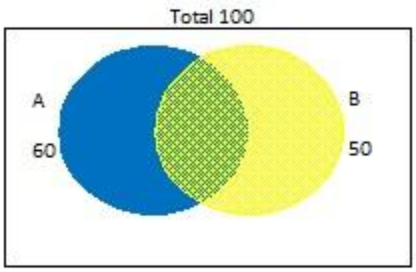
40.Nuances of Sets

We will start with sets today. Your Veritas Prep GMAT book explains you the basics of sets very well so I am not going to get into those. If you have gone through the concepts, you know that we can use Venn diagrams to solve the sets questions.

First, let's look at why we should focus on terminology in sets question. Thereafter, we will put up a very nice question from our very own book which is simple but takes down many people (just like a typical GMAT question):

Say, there are a total of 100 people in a housing society. There are two clubs close to the society – A and B. You are given that of the 100 people of the housing society, 60 people are members of club A and 50 people are members of club B.

Quarter Wit_Quarter Wisdom- Part-2



Question 1: How many people are members of both the clubs?

We are looking for the number of people in the green region. The answer here is not 10. It is 'cannot be determined' i.e. you cannot say how many people are members of both the clubs. The reason is that you do not know how many people belong to neither club.

Say, for all future questions (unless mentioned otherwise), you are given that 20 people belong to neither club. What can you say about the number of people who belong to both the clubs? Now, out of the pool of 100, 20 are out. Only 80 people are club members. Since 60 are members of club A and 50 people are members of club B which gives us a total of 110, there must be an overlap of 30 people i.e. 30 people must belong to both the clubs (80 = 60 + 50 – Both)

Question 2: How many people belong to only one club?
We found above that 30 people belong to both the clubs. So out of the 60 people of club A, 30 belong to only club A. Out of the 50 people of club B, 20 belong to only club B. So a total of 30+20 = 50 people belong to only one club, either A or B but not both. (60 – Both + 50 – Both = 30 + 20 = 50)

Question 3: Say, you don't know the number of people who belong to neither club. What is the minimum number of people who must belong to both the clubs?

We know that there are a total of 100 people. 60 belong to club A and 50 belong to club B which adds up to 110. Therefore, AT LEAST 10 people must have membership of both the clubs. Now if you increase the number of people who do not belong to either club, the number of people who belong to both will increase by the same number. Think in terms of the Venn diagram. If the 'Neither' number increases, the number of people who are members decreases. Hence, the overlap increases to keep A = 60 and B = 50.

Let's look at the promised question which will make this concept clear.

Question: Of the 400 members at a health club, 260 use the weight room and 300 use the pool. If at least 60 of the members do not use either, then the number of members using both the weight room and the pool must be between:

(A) 40 to 100
(B) 80 to 140
(C) 160 to 260
(D) 220 to 260
(E) 220 to 300

Solution: When we minimize “number of members who do not use either”, we are minimizing the “number of members who use both” as well.
Look at the equation:

Total = A + B – Both + Neither

Since the total sum 400 is constant, if we increase the ‘Neither’ i.e. 60, we will have to increase the ‘Both’ term too to maintain the sum of 400 (Assuming A and B are constant which they are since they are given to us).
Least value of ‘number of members who use neither’ is 60. We will get the least value of ‘number of members who use both’ when we put ‘Neither’ = 60.
400 = 260 + 300 – ‘Minimum value of both’ + 60

Minimum value of both = 220

On the same lines, if we maximize “number of members who use neither”, we are maximizing the “number of members who use both” as well.

What is the maximum number of people who use neither? Out of a total of 400 people, 300 people use the pool. Hence at least 300 people use at least one of the two facilities. This means that there can be AT MOST 100 people (total 400 – 300 who use pool) who use neither facility.

400 = 260 + 300 – ‘Maximum value of both’ + 100

Maximum value of both = 260

Answer (D)

Hope today’s post has helped improve your sets concepts!

41. A sets questions that upsets many

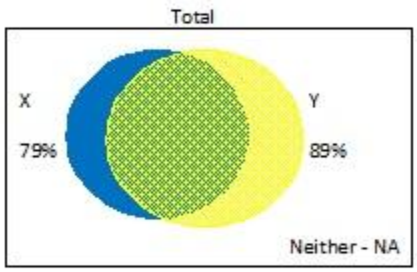
We hope last week’s discussion improved your understanding of sets and showed you how you can come across some tricky sets questions even though the concepts seem very simple. Today, let’s further build up on what we learned last week with the help of an example.

Example: A group of people were given 2 puzzles. 79% people solved puzzle X and 89% people solved puzzle Y. What is the maximum and minimum percentage of people who could have solved both the puzzles?

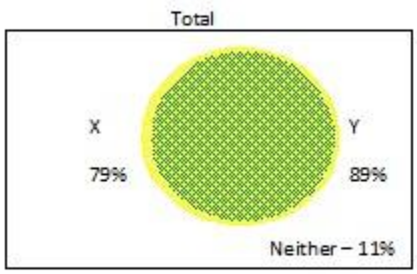
(A) 11%, 0%
(B) 49%, 33%
(C) 68%, 57%
(D) 79%, 68%
(E) 89%, 79%

Solution: The first thing to note here is that we do not know the % of people who could not solve either puzzle. All we know is that puzzle X was solved by 79% of the people and puzzle Y was solved by 89% of the people.

Quarter Wit_Quarter Wisdom- Part-2

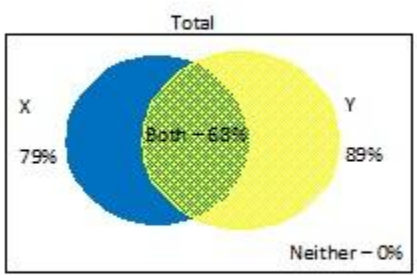


Let's first try to maximize the % of people who solved both the puzzles. We want to make these two sets overlap as much as possible i.e. we need to get them as close to each other as possible. Region of overlap can be 79% at most since we know that only 79% people solved puzzle X. In this case, the venn diagram will look something like this.



Hence, the maximum % of people who could have solved both the puzzles is 79%.

Now, let's try to minimize the % of people who solved both the puzzles. We want the sets to be as far apart as possible. In this case, the % of people who solved neither puzzle must be 0. Only then will the overlap of the sets be as little as possible.



In this case, 68% people must have solved both the puzzles.

Hence, the **answer is (D)**

Note: If the question instead gave you the % of people who did not solve either puzzle (e.g. by giving you that everyone solved at least one puzzle), then there is no question of maximizing/minimizing the % of people who solved both the puzzles. Consider this:

(For ease, let's drop the percentage and work with just numbers.)

Total no. of people = No. of people who solved X + No. of people who solved Y – No. of people who solved both + No. of people who solved neither

100 = 79 + 89 – No. of people who solved both + No. of people who solved neither

No. of people who solved both – No. of people who solved neither = 68

We can maximize/minimize the two numbers by adjusting them against each other. If one increases, the other increases too. If one decreases, the other decreases too.

If no. of people who solve both = 68, no. of people who solve neither = 0

If no. of people who solve both = 69, no. of people who solve neither = 1

If no. of people who solve both = 79, no. of people who solve neither = 11

If you are given the number of people who solved neither, you have a fixed number of people who solved both. Hence, maximizing or minimizing becomes pointless.

You can also work on this concept logically –

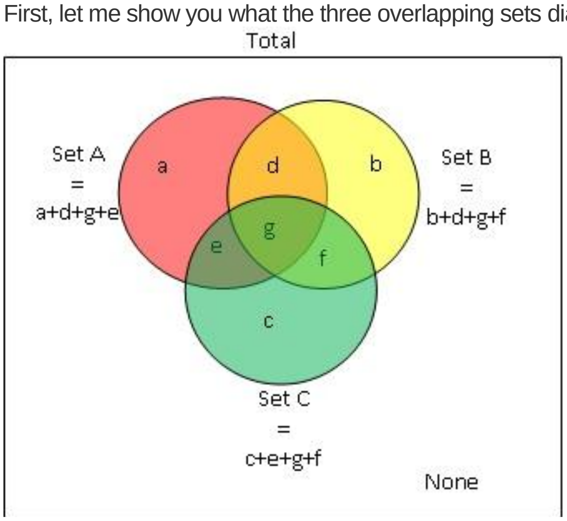
Say, every solution of every puzzle was written down on a separate sheet of paper. Then there would be 168 sheets of paper in all (79 + 89). If everyone solved 1 puzzle, then we have accounted for 100 sheets. The other 68 sheets were made by 68 people who have already solved one puzzle each. Hence, 68 people would have solved both the puzzles. If instead, 99 people solve at least one puzzle and one person solves no puzzle, then 69 (obtained by 168 – 99) sheets would have been made by people who have already solved one. Hence 69 people would have solved both the puzzles. Note here that the number of people who solved neither and the number of people who solved both are not independent of each other. One number depends on the other. It will be good if you make a note of this in your log book.

Next week, let's tackle three overlapping sets.

42. Three overlapping sets

Today, let's take a look at how to use Venn diagrams to solve questions involving three overlapping sets.

First, let me show you what the three overlapping sets diagram looks like.



Notice that the total comprises of the elements that do not fall in any of the three sets and the elements that are a part of at least one of the three sets.

Quarter Wit_Quarter Wisdom- Part-2

The elements falling in the red, yellow or green region (region a, b or c) fall in only one set. The elements falling in region d, e or f fall in exactly two sets and the elements falling in region g fall in all three sets.

Now some quick questions to get a clear picture:

Question 1: Which regions represent the elements that belong to at least 2 sets?

Answer 1: $d + e + f + g$

Question 2: Which regions represent the elements that belong to at least 1 set?

Answer 2: $a + b + c + d + e + f + g = \text{Total} – \text{None}$

Question 3: Which regions represent the elements that belong to at most 2 sets?

Answer 3: $\text{None} + a + b + c + d + e + f = \text{Total} – g$

Hope there are no doubts up till now. Let’s look at a question to see how to apply these concepts.

Question: Three table runners have a combined area of 200 square inches. By overlapping the runners to cover 80% of a table of area 175 square inches, the area that is covered by exactly two layers of runner is 24 square inches. What is the area of the table that is covered with three layers of runner?

(A) 18 square inches

(B) 20 square inches

(C) 24 square inches

(D) 28 square inches

(E) 30 square inches

Solution: Let’s first try to understand what exactly is given to us. The area of all the runners is equal to 200 square inches.

Runner 1 + Runner 2 + Runner 3 = 200

In our diagram, this area is represented by

$(a + d + g + e) + (b + d + g + f) + (c + e + g + f) = 200$

(We need to find the value of g i.e. the area of the table that is covered with three layers of runner.)

Area of table covered is only 80% of 175 i.e. only 140 square inches. This means that if each section is counted only once, the total area covered is 140 square inches.

$a + b + c + d + e + f + g = 140$

So the overlapping regions are obtained by subtracting second equation from the first. We get $d + e + f + 2g = 60$

But $d + e + f$ (area with exactly two layers of runner) = 24

So $2g = 60 – 24 = 36$

$g = 18$ square inches

Note that you don’t need to make all these equations and can directly jump to $d + e + f + 2g = 60$. We wrote these equations down only for clarity. It is a matter of thinking vs solving. If we think more, we have to solve less. Let’s see how.

Combined area of runners is 200 square inches while area of table they cover is only 140 square inches. So what does the extra 60 square inches of runner do? It covers another runner!

Wherever there are two runners overlapping, one runner is not covering the table but just another runner. Wherever there are three runners overlapping, two runners are not covering the table but just the third runner at the bottom.

So can we say that $(d + e + f)$ represents the area where one runner is covering another runner and g is the area where two runners are covering another runner?

Put another way, can we say $d + e + f + 2g = 60$?

We know that $d + e + f = 24$ giving us $g = 18$ square inches

This entire ‘thinking process’ takes ten seconds once you are comfortable with it and your answer would be out in about 30 sec!

43.Get the full picture

Today we will discuss why 'understanding' rather than just 'learning' a concept is important. Most questions can be solved using different methods. Sometimes, a particular method seems really easy and quick and we tend to 'learn' it without actually knowing why we are doing what we are doing. We need to understand the strengths and the weaknesses of the method before we use it. Let me elaborate with an example.

Question: What is the probability that you will get a sum of 8 when you throw three dice simultaneously?

When you throw three dice simultaneously, you can obtain a total sum ranging from 3 (1 on each die) to 18 (6 on each die). Let's consider all the possible sums that we can obtain:

Sum of 3: This can happen in only 1 way. Each die shows 1 in this case

Sum of 4: This can happen in 3 ways. One die shows 2 and the other two show 1 each. Any of the three dice could show 2 so there are a total of 3 ways of getting a sum of 4.

Sum of 5: There are different ways of obtaining a sum of 5:

1, 1, 3 – Any of the three dice could show 3 so there are a total of 3 ways of obtaining 5 in this way.

1, 2, 2 – Any of the three dice could show 1 so there are a total of 3 ways of obtaining 5 in this way.

Total number of ways of obtaining a sum of 5 = 3 + 3 = 6

Sum of 6: There are different ways of obtaining a sum of 6:

1, 1, 4 – Any of the three dice could show 4 so there are a total of 3 ways of obtaining 6 in this way.

1, 2, 3 – These 3 numbers could be arranged among the 3 dice in 3! ways (using basic counting principle). There are a total of 3! = 6 ways of obtaining 6 in this way.

2, 2, 2 – This can happen in only one way. All dice show 2.

Total number of ways of obtaining a sum of 6 = 3 + 6 + 1 = 10

Similarly, we can find the number of ways in which all other sums can be obtained. Below I will list the number of ways in each case.

Sum of 3: 1 way	Sum of 18: 1 way
Sum of 4: 3 ways	Sum of 17: 3 ways
Sum of 5: 6 ways	Sum of 16: 6 ways
Sum of 6: 10 ways	Sum of 15: 10 ways
Sum of 7: 15 ways	Sum of 14: 15 ways
Sum of 8: 21 ways	Sum of 13: 21 ways
Sum of 9: 25 ways	Sum of 12: 25 ways
Sum of 10: 27 ways	Sum of 11: 27 ways

Notice the symmetry here. There is only 1 way of obtaining 3 and only one way of obtaining 18. Similarly, there are 3 ways in which you can obtain 4 and 3 ways in which you can obtain 17. Is it a co-incidence? No. The second half of the column can be obtained by replacing 1 by 6, 2 by 5, 3 by 4, 4 by 3, 5 by 2 and 6 by 1. The number of ways of obtaining a sum is symmetrical about the center.

We have simplified one aspect of this problem. If we need to find the number of ways of obtaining 15, we can instead find the number of ways of obtaining a sum of 6 (which is psychologically easier to handle). Now the problem is whether there is an easier way of obtaining the number of ways in which you can get a sum of 6 or do we have to enumerate it in every case.

You might have come across something like this:

Sum of 3: 2C2 = 1 way

Sum of 4: 3C2 = 3 ways

Sum of 5: 4C2 = 6 ways

Sum of 6: 5C2 = 10 ways

Sum of 7: 6C2 = 15 ways

Sum of 8: 7C2 = 21 ways

Perfect till now! Matches the numbers we obtained above. It seems like a good method to use instead of writing down all the cases, doesn't it? But what happens further on?

Sum of 9: 8C2 = 28 ways

Sum of 10: 9C2 = 36 ways

These don't match! The key to understanding this is to understand why it works in the above given cases. First review this post.

Focus on method II of question 2. Notice how you divide n identical objects among m distinct groups. Let's take the example of a sum of 7. You have to divide 7 among 3 dice such that each die must have at least 1 (no die face can show 0). First step is to take 3 out of the 7 and give one each to the three dice. Now you have 4 left to distribute among 3 distinct groups such that it is possible that some groups may get none of the four. Think of partitioning 4 in 3 groups. This can be done in (4+2)!/4!*2! = 6C2 ways (check out the given link if you do not understand this)

This is how you obtain 6C2 for the sum of 7.

The concept works perfectly till the sum of 8. Thereafter it fails. Think why. I will explain it next week.

44.Get the full picture-II

Let's refer back to last week's post. We discussed why it is important to fully understand what you are doing and why you are doing it, especially while using an innovative method. We talked about it using a combinatorics example.

Let's revisit it here:

Question 1: What is the probability that you will get a sum of 8 when you throw three dice simultaneously?

We discussed the ways of obtaining various sums. The regular way of obtaining a sum of 8 is enumerating all the possibilities. An innovative way was using 7C2 (as discussed last week).

Now the question we raised last week was this: Why does this method fail for the sum of 9?

Think about why this method works for the sum of 8 (and others before it) – when you split 8 in 3 groups, you first give 1 each of the three dice (since each dice must show at least 1) and then split the rest of the 5 in 3 groups. What happens when we try to do the same for the sum of 9? When we give 1 to each die, we are left with 6. Now when we split 6 among 3 groups, three of the cases will look like this:

6, 0, 0

0, 6, 0

0, 0, 6

What does this imply? It implies that 2 dice show 1 each and one die shows 7! Of course, no die can show 7 so these 3 cases need to be removed. So out of 8C2 = 28 cases we need to remove 3 and we get 25 cases. That is the correct answer!

Similarly, how will you adjust for the sum of 10? There will be cases where the split is like this: (7, 0, 0) or (6, 1, 0). These do not work since the maximum a die can show is 6. So you need to remove 9 cases (3 arrangements of 7, 0, 0 and 6 arrangements of 6, 1, 0) from the obtained sum of 9C2 = 36.

So there are 36 – 9 = 27 ways in which you can obtain a sum of 10.

Let's answer the original question now:

No. of cases in which the sum will be 8 = 21

Total number of cases = 6*6*6 = 216 (since the 3 dice can show any one of the 6 numbers)

Probability of obtaining a sum of 8 on a throw of three dice = 21/216 = 7/72.

Let's see if this method works for 4 dice.

Question 2: If four dice are thrown together, what is the probability that the sum on them together is either 18 or 22?

Solution: Let's try and use the same method to find the sum in case of four dice.

First of all, we see that the minimum sum will be 1*4 = 4 and the maximum sum will be 6*4 = 24. The number of ways will be symmetrical about the mid-point i.e. 14.

<u>4, 5, 6 … 12, 13</u>, 14, <u>15, 16 … 22, 23, 24</u>

So number of ways in which you can obtain 13 is the same as the number of ways in which you can obtain 15. The number of ways in which you can obtain 12 is the same as the number of ways in which you can obtain 16 and so on.

No. of ways in which you can obtain 18 will be the same as the number of ways in which you can obtain 10. To get a sum of 10, we give 1 to each die and then split the leftover 6 among 4 groups. We can do that in 9C3 ways [As discussed last week, we obtain (6+3)!/3!*6! by using method II of question 2 discussed in this post.]

But 9C3 includes 4 cases which look like this:

6, 0, 0, 0

0, 6, 0, 0

0, 0, 6, 0

0, 0, 0, 6

This is not acceptable since no die can show more than 6. Hence, number of ways of obtaining a sum of 10 = 9C3 – 4 = 80 ways

No. of ways in which you can obtain a sum of 22 is the same as the number of ways in which you can obtain a sum of 6. To obtain a sum of 6, give 1 to each of the 4 dice and split the remaining 2 in 4 groups in 5C3 = 10 ways

No. of ways in which you can obtain 18 or 22 = 80 + 10 = 90

Total no. of cases = 6*6*6*6

Probability of obtaining a sum of 18 or 22 = 90/6*6*6*6 = 5/72

To sum it all up, it's great to use innovative methods/shortcuts but there is a caveat – ensure that you fully understand the reason the shortcut works.

45. Working on Getting the Full Picture Again!

Today, we will continue our discussion on why it is important to understand the workings behind seemingly miraculous shortcuts. We will use another example from probability.

Question 1: A bag contains 4 white balls, 2 black balls & 3 red balls. One by one three balls are drawn out with replacement (i.e. a ball is drawn and then put back. Thereafter, another ball is drawn). What is the probability that the third ball is red?

Solution:

The question is simple, isn't it? When you draw balls with replacement, the probability stays the same at the beginning of every cycle. On first draw, the probability of drawing a red ball is 3/9 (since there are 3 red balls and 9 balls in all). When you draw the ball and put in back, the probability of drawing a red ball again stays the same i.e. 3/9 (since again there are 3 red balls and 9 balls in all). The situation at the beginning of every draw is the same.

Right, let's move on to the question we actually wanted to discuss!

Question 2: A bag contains 4 white balls, 2 black balls & 3 red balls. One by one three balls are drawn out without replacement (the balls are not put back). What is the probability that the third ball is red?

Solution: This question differs from the previous one. The balls are not replaced here so every pick is not the same.

I guess, intuitively, you will say that the probability of getting a red ball in the third draw will now change. Now here is a fun fact: the probability that the third ball is red is still 3/9. Without knowing the other results, the probability of drawing a red ball will not change for the successive drawings. The probability will remain the same as it is on the first draw. It's good to remember this i.e. it's a shortcut. Rather than working on all the cases e.g. you pick a red first, then a red and a red again or you pick a red first, then a non red and then a red again etc, you know that the probability will not change in any successive draw.

Let's try to understand why this is so. We will take the example of the second draw and see why the probability of picking a red stays the same as the probability of picking a red in the first draw.

Probability of picking a red in the second draw:

Let's discuss the two cases: 'First draw is red' and 'first draw is not red'

Case 1: First draw is red and the second draw is red.

Probability of first draw being red = (3/9)

Probability of second draw being red = (2/8)

The probability of picking a red first and then a red again = (3/9)*(2/8)

Case 2: First draw is non red and the second draw is red.

Probability of first draw being non red = (6/9)

Probability of second draw being red = (3/8)

The probability of picking non red first and then red = (6/9)*(3/8)

(6/9)*(3/8) = (3/9)*(6/8).

(3/9)*(6/8) is the probability of picking a red ball first and then a non red ball.

Think about it: Probability of picking a non red ball first and then a red ball will be the same as the probability of picking a red ball first and then a non red ball. Note here that we do not need to deal with white and black balls separately. They are just non-red since we don't care about the balls of other colors.

Total probability of second draw being red = Probability in case 1 + Probability in case 2

= (3/9) * (2/8) + (3/9) * (6/8) = 3/9 * (2/8 + 6/8) = 3/9 * 1

This is just the probability of picking a red ball first and then any ball (non red or red). Probability of picking ANY ball will be 1. Hence, the probability of picking a red in the second draw will be the same as the probability of picking a red in the first draw.

Using the same concept, you can see that the probability of picking a red in any draw will be the same if the balls obtained in the previous draws are not known.

46.Successive Division

We discussed divisibility and remainders many weeks ago. Today, we will use those concepts and discuss another type of question – successive division. But before we do, you need to go through the previous related posts on division if you haven't read them already:

Divisibility Unraveled

Divisibility Applied on the GMAT

Divisibility Applied to Remainders

Now, let's start working on today's topic.

What is meant by – 'a number when divided successively by 4 and 5 leaves remainder 1 and 4 respectively'?

Does it mean that when you divide the number by 4, the remainder is 1 and when you divide it by 5 the remainder is 4? No. It means that when you divide the number by 4, the remainder is 1 and then when you divide the quotient obtained (from the first division) by 5, the remainder is 4.

e.g., when you divide 37 by 4, the remainder you get is 1 and the quotient you get is 9. When you divide 9 (not 37 here) by 5, the remainder you get is 4.

This is what we mean by successive division i.e. you keep dividing quotients you get instead of starting with the original number again. Now that we understand what successive division is, let's look at what kind of questions appear on successive division.

Question 1: A number when divided successively by 4 and 5 leaves remainders 1 and 4 respectively. What will be the remainder when this number is divided by 20?

(A) 0
(B) 3
(C) 4
(D) 9
(E) 17

Solution: Let's find one number, say n, which when divided successively by 4 and 5 leaves remainder 1 and 4 respectively. When you divide n by 4, you get a quotient and remainder 1. When you divide the quotient by 5, you get the remainder 4. What can the quotient be (when you divide by 5) for the remainder to be 4? The first one that comes to mind is that the quotient could be 4 itself. If you divide 4 by 5, the remainder is 4. Now, if in the previous step when you divided n by 4, if the quotient was 4 and the remainder was 1, then the number n must have been 4*4 + 1 = 17 (n = Quotient*Divisor + Remainder). Now, what will be the remainder when you divide 17 by 20? It will be 17.

Answer (E)

Basically, you multiplied the second remainder with the first divisor and added the first remainder to get the first such number.

Two divisors: 4, 5

Two remainders: 1, 4

Start from the bottom right corner and go up diagonally: 4*4.

Then go down and add 1: 4*4 + 1

The first such number is 17.

If you want to see the algebra approach, let us show you that too.

n = 4a + 1 (When n is divided by 4, quotient is 'a')

a = 5b + 4 (When a is divided by 5, quotient is 'b')

n = 4(5b + 4) + 1 = 20b + 17

When n is divided by 20, we see that the remainder will be 17.

The problem with the algebra approach is that it gets a little cumbersome when dealing with 3 or more successive divisions. Let's look at a question involving 3 divisions now.

Question 2: On dividing a certain number by 5, 7 and 8 successively, the remainders obtained are 2, 3 and 4 respectively. When the order of division is reversed and the number is successively divided by 8, 7 and 5, the respective remainders will be:

(A) 3, 3, 2
(B) 3, 4, 2
(C) 5, 4, 3
(D) 5, 5, 2
(E) 6, 4, 3

Solution:

Three given divisors: 5, 7, 8
Three given remainders: 2, 3, 4
We start by considering the last step first. At the end, when we divide by 8, we want remainder to be 4. This means that after division by 7, we should get 4 as the quotient (to get the first value of n). When we divide by 7, the quotient must be 4 and the remainder must be 3. At this step, the number becomes 7*4 + 3 = 31

Now, when we divide by 5, we need the quotient to be 31 and remainder to be 2 so number must be 5*31 + 2 = 157

157 is the first such number. Divide 157 by 8, 7 and 5, in that order:
157/8 gives quotient = 19 and remainder = 5
19/7 gives quotient = 2 and remainder = 5
2/5 gives quotient = 0 and remainder = 2

Remainders are 5, 5, 2

Answer (D)

The steps followed are these:

Three Divisors: 5, 7, 8
Three Remain: 2, 3, 4

Start from the bottom of the last column i.e. from the third remainder:

Go up diagonally and multiply by the second divisor: 4*7 = 28

Go down and add the second remainder: 28 + 3 = 31

Go up diagonally and multiply by the first divisor: 31* 5 = 155

Go down and add the first remainder: 155 + 2 = 157

157 is the first such number. Now proceed as before to get the remainders when you divide 157 by 8, 7 and 5.

Now you can tackle any number of divisors and remainders easily!ility

47.Managing Factorials in Equations

A concept we have not yet covered in this series is factorials (though we used some factorials in the post Power in Factorials). Let's first discuss the basics of factorials. Once we do, we will see that most factorial expressions can be easily solved using a single method: taking common!

First of all, what is (n!)?

n! = 1*2*3*4*5*6*…*(n – 2)*(n -1)*n

Let's take some examples:

0! = 1 (mind you, it is not 0)

1! = 1

2! = 1*2

3! = 1*2*3

4! = 1*2*3*4

and so on…

Look carefully. Do you see any relation between 3! and 4!? Sure. 3! Appears in 4! too.

4! = 1*2*3*4 = (1*2*3)*4 = (3!)*4

Similarly, 2! is also a part of 3! as well as 4!

4! = 1*2*3*4 = (2!)*3*4 = (3!)*4

As a general note, we can say that:

n! = (n – 1)! * n

n! = (n – 2)! * (n – 1) * n

n! = (n – 3)! * (n – 2) * (n – 1) * n

and so on…

We can write n! in many different ways. We use whatever suits us best in the question. How does knowing this help us solve questions? Let's see:

Question: If (n-2)!= [n! + (n-1)!]/99 and n is a positive integer, how many distinct values can n take?

(A) 0
(B) 1
(C) 2
(D) 3
(E) Infinite

Solution: We need to solve this equation to find out the values of x which satisfy it. But how do we solve equations with factorials in them?

(n-2)!= [n! + (n-1)!]/99

It looks rather complicated, right? It is not, actually! Let's use what we have just learned. We need to separate the factorials from the rest of the equation. To do that, we need to take something common. The left hand side has (n-2)!

We know that n! = (n-2)!*(n-1)*n

and (n-1)! = (n-2)!*(n-1)

The equation becomes: (n-2)!= [(n-2)!*(n-1)*n + (n-2)!*(n-1)]/99

(n-2)! = (n-2)!*[(n-1)*n + (n-1)]/99

99(n-2)! = (n-2)!*[(n-1)*n + (n-1)]

(n-2)!*[(n-1)*n + (n-1) - 99] = 0

The product of two factors (n-2)! and [(n-1)*n + (n-1) - 99] must be 0 so at least one of them must be 0. Notice that factorial of a number cannot be 0 so the other factor i.e. [(n-1)*n + (n-1) - 99] must be 0.

[(n-1)*n + (n-1) - 99] = 0

n^2 = 100

n can take two values: 10 and – 10

But it is given to us that n is a positive integer so only 10 is acceptable.

Hence, there is only 1 value which satisfies this equation.

Answer (B)

Remember, when dealing with multiple factorials, all you can do is take something common. But then, that may be all you need to do!

48. Questions on Factorials

Last week we discussed factorials – how we can take something common when we have factorials in some equations. Today let's discuss a couple of questions based on factorials. They look intimidating but they are pretty simple. Factorial is all about multiplication and hence there is a high probability that you will be able to take something common and cancel something. These techniques reduce our work significantly. Hence, seeing a factorial in a question should bring a smile to your face!

Question 1: Given that x, y and z are positive integers, is y!/x! an integer?

Statement 1: (x + y)(x – y) = z! + 1

Statement 2: x + y = 121

Solution: First let's focus on what the question is asking.

Is y!/x! an integer? When will division of two factorials lead to an integer? Take examples to understand 5!/4! = 5 (an integer).

3!/5! is not an integer.

10!/5! = 6*7*8*9*10 (an integer)

We can see that if y is greater than or equal to x, y!/x! will be an integer. If y is less than x, then we will obtain a proper fraction of the form 1/n. Let's go on to the statements now.

Statement 1: (x + y)(x – y) = z! + 1

x and y are positive integers. This mean (x + y) must be positive.

z is a positive integer so z! must be positive too. This means the right hand side must be positive. So the left hand side should be positive too. This means (x – y) must be positive i.e. x – y > 0 or x > y.

If x > y, y!/x! will not be an integer (as discussed above).
This statement alone is sufficient to answer the question.

Statement 2: x + y = 121

This statement doesn't tell us whether x is greater than y. Hence, this statement alone is not sufficient to answer the question.

Answer (A)

Question 2: Given that k = [(17!)^16 - (17!)^8] / [(17!)^8 + (17!)^4], what is the units digit of k/(17!)^4

(A) 0
(B) 1
(C) 3
(D) 5
(E) 9

Solution: The given expression can be easily made manageable. (I would suggest you to write it down on paper since the tons of brackets and operators used here make it difficult to understand.) All we have to remember is that the only thing that works with factorials is 'taking common'

k = [(17!)^16 - (17!)^8] / [(17!)^8 + (17!)^4]
k = (17!)^8[(17!)^8 - 1] / [(17!)^4{(17!)^4 + 1}]
k = (17!)^4[(17!)^4 + 1] *[(17!)^4 - 1] / [(17!)^4 + 1]

(Using the identity a^2 – b^2 = (a – b)(a + b))

We get k = (17!)^4[(17!)^4 - 1]

Then, k/(17!)^4 = (17!)^4[(17!)^4 - 1]/(17!)^4

k/(17!)^4 = (17!)^4 – 1

49.Rates Revisited

People often complain about getting stuck in work-rate problems. Hence, I would like to take some 700+ level questions on rate today. I have discussed the basic concepts of work-rate (using ratios) in a previous post:

Cracking the Work Rate Problems

You might want to go through that post before you set out to work on these problems. Ensure that you are very comfortable with the relation: Work = Rate*Time and its implications: If rate doubles, work done doubles too if the time remains constant; if one work is done, rate = 1/time etc. Thorough understanding of these implications is fundamental to 'reasoning out' the answer.

Question 1: Machine A and Machine B can produce 1 widget in 3 hours working together at their respective constant rates. If Machine A's speed were doubled, the two machines could produce 1 widget in 2 hours working together at their respective rates. How many hours does it currently take Machine A to produce 1 widget on its own?

(A) 1/2
(B) 2
(C) 3
(D) 5
(E) 6

Solution: Tricky, eh? It is a little cumbersome if you get into variables. If you just try to reason it out, it could be done rather quickly and easily. Let's see!

Machine A and B together complete 1 work in 3 hrs i.e. together, they do $1/3^{rd}$ work every hour.

If machine A's speed were double, they would do 1/2 work in 1 hour together. How come they do (1/2 – 1/3 =) $1/6^{th}$ work extra in 1 hour now? Because machine A's speed is double the previous speed. The extra speed that machine A has allows it to do 1/6th work extra. This means, at normal speed, machine A used to do 1/6 work in an hour (its speed had doubled so work had doubled too). Hence, at usual speed, it will take 6 hrs to produce 1 widget.

Answer (E)

Consider the amount of time and effort you would have spent on this question had you tried to use two variables to figure out the answer. You would have made equations like this: 1/a + 1/b = 1/3 and 2/a + 1/b = 1/2 and then you would have solved them simultaneously to get the value of a. Whereas in the solution above, we have done all the work orally!

Question 2: One woman and one man can build a wall together in two hours, but the woman would need the help of two girls in order to complete the same job in the same amount of time. If one man and one girl worked together, it would take them four hours to build the wall. Assuming that rates for men, women and girls remain constant, how many hours would it take one woman, one man, and one girl, working together, to build the wall?

(A) 5/7
(B) 1
(C) 10/7
(D) 12/7
(E) 22/7

Solution: This question is certainly quite tricky but if you understand the relation between work and rate, you can still solve this question easily. Mind you, we are using variables here only because I don't want to write man, woman and girl again and again. Notice that there are no '=' signs i.e. we are not making equations so we are not doing any algebraic manipulations.

The question is long so take one line at a time and analyze it. We will keep condensing the information we get from each sentence and figuring out the implications of new and previous information as we go along.

“One woman and one man can build a wall together in 2 hrs,”
1w + 1m -> 2 hrs ……(I)

“but the woman would need the help of 2 girls in order to complete the same job in the same amount of time.”
1w + 2g -> 2 hrs …..(II)

From (I) and (II), we can say that 1m is equivalent to 2g (i.e. 1 man does the same work as 2 girls do in the same amount of time; 1m ? 2g)

“If 1 man and 1 girl worked together, it would take them four hours to build the wall.”
1m + 1g -> 4hrs (Since 1m ? 2g, we can say that 3g will take 4 hrs to build the wall.)
or 2m + 2g -> 2 hrs …..(III) (If number of workers double, time taken to do the work becomes half)

From (II) and (III), 1w ? 2m (i.e. 1 woman does the same work as 2 men do in the same amount of time)
Hence, 1w ? 2m ? 4g

“Assuming that rates for women, men and girls remain constant, how many hours would it take 1 woman, 1 man and 1 girl working together to build the wall?”
1w + 1m + 1g ? 4g + 2g + 1g ? 7g. Since 3g take 4 hrs to build the wall, 7g will take 3*4/7 = 12/7 hrs to complete the wall.

Answer (D)

We have done most of the work while reading the question only. Had we tried to solve it algebraically, we would have made 3 equations using 3 variables and then tried to solve them.

50. Have a Game Plan

I have been meaning to discuss a question for a while. We can easily solve it by plugging in the right values. The only issue is in figuring out the right values quickly. The point we are going to discuss is that there has to be a plan.

Question 1: Six countries in a certain region sent a total of 75 representatives to an international congress. No two countries sent the same number of representatives. Of the six countries, if Country A sent the second greatest number of representatives, did Country A send at least 10 representatives?

Statement 1: One of the six countries sent 41 representatives to the congress.

Statement 2: Country A sent fewer than 12 representatives to the congress.

Solution:

There is no straight and simple algebra method here. You need to plug in values and understand the different possible combinations. At the same time, it is not as hard as you might expect. Work with a plan and you might get your answer quickly. You should just know how to manipulate numbers and examples.

Let’s elaborate a little.

6 countries, 75 people.

No two countries sent the same number of people.

Statement 1: One of the six countries sent 41 representatives to the congress.

If one country sent 41 people, the other 5 together sent 75 – 41 = 34 people. Since this country sent more people than all the other countries put together, we can say that this country sent the maximum number of people.

Country ‘A’ sent the second greatest number of representatives so it sent the most number of people from the remaining 5 countries. Does country A need to send at least 10 people?

Here some number manipulation helps us.
34 divided by 5 is approximately 7. On average, every country sent 7 people. No two countries sent the same number of people. We can split 34 around 7 in various ways e.g. 4, 6, 7, 8, 9 (try and split 34 around 7 so that the average stays approximately 7). Country A sent 9 people in this case. Country A could have sent less than 10 people.
We can easily see that country A could have sent more than 10 people (Say if the other 4 countries sent 1, 2, 3 and 4 people).
Since country A could have sent more than 10 or less than 10 people, this statement alone is not sufficient.
Statement 2: Country A sent fewer than 12 representatives to the congress.
There is no condition on the number of people sent by the country that sent the maximum number of people. It could be anything 20, 30, 40 or even 41. There are many cases possible. All we have to show is that there are at least two cases – one in which country A sends less than 10 people and another in which it sends more than 10 people. It's best to work with the 41 example since we are already familiar with it. We already know that country A could have sent 9 people (as shown while analyzing statement 1 above). Let's find out whether the number of people sent by country A can be 11 (keeping in mind the less than 12 condition) in this case.
We need to figure out whether country A can send 11 people. For 9, we split 34 as 4, 6, 7, 8, 9. Try to make minimum changes to get what you want so that you can minimize the chances of error. Since we want to increase the last number, we just reduce the first one appropriately. The split could be 2, 6, 7, 8, 11. In this case, country A could have sent 11 people.
Again, since country A could have sent more than 10 or less than 10 people, this statement alone is not sufficient.
Using both statements together, A could have sent 9 or 11 people so both statements together are not sufficient.
Answer (E)
As we said before, the question is simple. You just need to keep in mind that you should not get lost in the values you put in. Try to make minimum changes to get the desired result.

51.Some GCF Concepts

Sometimes students come up looking for explanations of concepts they come across in books. Actually, in Quant, you can establish innumerable inferences from the theory of any topic. The point is that you should be comfortable with the theory. You should be able to deduce your own inferences from your understanding of the topic. If you come across some so-called rules, you should be able to say why they hold. Let's discuss a couple of such rules from number properties regarding GCF (greatest common factor). Many of you might read them for the first time. Stop and think why they must hold.
Rule 1: Consecutive multiples of 'x' have a GCF of 'x'
Explanation: What do we mean by consecutive multiples of x? They are the consecutive terms in the multiplication table of x. For example, 4x and 5x are consecutive multiples of x. So are 18x and 19x…
What will be the greatest common factor of 18x and 19x? We know that x is their common factor. Do 18 and 19 have any common factors (except 1)? No. So greatest common factor will be x. Take any two consecutive numbers. They will have no common factors except 1. Hence, if we have two consecutive factors of x, their GCF will always be x.
For more on common factors of consecutive numbers, check:
http://www.veritasprep.com/blog/2011/09 … c-or-math/
http://www.veritasprep.com/blog/2011/09 … h-part-ii/
Can you derive some of your own 'rules' based on this now? Let's give you some ideas:
Two consecutive integers have GCF of 1.
Two consecutive odd multiples of x have GCF of x.
Rule 2: The G.C.F of two distinct numbers cannot be larger than the difference between the two numbers.
Explanation: GCF is a factor of both the numbers. Say, the GCF of two distinct numbers is x. This means the two numbers are mx and nx where m and n have no common factor. What can be the smallest difference between m and n? m and n

cannot be equal since the numbers are distinct. The smallest difference between them can be 1 i.e. they can be consecutive numbers. In that case, the difference between mx and nx will be x which is equal to the GCF. If m and n are not consecutive integers, the difference between them will be much larger than x. The difference between mx and nx cannot be less than x. Say, GCF of two numbers is 6. The numbers can be 12 and 18 (GCF = 6) or 12 and 30 (GCF = 6) etc but they cannot be 12 and 16 since both numbers must have 6 as a factor. So after a multiple of 6, the other multiple of 6 must be at least 6 away.

Let's look at a third party question based on these concepts now.

Question 1: What is the greatest common factor of x and y?

Statement 1: Both x and y are divisible by 4.

Statement 2: x – y = 4

Solution:

Statement 1: Both x and y are divisible by 4

We know that 4 is a factor of both x and y. But is it the highest common factor? We do not know. There could be another factor common between x and y and hence highest common factor could be greater than 4. e.g. 4 and 16 have 4 as the highest common factor but 12 and 36 have 12 as the highest common factor though both pairs have 4 as a common factor.

Statement 2: x – y = 4

We know that x and y differ by 4. So their GCF cannot be greater than 4 (as discussed above). The GCF could be any of 1/2/4 e.g. 7 and 11 have GCF of 1 while 2 and 6 have GCF of 2.

Taking both statements together: From statement 1, we know that x and y have 4 as a common factor. From statement 2, we know that x and y have one of 1/2/4 as highest common factor. Hence 4 is the highest common factor.

Answer (C)

52. GCF and LCM of Fractions

Last week we discussed some concepts of GCF. Today we will talk about GCF and LCM of fractions.

LCM of two or more fractions is given by: LCM of numerators/GCF of denominators

GCF of two or more fractions is given by: GCF of numerators/LCM of denominators

Why do we calculate LCM and GCF of fractions in this way? Let's look at the algebraic explanation first. Then we will look at a more intuitive reason.

Algebraic Approach:

Consider 2 fractions a/b and c/d in their lowest form, their LCM is Ln/Ld and GCF is Gn/Gd, also in their lowest forms.

Let's work on figuring out the LCM first.

LCM is a multiple of both the numbers so Ln/Ld must be divisible by a/b. This implies (Ln/Ld)/(a/b) is an integer. We can re-write this as:

Ln*b/Ld*a is an integer.

Since a/b and Ln/Ld are in their lowest forms, Ln must be divisible by a; also, b must be divisible by Ld (because a and b have no common factors and Ln and Ld have no common factors).

Using the same logic, Ln must be divisible by c; also d must be divisible by Ld.

Ln, the numerator of LCM, must be divisible by both a and c and hence should be the LCM of a and c, the numerators. Ln cannot be just any multiple of a and c; it must be the lowest common multiple so that Ln/Ld is the *lowest* multiple of the two fractions.

b and d both must be divisible by Ld, the denominator of LCM, and hence Ld must be their highest common factor. Mind you, it cannot be just any common factor; it needs to be the *highest* common factor so that Ln/Ld is the lowest multiple possible.

This is why LCM of two or more fractions is given by: LCM of numerators/GCF of denominators.

Using similar reasoning, you can figure out why we find GCF of fractions the way we do.

Now let me give you some feelers. They are more important than the algebraic explanation above. They build intuition.
Intuitive Approach:
Let me remind you first that LCM is the lowest common multiple. It is that smallest number which is a multiple of both the given numbers.
Say, we have two fractions: 1/4 and 1/2. What is their LCM? It's 1/2 because 1/2 is the smallest fraction which is a multiple of both 1/2 and 1/4. It will be easier to understand in this way:
1/2 = 2/4. (Fractions with the same denominator are comparable.)
LCM of 2/4 and 1/4 will obviously be 2/4.
If this is still tricky to see, think about their equivalents in decimal form:
1/2 = 0.50 and 1/4 = 0.25. You can see that 0.50 is the lowest common multiple they have.
Let's look at GCF now.
What is GCF of two fractions? It is that greatest factor which is common between the two fractions. Again, let's take 1/2 and 1/4. What is the greatest common factor between them?
Think of the numbers as 2/4 and 1/4. The greatest common factor between them is 1/4.
(Note that 1/2 and 1/4 are both divisible by other factors too e.g. 1/8, 1/24 etc but 1/4 is the greatest such common factor)
Now think, what will be the LCM of 2/3 and 1/8?
We know that 2/3 = 16/24 and 1/8 = 3/24.
LCM = 16*3/24 = 48/24 = 2
LCM is a fraction greater than both the fractions or equal to one or both of them (when both fractions are equal). When you take the LCM of the numerator and GCF of the denominator, you are making a fraction greater than (or equal to) the numbers.
Also, what will be the GCF of 2/3 and 1/8?
We know that 2/3 = 16/24 and 1/8 = 3/24.
GCF = 1/24
GCF is a fraction smaller than both the fractions or equal to one or both of them (when both fractions are equal). When you take the GCF of the numerator and LCM of the denominator, you are making a fraction smaller than (or equal to) the numbers.
We hope the concept of GCF and LCM of fractions makes sense to you now.

53. Pattern Recognition

If you are hoping for a 700+ in GMAT, you need to develop the ability to recognize patterns. GMAT does not test advanced concepts but you can certainly get advanced questions on simple concepts. For such questions, the ability to quickly observe patterns can come in quite handy. We will discuss a complicated question today which can be easily solved by observing the pattern.
Question: If a and b are distinct integers and a^b = b^a, how many solutions does the ordered pair (a, b) have?

(A) None
(B) 1
(C) 2
(D) 4
(E) Infinite
Solution: Given a^b = b^a (a and b are distinct integers).
First thing that comes to mind is that if we didn't need *distinct* integers then the answer would have simply been infinite since 1^1 = 1^1; 2^2 = 2^2; 3^3 = 3^3 and so on…

Next, integers include positive and negative numbers. If a result is true for positive a and b, it will also be true for negative a and b and vice versa. Note that 'a' and 'b' must have the same sign. It is not possible that 'a' is a positive integer while 'b' is a negative integer or vice versa because then the side of the equation that has negative power will get flipped and will not be an integer (except if the base is 1 but in that case one side will be positive and the other will be negative)

So basically, we need to consider only positive integers (we can mirror them on to the negative side subsequently). Also, we need to consider only numbers where a < b because the equation is symmetrical in a and b. So if we get a solution of two distinct integers satisfying the equation (e.g. a = 2 and b = 4), it will give us one more solution a = 4 and b = 2.

Let us take a look at 0 separately. 'a' cannot be 0 since it will lead to 0^b = b^0. The left hand side is 0 and the right hand side is 1. This will not hold for any value of b.

Next, let's consider a = 1; 1^b = b^1; again, this is not possible because left hand side is 1 while the right hand side cannot be 1 (recall that 'b' should not be equal to 'a')

Let us consider a = 2 now. We need to look for values where 2^b = b^2. Let's put in some values of b now.

2^3 < 3^2;

2^4 = 4^2 (our first solution);

2^5 > 5^2;

2^6 > 6^2

and the difference between the left hand side and the right hand side will keep widening.

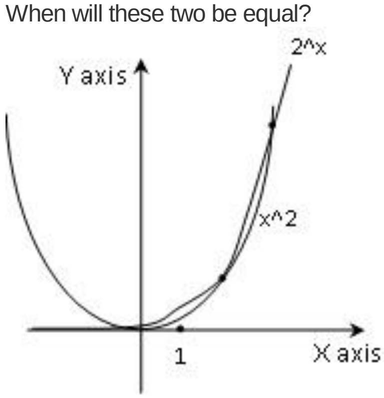
This is where pattern recognition comes in the picture – observe that the gap will keep widening.

Another approach is to look at the graphs of b^2 and 2^b.

b^2 is a quadratic so its graph will be a parabola with its vertex on (0, 0)

2^b is an exponential function. It intersects the x axis at 1.

When will these two be equal?



They intersect at two points x = 2 and x = 4 (the diagram is not to scale). After that the exponential function rises much faster than the quadratic function. So after intersecting at (2, 4), they will never intersect again. We ignore the point (2, 2) since a and b should not be equal.

Now consider a = 3.

3^4 > 4^3; 3^5 > 5^3 and the gap keeps widening (already, the left hand side is greater than the right hand side)

Here, the exponential function is already greater than the quadratic. So going further to the right, they will never intersect.

The pattern should be clear by now. 4^5 > 5^4, 5^6 > 6^5 and so on…

As the value of 'a' keeps increasing, the difference in the two terms will keep increasing.

So we have four solutions (2, 4), (4, 2), (-2, -4) and (-4, -2).

Answer (D)

Note: Generally, out of a^b and b^a, the term where the base is smaller will be the bigger term (discussing only positive integers). In very few cases will it be smaller or equal.
Now, we know that if the question did not have the word 'distinct', the answer would have been different, but what if the question did not have the word 'integer'? Would it make a difference? Something to think about…

54. Pattern Recognition-II

Last week we saw how to use pattern recognition. Today, let's take up another question in which this concept will help us. Mind you, there are various ways of solving a question. Most questions we solve using pattern recognition can be solved using another method. But pattern recognition is a method we can use in various cases. It is something that comes to our aid when we forget everything else. If you don't know from where to start on a question, try to give some values to the variables. You might see a pattern. You may not 'know' something. Even then, you can 'figure out' the answer because GMAT is not a test of your knowledge; it is a test of your wits. It is a test of whether you can keep your cool when faced with the unknown and use whatever you know to solve the question.

Let's look at a question now.

Question: What is the sum of the cubes of the first ten positive integers?
(A) 10^3
(B) 45^2
(C) 55^2
(D) 100^2
(E) 100^3
Solution:
You are obviously not expected to find the cubes and sum them.
There is a simple formula to find the sum of cubes of first n positive integers. It is given by [n(n+1)/2]^2
Put n = 10 in the formula. You get [10*11/2]^2 = 55^2.
Now, do we need to learn the formula?
No. Even if you didn't know it, you should have tried to look at the pattern.
1^3 = 1
1^3 + 2^3 = 9
1^3 + 2^3 + 3^3 = 36
1^3 + 2^3 + 3^3 + 4^3 = 100
We see that all the sums are squares.
1^3 = 1 = 1^2
1^3 + 2^3 = 9 = 3^2
1^3 + 2^3 + 3^3 = 36 = 6^2
1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2
The point is – what are they squares of? What are 1, 3, 6 and 10? If you notice carefully, you will see that we obtain these numbers when we add the first n numbers.
1 = 1
1 + 2 = 3
1 + 2 + 3 = 6
1 + 2 + 3 + 4 = 10

This is pattern recognition. It might seem a little hard initially, but once you get used to it, you realize that every number you get is there for a reason. E.g., very rarely will you see 81 if it has nothing to do with it being 3^4.

You can also use another method in this question – of averaging.

We need to find this sum: 1^3 + 2^3 + 3^3 + … + 8^3 + 9^3 + 10^3

The numbers on the right 8^3, 9^3, 10^3 will be much larger than those on the left which are small. The average would lie not in the middle at 5^3 but on the right somewhere between 6^3 and 7^3. I would say around 300. Since the average represents the number that can replace every number in the list, the sum will be around 300*10 = 3000. This leads us to 55^2 since 55^2 = 3025 (it is very easy to find squares of numbers ending with 5. We will discuss it soon.)

The only hitch in using this approximation is the 45^2.

45^2 = 2025

To convince yourself that the average is around 300 and not around 200, notice that from 10^3 = 1000, you can make five 200s. From 9^3, you can make about four 200s (using some extra). So overall, you can make many more 200s than the required 10. Therefore, the average must be around 300 and not around 200.

Answer (C).

This is just one example of the ways pattern recognition can help us. Keep an open mind and you will see patterns everywhere.

55.Pattern Recognition or Number property?

Continuing our quest to master 'pattern recognition', let's discuss a tricky little question today. It is best done using divisibility and remainders logic we discussed in some previous posts. We suggest you check out these divisibilityposts if you haven't yet.

We are first going to see how to solve the question conceptually. The interesting thing is – what do you do if you are under immense pressure during the test and are unable to remember anything you ever read on divisibility? It is a fairly common phenomenon – students have reported that they had blanked out during the test and couldn't think of an appropriate approach. Our suggestion is that in that case, you should lean on trying to figure out the pattern. Try out a couple of values and see what you get. You may not understand why you are getting what you are getting but that doesn't stop you from getting the correct answer. Let's jump on to the question – we will first discuss the ideal approach and then go on to what happens if you don't have a clue of what to do in the question.

Question: If p, m, and n are positive integers, n is odd, and p = m^2 + n^2, is m divisible by 4?

Statement 1: When p is divided by 8, the remainder is 5.

Statement 2: m – n = 3

Solution: Given that p, m and n are positive integers. If n is odd, n^2 must also be odd. How do you represent an odd integer? As (2k + 1)

n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1

Since one of k and (k+1) will definitely be even (out of any two consecutive integers, one is always even, the other is always odd – discussed in detail here), 4k(k+1) will be divisible by 8. Therefore, when n^2 is divided by 8, it will leave a remainder of 1.

Statement 1: When p is divided by 8, the remainder is 5.

When p (i.e. m^2 + n^2) is divided by 8, we get a remainder of 5. When n^2 is divided by 8, the remainder will be 1. To get a remainder of 5, when m^2 is divided by 8, we should get a remainder of 4.

m^2 = 8a + 4 (i.e. we can make 'a' groups of 8 and 4 will be leftover)

m^2 = 4(2a + 1)

This implies m = 2*?(Odd Number) because (2a+1) is an odd number. Square root of an odd number will also be odd. Therefore, we can say that m is not divisible by 4.

This statement alone is sufficient.

Statement 2: m – n = 3

The difference between m and n is 3 i.e. an odd number. Since n is odd, we can say that m will be even (Even – Odd = Odd). But whether m is divisible by 2 only or by 4 as well, we cannot say since here we have no constraints on p.

This statement alone is not sufficient.

Answer (A)

In this question, analyzing the question stem and statement 1 is a little complicated. Say we don't analyze the question stem and jump to statement 1. Let's see how we can use pattern recognition to make it easier.

Question: If p, m, and n are positive integers, n is odd, and p = m^2 + n^2, is m divisible by 4?

Statement 1: When p is divided by 8, the remainder is 5.

Statement 2: m – n = 3

Solution:

Statement 1: When p is divided by 8, the remainder is 5.

When p (i.e. m^2 + n^2) is divided by 8, we get a remainder of 5. We need a remainder of 5 when m^2 + n^2 is divided by 8. Let's try to find the remainders when m^2 and n^2 are divided by 8.

We are given that n is odd. Let's try to figure out what this implies.

n = 1; When n^2 (= 1) is divided by 8, the remainder is 1.

n = 3; When n^2 (= 9) is divided by 8, the remainder is 1.

n = 5; When n^2 (= 25) is divided by 8, the remainder is 1.

n = 7; When n^2 (= 49) is divided by 8, the remainder is 1.

There is a pattern here! Whenever you divide the square of an odd number by 8, you get the remainder 1. (We have discussed 'why' above in the logical explanation of statement 1)

This implies that when we divide m^2 by 8, we get 4 as remainder. If m^2 gives 4 as remainder, it means it is of the form m^2 = 8a + 4 = 4(2a + 1). So m must be of the form 2?(Odd Number). Hence m is not divisible by 4.

This statement is sufficient alone.

Through this example, you can see that pattern recognition is a very important tool (in easy as well as difficult questios)

56. Varying Directly

We can keep working on 'pattern recognition' questions for a long time and not run out of questions of different types on which it can be used. We hope you have understood the basic concepts involved. So let's move on to another topic now: Variation.

Basically, variation describes the relation between two or more quantities. e.g. workers and work done, children and noise, entrepreneurs and start ups. More workers means more work done; more children means more noise; more entrepreneurs means more start ups and so on… These are examples of direct variation i.e. if one quantity increases, the other increases proportionally. Then there are quantities that have inverse variation between them e.g. workers and time taken. If there are more workers, time taken to complete a work will be less.

Let's discuss **direct variation** today.

Formally, let's say x varies directly with y. If x takes values x1, x2, x3… and y takes values y1, y2, y3 … correspondingly, then x1/y1 = x2/y2 = x3/y3 = Some constant value

In other words, ratio of x and y stays the same in different instances.

(Notice that this is the same as x1/x2 = y1/y2)

It might seem a little cumbersome when put this way but the truth is that direct variation is quite intuitive. A couple of questions will make it clear.

Question 1: 20 workmen can make 35 widgets in 5 days. How many workmen are needed to make 105 widgets in 5 days?

(A) 7

(B) 20

(C) 25

(D) 30

(E) 60

Solution: Notice that the number of days stays the same so we can ignore it. Now think, how are workmen and widgets related? If the number of workmen increases, the number of widget made also increases proportionally. You need to find the new number of workmen required. The number of widgets has become thrice (105/35 = 3) so number of workmen needed will become thrice as well (remember, the number of workmen will increase in the same proportion).

We need 20*3 = 60 workmen

Answer (E)

The concept of variation is very intuitive. If the number of widgets required doubles, the number of workmen required to make them in the same amount of time will double too. If the number of widgets required becomes one fourth, the number of workmen required to make them in the same amount of time will become one fourth too.

A quantity can directly vary with some power of another quantity. Let's take an example of this scenario too.

Question 2: If the ratio of the volumes of two right circular cylinders is given by 64/9, what is the ratio of their radii? (Both the cylinders have the same height)

(A) 4/3

(B) 8/3

(C) 16/9

(D) 4/1

(E) 16/3

Solution: This question involves a little bit of geometry too. The volume of a right circular cylinder is given by Area of base * height i.e.

Volume of a right circular cylinder = pi*radius^2 * height

So volume varies directly with the square of radius.

Va/Vb = 64/9 = Ra^2/Rb^2

Ra/Rb = 8/3

Answer (B)

We hope this little concept is not hard to understand. We will work on inverse proportion next week and then work on problems involving both (that's where the good questions are!).

57.<u>Varying Inversely</u>

As promised, we will discuss inverse variation today. The concept of inverse variation is very simple – two quantities x and y vary inversely if increasing one decreases the other proportionally.

If x takes values x1, x2, x3… and y takes values y1, y2, y3 … correspondingly, then x1*y1 = x2*y2 = x3*y3 = some constant value

This means that if x doubles, y becomes half; if x becomes 1/3, y becomes 3 times etc. In other words, product of x and y stays the same in different instances.

Notice that x1/x2 = y2/y1; The ratio of x is inverse of the ratio of y.

The concept will become clearer after working on a few examples.

Quarter Wit_Quarter Wisdom- Part-2

Question 1: The price of a diamond varies inversely with the square of the percentage of impurities. The cost of a diamond with 0.02% impurities is $2500. What is the cost of a diamond with 0.05% impurities (keeping everything else constant)?

(A) $400

(B) $500

(C) $1000

(D) $4000

(E) $8000

Solution:

Price1*(% of Impurities1)^2 = Price2*(% of Impurities2)^2

2500*(.02)^2 = Price2*(.05)^2

Price = $400

Answer (A)

The answer is quite intuitive in the sense that if % of impurities in the diamond increases, the price of the diamond decreases.

There is an important question type related to inverse variation. It often uses the formula:

Total Price = Number of units*Price per unit

If, due to budgetary constraints, we need to keep the total money spent on a commodity constant, number of units consumed varies inversely with price per unit. If price per unit increases, we need to reduce the consumption proportionally.

Question 2: The cost of fuel increases by 10%. By what % must the consumption of fuel decrease to keep the overall amount spent on the fuel same?

(A) 5%

(B) 9%

(C) 10%

(D) 11%

(E) 20%

Solution: Do you think the answer is 10%? Think again.

Total Cost = Number of units*Price per unit

If the price per unit increases by 10%, it becomes 11/10 of its original value. To keep the total cost same, you need to multiply number of units by 10/11. i.e. you need to decrease the number of units by 1/11 i.e. 9.09%. In that case,

New Total Cost = (10/11)*Number of units*(11/10)*Price per unit

This new total cost will be the same as the previous total cost.

Answer (B)

Let's look at one more example of the same concept but this one is a little trickier.

Question 3: Recently, fuel price has seen a hike of 20%. Mr X is planning to buy a new car with better mileage as compared to his current car. By what % should the new mileage be more than the previous mileage to ensure that Mr X's total fuel cost stays the same for the month? (assuming the distance traveled every month stays the same)

(A) 10%

(B) 17%

(C) 20%

(D) 21%

(E) 25%

Solution: The problem here is 'how is mileage related to fuel price?'

Total fuel cost = Fuel price * Quantity of fuel used

Since the ‘total fuel cost’ needs to stay the same, ‘fuel price’ varies inversely with ‘quantity of fuel used’.
Quantity of fuel used = Distance traveled/Mileage
Distance traveled = Quantity of fuel used*Mileage
Since the same distance needs to be traveled, ‘quantity of fuel used’ varies inversely with the ‘mileage’.
We see that ‘fuel price’ varies inversely with ‘quantity of fuel used’ and ‘quantity of fuel used’ varies inversely with ‘mileage’. So, if fuel price increases, quantity of fuel used decreases proportionally and if quantity of fuel used decreases, mileage increases proportionally. Hence, if fuel price increases, mileage increases proportionally or we can say that fuel price varies directly with mileage.
If fuel price becomes 6/5 (20% increase) of previous fuel price, we need the mileage to become 6/5 of the previous mileage too i.e. mileage should increase by 20% too.
Another method is that you can directly plug in the expression for ‘Quantity of fuel used’ in the original equation.
Total fuel cost = Fuel price * Distance traveled/Mileage
Since ‘total fuel cost’ and ‘distance traveled’ need to stay the same, ‘fuel price’ is directly proportional to ‘mileage’.
Answer (C)
We hope you are comfortable with fundamentals of direct and inverse variation now. More next week!

58.Varying Jointly

Now that we have discussed direct and inverse variation, joint variation will be quite intuitive. We use joint variation when a variable varies with (is proportional to) two or more variables.
Say, x varies directly with y and inversely with z. If y doubles and z becomes half, what happens to x?
“x varies directly with y” implies x/y = k (keeping z constant)
If y doubles, x doubles too.
“x varies inversely with z” implies xz = k (now keeping y constant)
If z becomes half, x doubles.
So the overall effect is that x becomes four times of its initial value.
The joint variation expression in this case will be xz/y = k. Notice that when z is constant, x/y = k and when y is constant, xz = k; hence both conditions are being met. Once you get the expression, it’s very simple to solve for any given conditions.
x1*z1/y1 = x2*z2/y2 = k (In any two instances, xz/y must remain the same)
x1*z1/y1 = x2*(1/2)z1/2*y1
x2 = 4*x1
Let’s look at some more examples. How will you write the joint variation expression in the following cases?
1. x varies directly with y and directly with z.
2. x varies directly with y and y varies inversely with z.
3. x varies inversely with y^2 and inversely with z^3.
4. x varies directly with y^2 and y varies directly with z.
5. x varies directly with y^2, y varies inversely with z and z varies directly with p^3.
Solution: Note that the expression has to satisfy all the conditions.
1. x varies directly with y and directly with z.
x/y = k
x/z = k
Joint variation: x/yz = k
2. x varies directly with y and y varies inversely with z.
x/y = k

yz = k
Joint variation: x/yz = k
3. x varies inversely with y^2 and inversely with z^3.
x*y^2 = k
x*z^3 = k
Joint variation: x*y^2*z^3 = k
4. x varies directly with y^2 and y varies directly with z.
x/y^2 = k
y/z = k which implies that y^2/z^2 = k
Joint variation: x*z^2/y^2 = k
5. x varies directly with y^2, y varies inversely with z and z varies directly with p^3.
x/y^2 = k
yz = k which implies y^2*z^2 = k
z/p^3 = k which implies z^2/p^6 = k
Joint variation: (x*p^6)/(y^2*z^2) = k
Let's take a GMAT prep question now to see these concepts in action:
Question 1: The rate of a certain chemical reaction is directly proportional to the square of the concentration of chemical M present and inversely proportional to the concentration of chemical N present. If the concentration of chemical N is increased by 100 percent, which of the following is closest to the percent change in the concentration of chemical M required to keep the reaction rate unchanged?
(A) 100% decrease
(B) 50% decrease
(C) 40% decrease
(D) 40% increase
(E) 50% increase
Solution:
Rate/M^2 = k
Rate*N = k
Rate*N/M^2 = k
If Rate has to remain constant, N/M^2 must remain the same too.
If N is doubled, M^2 must be doubled too i.e. M must become ?2 times. Since ?2 = 1.4 (approximately),
M must increase by 40%.
Answer (D)
Simple enough?

59. Work Rate using Joint Variation

This week, let's look at some work-rate questions which use joint variation. Check out the last three posts of QWQW series if you are not comfortable with joint variation.
Question 1: A contractor undertakes to do a job within 100 days and hires 10 people to do it. After 20 days, he realizes that one fourth of the work is done so he fires 2 people. In how many more days will the work get over?
(A) 60
(B) 70
(C) 75

(D) 80

(E) 100

Solution: Can we say that 10 people can finish the work in 100 days? No. If that were the case, after 20 days, only 1/5th of the work would have been over. But actually 1/4th of the work is over. This means that '10 people can complete the work in 100 days' was just the contractor's estimate (which turned out to be incorrect). Actually 10 people can do 1/4th of the work in 20 days. The contractor fires 2 people. So the question is how many days are needed to complete 3/4th of the work if 8 people are working?

We need to find the number of days. How is 'no. of days' related to 'no. of people' and 'work done'?

If we have more 'no. of days' available, we need fewer people. So 'no. of days' varies inversely with 'no. of people'.

If we have more 'no. of days' available, 'work done' will be more too. So 'no. of days' varies directly with 'work done'.

Therefore,

'no. of days' * 'no. of people'/'work done' = constant

20*10/(1/4) = 'no. of days'*8/(3/4)

No. of days = 75

So, the work will get done in 75 days if 8 people are working.

We can also do this question using simpler logic. The concept used is joint variation only. Just the thought process is simpler.

10 people can do 1/4th of the work in 20 days.

8 people can do 3/4th of the work in x days.

Start with the no. of days since you want to find the no of days:

x = 20*(10/8)*(3/1) = 75

From where do we get 10/8? No. of people decreases from 10 to 8. If no. of people is lower, the no of days taken to do the work will be more. So 20 (the initial no. of days) is multiplied by 10/8, a number greater than 1, to increase the number of days.

From where do we get (3/1)? Amount of work increases from 1/4 to 3/4. If more work has to be done, no. of days required will be more. So we further multiply by (3/4)/(1/4) i.e. 3/1, a number greater than 1 to further increase the number of days.

This gives us the expression 20*(10/8)*(3/1)

We get that the work will be complete in another 75 days.

Answer (C)

Let's take another question to ensure we understand the logic.

Question 2: A company's four cars running 10 hrs a day consume 1200 lts of fuel in 10 days. In the next 6 days, the company will need to run 9 cars for 12 hrs each so it rents 5 more cars which consume 20% less fuel than the company's four cars. How many lts of fuel will be consumed in the next 6 days?

(A) 1200 lt

(B) 1555 lt

(C) 1664 lt

(D) 1728 lt

(E) 4800 lt

Solution: First let's try to figure out what is meant by 'consume 20% less fuel than the company's cars'. It means that if company's each car consumes 1 lt per hour, the hired cars consume only 4/5 lt per hour. So renting 5 more cars is equivalent to renting 4 cars which are same as the company's cars. Hence, the total number of cars that will be run for the next 6 days is 8 company-equivalent cars.

4 cars running 10 hrs for 10 days consume 1200 lt of fuel

8 cars running 12 hrs for 6 days consume x lt of fuel

x = 1200*(8/4)*(12/10)*(6/10) = 1728 lt

We multiply by 8/4 because more cars implies more fuel so we multiply by a number greater than 1.

We multiply by 12/10 because more hours implies more fuel so we multiply by a number greater than 1.
We multiply by 6/10 because fewer days implies less fuel so we multiply by a number smaller than 1.
Answer (D)

60.The efficiency of using Variations

Today, we would like to discuss one of our own work questions. The intent is to show you how simple your calculations can get when you use the methods we discussed in the last few weeks. I couldn't say it enough – develop a love for ratios. You will save a huge amount of time in lots of questions. If you haven't been following the last few weeks' posts, take a look at this link before checking out the question. Otherwise the method may not make sense to you.

Question: 16 horses can haul a load of lumber in 24 minutes. 12 horses started hauling a load and after 14 minutes, 12 mules joined the horses. Will it take less than a quarter-hour for all of them together to finish hauling the load?

(1) Mules work more slowly than horses.

(2) 48 mules can haul the same load of lumber in 16 minutes.

Solution: First do this question on your own and see the calculations involved. Thereafter, check out the solution given below to know how we can solve the question using our joint variation method.

We are given that 16 horses can complete the work in 24 mins. Let's find out how much work is done by 12 horses in 14 mins (before the mules join in)

16 horses ……… 24 mins ………. 1 work

12 horses ……… 14 mins ………. ?? work

Work done = 1*(14/24)*(12/16) = 7/16 work (if you don't know how we arrived at this, seriously, check out last week's post first)

So in 14 mins, the 12 horses can complete 7/16 of the work i.e. they do 1/16 of the work every 2 mins.

How much work is leftover for the mules and horses to do together? 1 – 7/16 = 9/16

Leftover work = 9/16

This makes us think that 12 horses alone will take 9*2 = 18 mins to finish the work. When 12 mules join in, depending on the rate of work of mules, time taken to complete this work could be less than or more than 15 mins.

Statement 1: Mules work more slowly than horses.

This statement doesn't give us enough information. It just tells us that mules work slower than horses. Say if they work very slowly so that, effectively, they are not adding much to the work done, the work will get done in approximately 18 mins. If they work faster, time taken will keep decreasing. If they work as fast as the horses, the rate at which the work will be done will double (because we already have 12 horses and we will add 12 mules which will be equivalent to 12 horses) and time taken will become half i.e. it will be 9 mins. So the time taken will vary in the range 9 mins to 18 mins depending on the rate of work of mules. This statement alone is not sufficient.

Statement 2: 48 mules can haul the same load of lumber in 16 minutes.

We now know the rate of work of mules. The point is that now we can easily calculate the exact time taken by 12 horses and 12 mules to complete 9/16 of the work. Once we calculate the exact time, we will be able to say whether the time taken will be less than or more than 15 mins. Hence this statement alone is sufficient to answer the question. We don't really need to find out exactly how much is taken by the 24 animals together since it is a DS question. Ideally, we should mark the answer as (B) and move on.

Nevertheless, let's do the calculations if only to practice application of work concepts.

Let's try to find the equivalence of mules and horses (the way we did with cars in the previous post)

We know that 16 horses can haul a load of lumber in 24 minutes. Let's find out the number of mules that are needed to complete the work in 24 mins.

48 mules …….16 mins.
?? mules ……..24 mins
No. of mules required = 48*16/24 = 32 mules
So, 32 mules do the same work in the same time as done by 16 horses. Or we can say that 2 mules are equivalent to 1 horse. Hence, 12 mules are equivalent to 6 horses. When 12 mules join the 12 horses, equivalently we get 12+6 = 18 horses.
16 horses ……… 24 mins ………. 1 work
18 horses ……….. ?? mins ……….. 9/16 work
Time taken by 18 horses (i.e. 12 horses and 12 mules) = 24*(9/16)*(16/18) = 12 mins
Yes, the horses and mules together will take less than a quarter-hour to finish hauling the load.
Answer (B)

61. Evading Calculations

We have discussed before how GMAT is not a calculation intensive exam. Whenever you land on an equation which looks something like this: 60/(n – 5) – 60/n = 2, you probably think that we don't know what we are talking about! You obviously need to cross multiply, make a quadratic and finally, solve the quadratic to get the value of n. Actually, you usually don't need to do any of that for GMAT questions. You have an important leverage – the options. Even if the options don't directly give you the values of n or n-5, you can use the knowledge that every GMAT question is do-able in 2 mins and that the numbers fit in beautifully well.
Let' see whether we can get a value of n which satisfies this equation without going the whole nine yards. We will not use any options and will try to rely on our knowledge that GMAT questions don't take much time.
60/(n – 5) – 60/n = 2
So, the difference between the two terms of the left hand side is 2. Try to look for values of n which give us simple numbers i.e. try to plug in values which are factors of the numerator.
Say, if n = 10, you get 60/5 – 60/10 = 12 – 6 = 6. The difference between them is much more than 2. 60/n and 60/(n – 5) need to be much closer to each other so that the difference between them is 2. The two terms should be smaller to bring them closer together. So increase the value of n.
Put n = 15 since it is the next number such that (15 – 5 =) 10 as well as 15 divide 60 completely. You get 60/10 – 60/15 = 6 – 4 = 2. It satisfies and you know that a value that n can take is 15. Usually, you will get a solution within 2-3 iterations. This is enough for a PS question. Notice that this equation gives us a quadratic so be careful while working on DS questions. You might need to manipulate the equation a little to figure out whether the other root is a possible solution as well. Anyway, today we will focus on the application of such equations in PS questions only. Let's take a question now to understand the concept properly:
Question: Machine A takes 2 more hours than machine B to make 20 widgets. If working together, the machines can make 25 widgets in 3 hours, how long will it take machine A to make 40 widgets?
(A) 5
(B) 6
(C) 8
(D) 10
(E) 12
Solution: We need to find the time taken by machine A to make 40 widgets. It will be best to take the time taken by machine A to make 40 widgets as the variable x. Then, when we get the value of x, we will not need to perform any other calculations on it and hence the scope of making an error will reduce. Also, value of x will be one of the options and hence plugging in to check will be easy.

Machine A takes x hrs to make 40 widgets.
Rate of work done by machine A = Work done/Time taken = 40/x
Machine B take 2 hrs less than machine A to make 20 widgets hence it will take 4 hrs less than machine B to make 40 widgets. Think of it this way: Break down the 40 widgets job into two 20 widget jobs. For each job, machine B will take 2 hrs less than machine A so it will take 4 hrs less than machine A for both the jobs together.
Time taken by machine B to make 40 widgets = x – 4
Rate of work done by machine B = Work done/Time taken = 40/(x – 4).
We know the combined rate of the machines is 25/3
So here is the equation:
40/x + 40/(x – 4) = 25/3
The steps till here are not complicated. Getting the value of x poses a bit of a problem.
Notice here that that the right hand side is not an integer. This will make the question a little harder for us, right? Wrong! Everything has its pros and cons. The 3 of the denominator gives us ideas for the values of x (as do the options). To get a 3 in the denominator, we need a 3 in the denominator on the left hand side too.
x cannot be 3 but it can be 6. If x = 6, 40/(6 – 4) = 20 i.e. the sum will certainly not be 20 or more since we have 25/3 = 8.33 on the right hand side.
The only other option that makes sense is x = 12 since it has 3 in it.
40/12 + 40/(12 – 4) = 10/3 + 5 = 25/3
Answer (E)
If we did not have the options, we might have tried x = 9 too before landing on x = 12. Nevertheless, these calculations are not time consuming at all since you can get rid of the incorrect numbers orally. Making a quadratic and solving it is certainly much more time consuming.
Another method could be to bring 3 to the left hand side to get the following equation:
120/x + 120/(x – 4) = 25
This step doesn't change anything but it helps if you face a mental block while working with fractions. Try to practice such questions using these techniques – they will save you a lot of time.

62. Evading Calculations- II

Last week we discussed how to solve equations with the variable in the denominator. We also said that the technique generally works for PS questions but you need to be careful while working on DS questions. Today, let's look at the reason behind the caveat.
Say, the question stem of a DS question asks you to find the value of n, the number of people in the room. Statement 1 of the question gives you the following equation:
60/(n – 5) – 60/n = 2
We can easily figure out that a value of n that satisfies this equation is 15. Now, is that enough to say that statement 1 is sufficient alone? No! It could be a trap! The equation, when manipulated, gives us a quadratic. It is important to find out whether the second solution of the quadratic works for us. When n is the number of people, it must be positive. So one extra step that we should take is re-arrange the equation to get the quadratic. If the constant term i.e. the product of the roots is negative, it means one root is positive and one is negative. Since we have already found the positive root, it is the only answer and hence we can say that the statement 1 is sufficient alone.
60/(n – 5) – 60/n = 2
60*n – 60*(n – 5) = 2*n*(n – 5)
n^2 – 5n – 150 = 0

The constant term, -150, is negative so the product of the roots must be negative. This means one root must be negative and the other must be positive. Since we have already found the positive root i.e. the number of people in the room, we can say that statement 1 is sufficient alone.

Let's look at an example where we could fall in the trap.

Say statement 1 gives us an equation which looks like this:

60/(n +5) – 10/(n – 5) = 2

As discussed last week, we will easily see that n = 10 satisfies this equation. So should we move on now and say that statement 1 is sufficient alone? No, not so fast! Let's try to manipulate the equation to get the quadratic.

60/(n +5) – 10/(n – 5) = 2

60*(n – 5) – 10*(n + 5) = 2*(n – 5)(n + 5)

n^2 – 25n + 150 = 0

n = 10 or 15

So actually, there are two values of n that satisfy this equation. In PS questions, since we have a single answer, there would be only one solution so once you get one, you are done. In DS questions, you need to be certain that only one value satisfies. There is a possibility that both values satisfy your constraints in which case your answer would change.

Therefore, it may not be necessary to solve the equation for the PS question, but it is certainly necessary to solve it for DS. That's counter intuitive, isn't it? We hope you understand the reason.

Another related trap in DS questions: Statement 1 gives you a quadratic and asks you for the value of x (no constraints that x must be an integer or positive number etc). You know that it is a quadratic and it will give you two values of x so you say that statement 1 is not sufficient alone and move on. But hold it! What if both the roots of the equation are same? It may not apparent to you when you look at the equation. When you solve it, you realize that the roots are the same. Hence, ensure that you solve the equation in DS questions before you decide on the sufficiency.

63. And Now, Evading Formula's

Today, we again pay homage to the lazy bum within each one of us in our QWQW series. If you are wondering what we mean by 'again', check out our last two posts of the QWQW series. We have been discussing how to avoid calculations. Today let's learn why it is advisable to avoid learning formulas too!

You really don't need to know many formulas for GMAT – just the basic ones e.g. Distance = Speed*Time, Work = Rate*Time (which are actually the same if you look at them closely) etc. If a Time-Distance-Speed question pertains to GMAT, rest assured it can be solved using just the formula given above and that too, within 1-2 mins. Then, do you need to learn the many formulas that people claim speed up question solving? No! In fact, the more specific the formula, the more constraints it has. It can be used in only particular circumstances and hence when the situation differs even a little bit from the ideal, you could end up using the formula incorrectly. Therefore, we recommend our students to stay away from the umpteen, less generic formulas until and unless they have already used them extensively. Let's discuss this point with an example:

Question: A and B start from Opladen and Cologne respectively at the same time and travel towards each other at constant speeds along the same route. After meeting at a point between Opladen and Cologne, A and B proceed to their destinations of Cologne and Opladen respectively. A reaches Cologne 40 minutes after the two meet and B reaches Opladen 90 minutes after their meeting. How long did A take to cover the distance between Opladen and Cologne?

(A) 1 hour

(B) 1 hour 10 minutes

(C) 2 hours 30 minutes

(D) 1 hour 40 minutes

(E) 2 hours 10 minutes

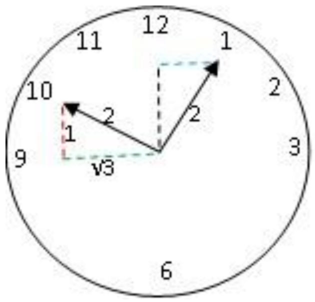
Solution: People often like to use a formula for this situation. Let's quickly discuss that first.
If two objects A and B start simultaneously from opposite points and, after meeting, reach their destinations in 'a' and 'b' hours respectively (i.e. A takes 'a hrs' to travel from the meeting point to his destination and B takes 'b hrs' to travel from the meeting point to his destination), then the ratio of their speeds is given by:
Sa/Sb = sqrt(b/a)
i.e. Ratio of speeds is given by the square root of the inverse ratio of time taken.
Sa/Sb = sqrt(90/40) = 3/2
This gives us that the ratio of the speed of A : speed of B as 3:2. We know that time taken is inversely proportional to speed. If ratio of speed of A and B is 3:2, the time taken to travel the same distance will be in the ratio 2:3. Therefore, since B takes 90 mins to travel from the meeting point to Opladen, A must have taken 60 (= 90*2/3) mins to travel from Opladen to the meeting point
So time taken by A to travel from Opladen to Cologne must be 60 + 40 mins = 1 hr 40 mins
Now let's see how we can solve the question without using the formula.
Think of the point in time when they meet:

A starts from Opladen and B from Cologne simultaneously. After some time, say t mins of travel, they meet. Since A covers the entire distance of Opladen to Cologne in (t + 40) mins and B covers it in (t + 90) mins, A is certainly faster than B and hence the Meeting point is closer to Cologne.
Now think, what information do we have? We know the time taken by A and B to reach their respective destinations from the meeting point. We also know that they both traveled the same distance i.e. the distance between Opladen and Cologne. So let's try to link distance with time taken. We know that 'Distance' varies directly with 'Time taken'. (Check out this post if you don't know what we are talking about here.)
Distance between Opladen and Meeting point /Distance between Meeting point and Cologne = Time taken to go from Opladen to Meeting point/Time taken to go from Meeting point to Cologne = t/40 (in case of A) = 90/t (in case of B)
t = 60 mins
So A takes 60 mins + 40 mins = 1 hr 40 mins to cover the entire distance.
Answer (D)
We could easily solve the question without using any specific formula. So stick to your basics and kick those little grey cells to get to the answer!

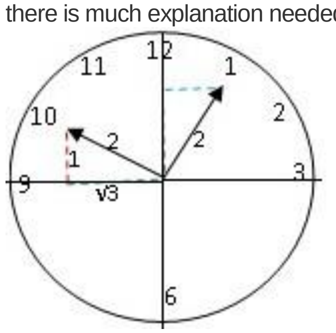
64. Graphs of Geometry- Part-1

Let's start with geometry today. It has some very interesting and intuitive concepts. We will discuss one of them today. It's surprising how a little bit of imagination can go a long way in helping you solve questions. Let's discuss the concept first. We will look at a question later.
Imagine a clock face. Think of the minute hand on 10. Ignore the hour hand for our discussion today. Say, the length of the minute hand is 2 cm. Its distance from the vertical and horizontal axis is shown in the diagram below (using the green and the red dotted lines). Let's say the minute hand moves to 1. Can you say something about the lengths of the dotted black and dotted blue lines?

Quarter Wit_Quarter Wisdom- Part-2



Isn't it apparent that when the minute hand moved by 90 degrees, the dotted green line became the dotted black line and the dotted red line became the dotted blue line. So can we say that the dotted black line is ?3 cm in length and the dotted blue line is 1 cm in length? The same thing will happen when the minute hand goes to 4 and to 7. We don't think there is much explanation needed here, right? The diagram makes it all clear.

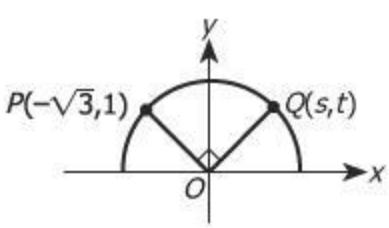


Let's look at the clock from coordinate geometry perspective. Let's say the center of the clock is the origin (0, 0). What are the x and y coordinates of the "10'o clock point" i.e. the tip of the minute hand before it moves to 1? Notice that the x coordinate will be -?3 (since the point is in the second quadrant, x coordinate will be negative) and y coordinate will be 1. What are the x and y coordinates of the "1'o clock point" i.e. the tip of the minute hand after it moves to 1. Notice that the absolute values of x coordinate and y coordinate have switched because the hand has turned 90 degrees. The x coordinate is 1 now and the y coordinate is ?3. Since it is the first quadrant, both the coordinates will be positive.

Now, think, what will be the coordinates of the "4'o clock point", "7'o clock point"? What about the "11'o clock point", "12'o clock point" etc?

Using this concept, we can solve a very tough GMAT Prep question in a few seconds.

Question 1:



In the figure above, points P and Q lie on the circle with center O. What is the value of s?

(A) 1/2

(B) 1

(C) ?2

(D) ?3

(E) 1/?2

Solution: You might be tempted to think on the lines of 'slope of a line' or '30-60-90' triangle (because of the presence of ?3) etc. But we should be able to arrive at the answer without using any of those.

Point P is (-?3, 1). O is the center of the circle at (0, 0). When OP is turned 90 degrees to give OQ, the x and y co-ordinates get interchanged. Also both x and y co-ordinates will be positive in the first quadrant. Hence s, the x co-ordinate of Q will be 1 (and y co-ordinate of Q will be ?3).

Answer (B)

The question doesn't seem difficult now (after understanding the concept); actually, it is a 700+ level.

Try another question using the same concept:

Question 2: On the coordinate plane (6, 2) and (0, 6) are the endpoints of the diagonal of a square. What are the coordinates of the point on the corner of the square which is closest to the origin?

(A) (0, 1)

(B) (1, 0)

(C) (1, 1)

(D) (2, 0)

(E) (2, 2)

65.Graphs of Geometry- Part-2

Let's pick up from where we left last week. We had discussed a coordinate geometry concept using clock faces and had left you with a tough question. Today we will see how you can solve that question using the concepts discussed last week.

You might wonder whether we can expect such a question in actual GMAT. The question we discussed in the last post was an official question and we could solve it easily using this concept. Of course there are many other ways of solving it but this is simplest (or trickiest depending on how you look at it), and it certainly is the fastest, no two ways about it! It is a very logical big-picture approach and people who get Q50-51 often use such methods. The question we will discuss today can also be solved in other ways but we will use the last week's 'turning minute hand 90 degrees' approach.

Question: On the coordinate plane (6, 2) and (0, 6) are the endpoints of the diagonal of a square. What are the coordinates of the point on the corner of the square which is closest to the origin?

(A) (0, 1)

(B) (1, 0)

(C) (1, 1)

(D) (2, 0)

(E) (2, 2)

Solution:

First rule of coordinate geometry – draw what you can.

So we make the xy axis and plot the given points, (6, 2) and (0, 6) on it. Let's say the square is denoted by points ABCD.

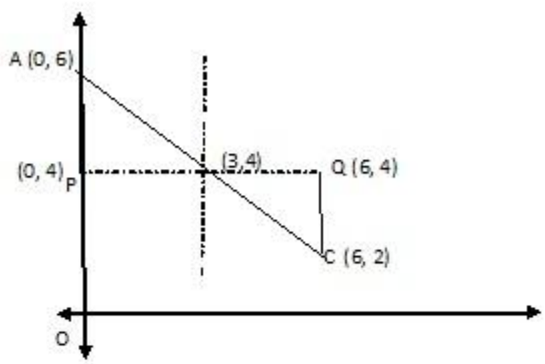
Say, A is (6, 2) and C is (0, 6). We see that AC is a sloping line. Its two end points are two vertices of the square. We need to find the other two vertices of the square. One of them will lie closest to the origin. The other two vertices will be the end points of the diagonal BD. BD will be perpendicular to AC at the mid-point of AC since a square's diagonals bisect each other and are perpendicular. So the question is, how do we obtain the end points, B and D? Let's try to figure out what information we need to draw BD. BD must pass through the mid-point of AC.

How will we obtain the mid-point of AC? By averaging x and y co-ordinates of the points A and C:

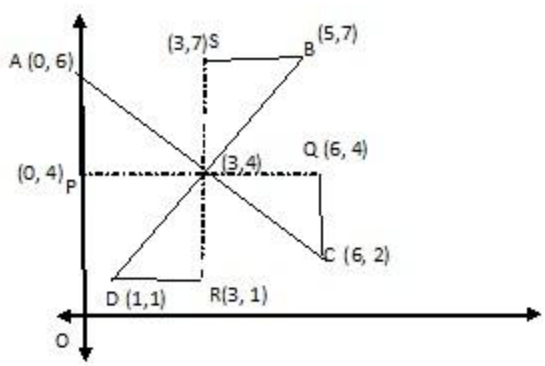
x coordinate of mid-point is (0 + 6)/2 = 3

y coordinate of mid-point is (6 + 2)/2 = 4

So BD must pass through (3, 4). When AC turns by 90 degrees, with point (3, 4) as the axis, we get the diagonal BD. So how do the coordinates of AC change when it turns by 90 degrees? Go back to last week's post and look at the clock face again. Think of a horizontal line PQ passing through (3, 4).



P coordinate will be given by (0, 4) and Q coordinate will be given by (6, 4) since length of P to mid point is 3 and length of mid-point to Q will also be 3. P shifts up by 2 units to give the point A and Q shifts down by 2 units to give the point C.
Now rotate PQ by 90 degrees and you get RS. We know the coordinates of a line perpendicular to PQ. R will be (3, 1) and S will be (3, 7). This is because R and S will have the same x coordinates as the mid-point (3, 4) and S is 3 units above the mid-point and R is 3 units below the mid-point. We are assuming that you can intuitively see these values on the graph. If not, it may be too soon to spend time on this post.



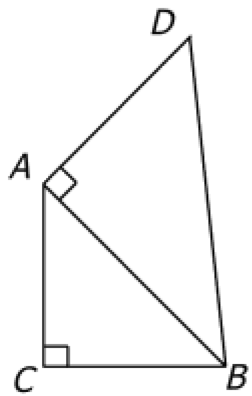
Now, can we obtain the diagonal BD using RS as reference? If you move S two units to the right, you will get point B (just like A was obtained by moving P two units up) and if you move R two units to the left, you will get point D. Notice that we are using PQ and RS as reference lines. It is easy to calculate vertical/horizontal distances. So B will be given by (5, 7) and D will be given by (1, 1).
The closest co-ordinate to (0, 0) is (1, 1).
Answer (C).
Take a few minutes to review the logic discussed here. The ability to 'see' such symmetry makes GMAT Quant very simple.

66. Diagrams of Geometry- Part-1

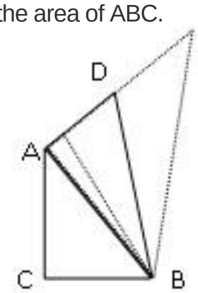
Let's continue with geometry today. We would like to discuss how drawing extreme diagrams can help you solve questions. Most GMAT questions are quite intuitive and hence our non-traditional methods are perfect for them. They

are not typical MATH problems per se; instead, they are logical puzzles. If you can prove why some things will not work, it means whatever is left will work.
Let me explain with the help of an official Data Sufficiency question.

Question:



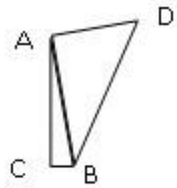
In the figure above, is the area of the triangle ABC equal to the area of the triangle ADB?
Statement 1: (AC)^2=2(AD)^2
Statement 2: ?ABC is isosceles.
Solution:
When presented with this question, people see right triangles and jump to Pythagorean theorem, isosceles triangles and then wage a war on AC, AB, CB and AD relations. Well, that is our traditional approach. But what do we do if making equations and solving for relations isn't our style?
We make diagrams and figure out the relations! One thing that is apparent the moment we read statement 1 is that the figure is not to scale. From the figure it looks as if AD is greater than or at best, equal to AC. That itself is an indication that if you draw the figure on your own, you could see something that will make this question very simple. The question setter doesn't want to show you that and hence he made the distorted figure.
Anyway, let's first analyze the question. Then we will look at the statements.
We need to compare areas of ABC and ADB. Both are right angled triangles.
Area of ABC = (1/2) * AC * BC
Area of ADB = (1/2) * AD * AB
We need to figure out whether these two are the same.
Think about it this way – we are given a triangle ABC with a particular area. So the length of AD must be defined. If AD is very small, (shown by the dotted lines in the diagram given below) the area of ADB will be very close to 0. If AD is very large, the area will be much larger than the area of ABC. So for only one value of AD, the area of DAB will be equal to the area of ABC.

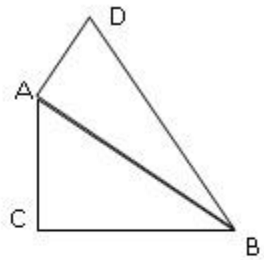


We need to figure out whether for the given relations, the triangles have equal area.

Statement 1: (AC)^2=2(AD)^2

This gives us AD = AC/?2. Let's draw AC and AD such that AD is somewhat shorter than AC. Now can we say that the areas of the two triangles are the same? No. The area of ABC is decided by AC and BC both not just AC. We can vary the length of BC to see that the relation between AC and AD is not enough to say whether the areas will be the same (see the diagrams given below).





So this statement alone is not sufficient.

(2) ?ABC is isosceles.

This means that AB = BC. Notice that the triangle is right angled so the hypotenuse must be the largest side. If ABC is isosceles, it means that the two legs of the triangle must be equal. Hence sides of ABC must be in the ratio 1:1:?2 = AC:BC:AB. Since we only need to consider relative length of the sides, let's say that AC = 1, BC = 1 and AB = ?2 or some multiple thereof.

We have no idea about the length of AD so this statement alone is also not sufficient.

Let's consider both statements together now:

AD = AC/?2 = 1/?2 (Since AC = 1)

Area of ABC = (1/2) * AC * BC = (1/2) * 1 * 1 = 1/2

Area of ADB = (1/2) * AD * AB = (1/2) * (1/?2) * ?2 = 1/2

Both triangles have the same area. Sufficient!

Answer (C)

Now compare this approach with your Pythagorean approach. Is this simpler?

67.<u>Diagrams of Geometry- part-2</u>

Last week, we discussed how drawing extreme diagrams can help solve Geometry questions. Today we will see how to solve another Geometry question by making diagrams. The diagram can help you understand exactly what it is that you need to do; doing it will be quite straightforward.

Question: If 10, 12 and 'x' are sides of an acute angled triangle, how many integer values of 'x' are possible?

(A) 7

(B) 12

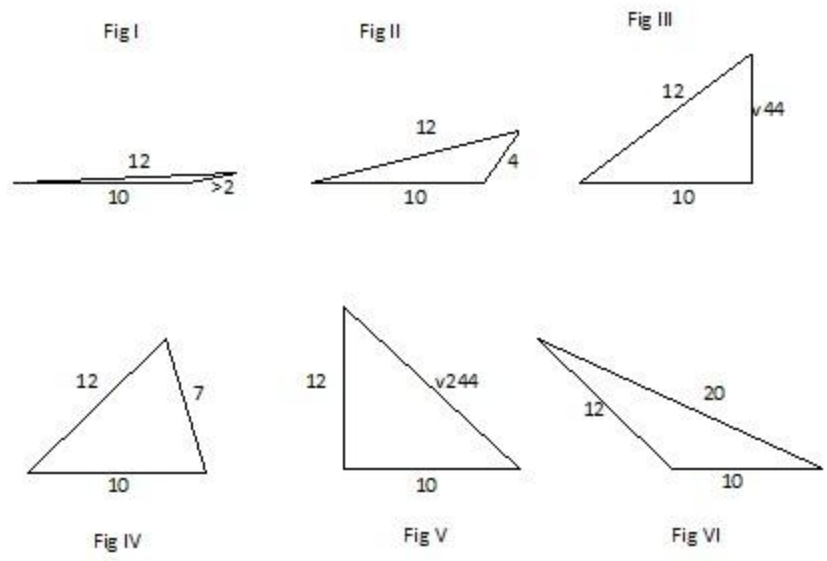
(C) 9

(D) 13

(E) 11

Solution: The question is very interesting. It asks you for an acute triangle i.e. a triangle with all angles less than 90 degrees. It's a little hard to wrap your head around it, isn't it? We know that the third side of a triangle can take many values. Right from a little more than the difference of the other two sides to a little less than the sum of the other two sides (Since we know that the sum of any two sides of a triangle is always greater than the third side). So x can be anything from a little more than 2 to a little less than 22. But how do we find out the values for which all the angles will be less than 90?

We want no obtuse or right angles. An obtuse angled triangle has one angle more than 90. So the thought here is that before one of the angles reaches 90, find out all the values that x can take.



Look at the figure given above. The value of x in the first figure is very small – slightly more than 2 – minimum required to make a triangle. There is an obtuse angle in that triangle. We keep making x bigger and bigger and the angle keeps becoming smaller till it reaches 90 (Fig III). We use Pythagorean theorem to get the value of x in that case:

x = ?(12^2 – 10^2)

x = ?44 which is 6.something

x should be greater than 6.something because the angle cannot be 90.

We further keep increasing x and all the angles are acute now. We reach Fig V where we hit another right triangle. We use Pythagorean theorem again to get the value of x (the hypotenuse) in this case:

x = ?(12^2 + 10^2)

x = ?244 which is 15.something

x should be less than 15.something so that the angle is not 90.

Further on, in Fig VI, we obtain an obtuse angle again.

We only need integral values of x so values that x can take range from 7 to 15 which is 9 values.

Answer (C).

Note: We made two angles 90 and found the values of x in between those two angles. The third angle cannot be 90 because that will make 10 the hypotenuse but hypotenuse is always the greatest side.

68. <u>Plug using Transitions points</u>

Let's take a break from Geometry today and discuss the concept of transition points. This is especially useful in questions where you are tempted to plug in values. A question often asked is: how do I know which values to plug and how do I know that I have covered the entire range in the 3-4 values I have tried? What transition points do is that they give you the ranges in which the relationships differ. All you have to do is try one value from each range. If you do, you would have figured out all the different relationships that can hold. We will discuss this concept using a GMAT Prep question. You can solve it using our discussion on inequalities too. But if number plugging is what comes first to your mind in this question, then it will be a good idea to get the transition points.

Let's begin:

If x is positive, which of the following could be correct ordering of $1/x$, $2x$ and x^2?

(I) $x^2 < 2x < 1/x$

(II) $x^2 < 1/x < 2x$

(III) $2x < x^2 < 1/x$

(A) none

(B) I only

(C) III only

(D) I and II

(E) I, II and III

Solution:

Notice that the question says "could be correct ordering". This means that for different values of x, different orderings could hold. We need to find the one (or two or three) which will not hold in any case. So what do we do? We cannot try every value that x can take so how do we know for sure that one or more of these relations cannot hold? What if we try 4-5 values and only one relation holds for all of those values? Can we say for sure that the other two relations will not hold for any value of x? No, we cannot since we haven't tried all values of x. So there are two options you have in this case:

1. Use logic to figure out which relations can hold and which cannot. This you can do using inequalities (but we will not discuss that today).

2. You can figure out the ranges in which the relationships are different and then try one value from each range. This is our transition points concept which we will discuss today.

Let's discuss the second option in more detail.

First of all, we are just dealing with positives so there is less to worry about. That's good.

To picture the relationship between two functions, we first need to figure out the points where they are equal.

$x^2 = 2x$

$x^2 – 2x = 0$

$x = 0$ or 2

x cannot be 0 since x must be positive so this equation holds when $x = 2$

So $x =2$ is the transition point of their relation. x^2 is less than $2x$ when x is less than 2 and it will be greater than $2x$ when x is greater than 2.

Let's try to figure out the relation between $1/x$ and $2x$ now.

$1/x = 2x$

$x = 1/\sqrt{2}$

Since $1/x$ is less than $2x$ when x is greater than 1/?2, it will be more than $2x$ when x is less than $1/\sqrt{2}$.

Move on to the relation between $1/x$ and x^2.

$1/x = x^2$

$x^3 = 1$ (notice that since x must be positive, we can easily multiply/divide by x without any complications)

$x = 1$

So you have got three transition points: $1/\sqrt{2}$, 1 and 2.

Now all you need to do is try a number from each of these ranges:

(i) $x < 1/\sqrt{2}$

(ii) $1/\sqrt{2} < x < 1$

(iii) 1 < x < 2

(iv) x > 2

If a relation doesn't hold in any of these ranges, it will not hold for any value of x.

(i) For x < 1/sqrt(2), put x = a little more than 0 (e.g. 0.01)

1/x = 100, 2x = 0.02, x^2 = 0.0001

We get **x^2 < 2x < 1/x is possible. So (I) is possible**

(ii) For 1/sqrt(2) < x < 1, put x = a little less than 1 (e.g. 0.99)

1/x = slightly more than 1, 2x = slightly less than 2, x^2 = slightly less than 1

We get **x^2 < 1/x < 2x is possible. So (II) is also possible.**

(iii) For 1 < x < 2, put x = 3/2

1/x = 2/3, 2x = 3, x^2 = 9/4 = 2.25

We get **1/x < x^2 < 2x is possible.**

(iv) For x > 2, put x = 3

1/x = 1/3, 2x = 6, x^2 = 9

We get **1/x < 2x < x^2 is possible.**

We see that for no positive value of x is the third relation possible. We have covered all different ranges of values of x.

Answer (D)

Try using inequalities instead of number plugging to see if solving the question becomes easier in that case.

69. Or Just use Inequalities

If you are wondering about the absurd title of this post, just take a look at last week's title. It will make much more sense thereafter. This post is a continuation of last week's post where we discussed number plugging. Today, as per students' request, we will look at the inequalities approach to the same official question. You will need to go through our inequalities post to understand the method we will use here.

Recall that, given a < b, (x – a)(x – b) < 0 gives us the range a < x < b and (x – a)(x – b) > 0 gives us the range x < a or x > b.

Question: If x is positive, which of the following could be the correct ordering of 1/x, 2x and x^2?

(I) x^2 < 2x < 1/x

(II) x^2 < 1/x < 2x

(III) 2x < x^2 < 1/x

(A) none

(B) I only

(C) III only

(D) I and II

(E) I, II and III

Solution: The question has three complex inequalities. We will take each in turn. Note that each inequality consists of two more inequalities. We will split the complex inequality into two simpler inequalities e.g. x^2 < 2x < 1/x gives us x^2 < 2x and 2x < 1/x. Next we will find the range of values of x which satisfy each of these two inequalities and we will see if the two ranges have an overlap i.e. whether there are any values of x which satisfy both these simpler inequalities. If there are, it means there are values of x which satisfy the entire complex inequality too. Things will become clearer once we start working on it so hold on.

Let's look at each inequality in turn. We start with the first one:

(I) x^2 < 2x < 1/x

We split it into two inequalities:

(i) x^2 < 2x

We can rewrite x^2 < 2x as x^2 – 2x < 0 or x(x – 2) < 0.

We know the range of x for such inequalities can be easily found using the curve on the number line. This will give us 0 < x < 2.

(ii) 2x < 1/x

It can be rewritten as x^2 – 1/2 < 0 (Note that since x must be positive, we can easily multiply both sides of the inequality with x)

This gives us the range -1/?2 < x < 1/?2 (which is 0 < x < 1/?2 since x must be positive).

Is there a region of overlap in these two ranges i.e. can both inequalities hold simultaneously for some values of x? Yes, they can hold for 0 < x < 1/?2. Hence, x^2 < 2x < 1/x will be true for the range 0 < x < 1/?2. So this could be the correct ordering.

Let's go on to the next complex inequality.

(II) x^2 < 1/x < 2x

Again, let's break up the inequality into two parts:

(i) x^2 < 1/x

x^1 < 1/x is rewritten as x^3 – 1 < 0 which gives us x < 1.

(ii) 1/x < 2x

1/x < 2x is rewritten as x^2 – 1/2 > 0 which gives us x < -1/?2 (not possible since x must be positive) or x > 1/?2

Can both x < 1 and x > 1/?2 hold simultaneously? Sure! For 1/?2 < x < 1, both inequalities will hold and hence x^2 < 1/x < 2x will be true. So this could be the correct ordering too.

(III) 2x < x^2 < 1/x

The inequalities here are:

(i) 2x < x^2

2x < x^2 can be rewritten as x(x – 2) > 0 which gives us x < 0 (not possible) or x > 2.

(ii) x^2 < 1/x

x^2 < 1/x gives us x^3 – 1 < 0 i.e. x < 1

Can x be less than 1 and greater than 2 simultaneously? No. Therefore, 2x < x^2 < 1/x cannot be the correct ordering.

Answer (D)

Is this method simpler?

70. Geometry Diagrams on DS Questions

Let's go back to geometry now. We will discuss how to use diagrams to solve DS questions today. Though we discussed a DS question in a previous geometry post, we didn't discuss how the thought process used for a DS question is different from the thought process used for a PS question. To find whether a statement is sufficient to answer the question, you should try to prove that it is not sufficient. Try to make two cases which answer the question differently using the give information. If there are two or more different answers possible, it means the given information is not enough. Let's discuss this with the help of an official question.

Question: A circle and a line lie in the XY plane. The circle is centered at the origin and has a radius 1. Does the line intersect the circle?

Statement I: The x-Intercept of the line is greater than 2

Statement II: The slope of the line is -1/5

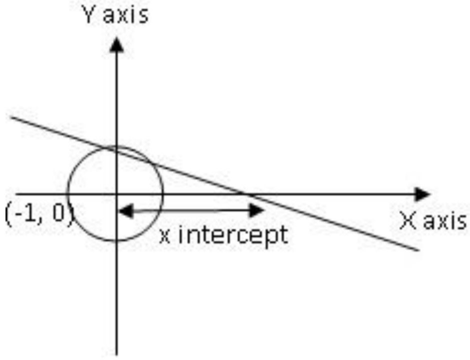
Solution:

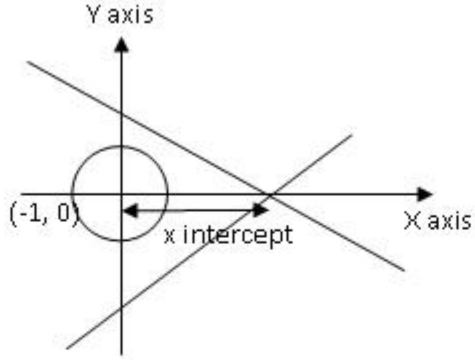
We are given that there is a circle and a line on the XY plane. The line can lie anywhere – it may or may not intersect the circle. The circle has radius 1 so it intersects the x axis at (1, 0) and (-1, 0).

Let's look at the information given in the two statements:

Statement I: The x-Intercept of the line is greater than 2.

If x intercept > 2, the line can be any of the following (and can be drawn in many more ways)

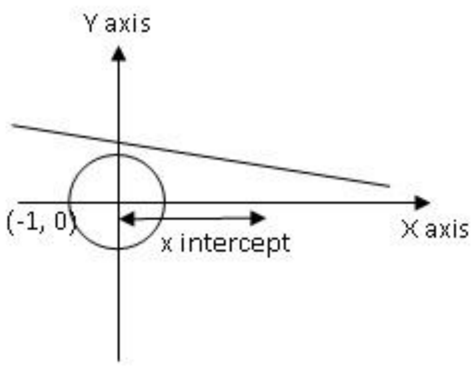


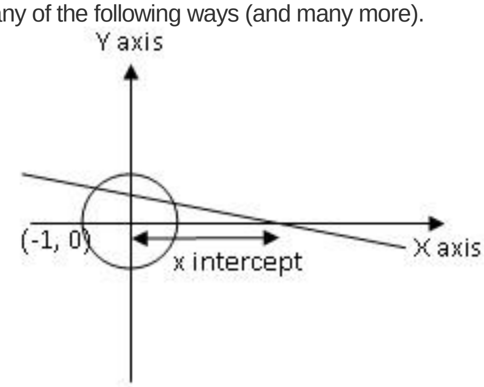


We found two cases – one in which the line intersects the circle and another in which it doesn't. The line could have different slopes and different x intercepts (as long as it is greater than 2) to get different cases. Hence we see that this information alone is not sufficient to answer the question.

Statement II: The slope of the line is -1/5.

If slope of the line is -1/5, the line can be drawn in any of the following ways (and many more).





Again, we found two cases – one in which the line intersects the circle and another in which it doesn't. The slope of the line stays the same but you can move it up or down to get the two different cases (and different x intercepts). Hence we see that this information alone is not sufficient to answer the question.

Using both together: Now the line has a defined slope = -1/5 but it has no defined x intercept. To get x intercept greater than 2, all we need is that y intercept must be greater than 2/5. If you are wondering how we arrived at this, recall from an earlier post:

Slope of a line = – (y intercept)/(x intercept) = – 1/5

y intercept = (1/5) * x intercept

Since x intercept must be greater than 2, y intercept will be greater than 2/5.

If the y intercept is much more than 2/5, it will not intersect the circle. If the y intercept is a little more than 2/5, the line will intersect the circle (as shown by the two diagrams in Fig 2). In one case, it will intersect the circle, in the other case, it will not. So both statements together are not sufficient.

Answer (E)

71. Regular Polygon & Irregular ones

Continuing our Geometry journey, let's discuss polygons today. Some years back, I used to often get confused in the polygon sum-of-the-interior-angles formula if I had to recall it after a gap of some months because I had seen two variations of it:

Sum of interior angles of a polygon = (n – 2)*180

Sum of interior angles of a polygon = (2n – 4)*90

Now, I don't want you to judge me. Of course, in the second formula, 2 has been removed from 180 and multiplied to the first factor. It is quite simple so why would anyone get tricked here, you wonder? The problem was that after a few months, I would somehow remember (2n – 4) and 180. So I was mixing up the two and I wasn't sure of the logic behind this formula. That is until I came across the simple explanation of this formula in our Veritas Prep Geometry book (the one which explains how you can divide every polygon with n sides into (n – 2) triangles and hence get the sum of (n – 2)*180). Now it made perfect sense! I couldn't believe that I had not come across that explanation before and had just learned up (well, tried to!) the formula blindly. So now I ensure that all my students understand every formula that I teach them.

Usually, we are given a regular polygon and we need to find the measure of interior angles or the number of sides. But what if we are given a polygon instead, not a regular polygon. Does this formula still apply? We wouldn't know if we didn't understand how the formula came into being. But since we know that we obtain the formula by dividing the polygon into (n-2) triangles, we know that the sum of all interior angles of a triangle is 180 irrespective of the kind of triangle. So it doesn't matter whether the polygon is regular or not. The sum of all interior angles will still be (n-2)*180.

Let's look at a question to see the application of this formula in *irregular* polygon scenario.

Question: The measures of the interior angles in a polygon are consecutive odd integers. The largest angle measures 153 degrees. How many sides does this polygon have?

A) 8

B) 9

C) 10

D) 11

E) 12

Solution:

The interior angles are: 153, 151, 149, 147 … and so on.

Now there are two ways to approach this question – one which is straight forward but uses algebra so is time consuming, another which makes you think but doesn't take much time. You can guess which one we are going to focus on! But before we do that let's take a quick look at the algebraic solution too.

Method 1: Algebra

Sum of interior angles of this polygon = 153 + 151 + 149 + … (153 – 2(n-1)) = (n – 2)*180

If there are n sides, there are n interior angles. The second largest angle will be 153 – 2*1. The third largest will be 153 – 2*2. The smallest will be 153 – 2*(n-1). This is an arithmetic progression.

Sum of all terms = [(First term + Last term)/2] * n = [(153 + 153 – 2(n-1))/2] * n

Equating, we get [(153 + 153 – 2(n-1))/2] * n = (n – 2)*180

Solving this you get, n = 10

But let's figure out a solution without going through this painful calculation.

Method 2: Capitalize on what you know

Angles of the polygon: 153, 151, 149, 147, 145, 143, 141, … , (153 – 2(n-1))

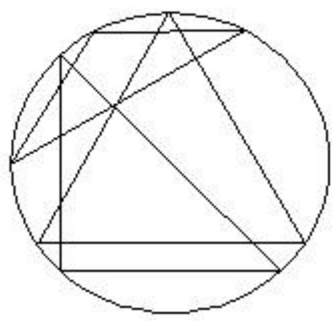
The average of these angles must be equal to the measure of each interior angle of a regular polygon with n sides since the sum of all angles is the same in both the cases.

Measure of each interior angle of n sided regular polygon = Sum of all angles / n = (n-2)*180/n

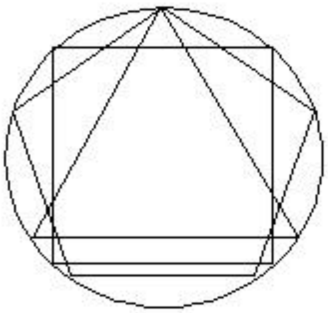
Using the options:
Measure of each interior angle of 8 sided regular polygon = 180*6/8 = 135 degrees
Measure of each interior angle of 9 sided regular polygon = 180*7/9 = 140 degrees
Measure of each interior angle of 10 sided regular polygon = 180*8/10 = 144 degrees
Measure of each interior angle of 11 sided regular polygon = 180*9/11 = 147 degrees apprx
and so on…
Notice that the average of the given angles can be 144 if there are 10 angles.
The average cannot be higher than 144 i.e. 147 since that will give us only 7 sides (153, 151, 149, 147, 145, 143, 141 – the average is 147 is this case). But the regular polygon with interior angle measure of 147 has 11 sides. Similarly, the average cannot be less than 144 i.e. 140 either because that will give us many more sides than the required 9.
Hence, the polygon must have 10 sides.
Answer (C).
Interesting, eh? Well, it will be when you understand method 2 well and can do it intuitively!

72.<u>Inscribing Polygons & Circles</u>

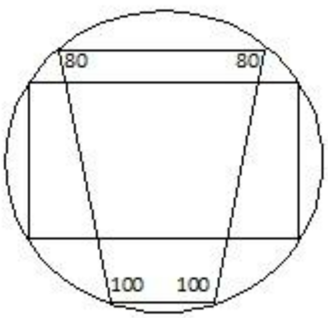
Last week we looked at regular and irregular polygons. Today, let's try to understand how questions involving one figure inscribed in another are done. The most common example of a figure inscribed in another is a polygon inscribed in a circle or a circle inscribed in a polygon. Let's see the various ways in which this can be done.
To inscribe a polygon in a circle, the polygon is placed inside the circle so that all the vertices of the polygon lie on the circumference of the circle.
There are a few points about inscribing a polygon in a circle that you need to keep in mind:
- Every triangle has a circumcircle so all triangles can be inscribed in a circle.



- All regular polygons can also be inscribed in a circle.

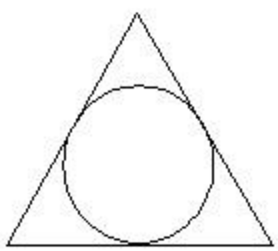


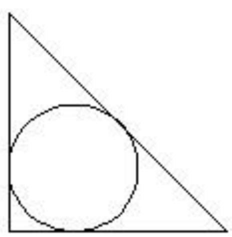
- Also, all convex quadrilaterals whose opposite angles sum up to 180 degrees can be inscribed in a circle.

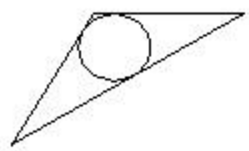


There are also a few points about inscribing a circle in a polygon that you need to keep in mind:

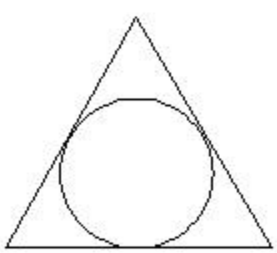
- All triangles have an inscribed circle (called incircle). When a circle is inscribed in a triangle, all sides of the triangle must be tangent to the circle.

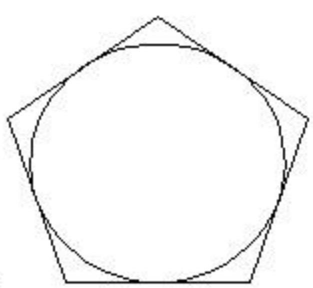


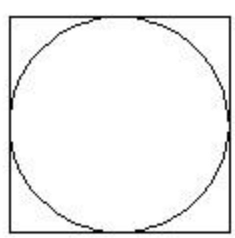




- All *regular* polygons have an inscribed circle.



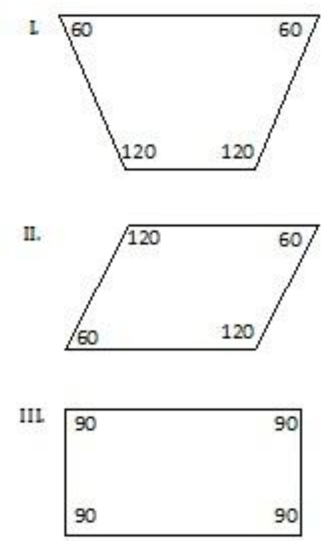




- Most other polygons do not have an inscribed circle

A simple official question will help us see the relevance of these points:

Question: Which of the figures below can be inscribed in a circle?



(A) I only
(B) III only
(C) I & III only
(D) II & III only
(E) I, II & III

Solution:

I think it will suffice to say that the **answer is (C).**

Next week, we will look at the relations between the sides of these polygons and the radii of the circles.

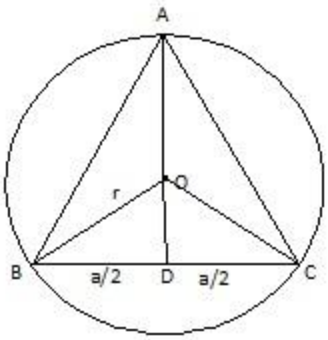
73. Circle and Regular polygon relations.

As promised last week, let's figure out the relations between the sides of various inscribed regular polygons and the radius of the circle.

We will start with the simplest regular polygon – an equilateral triangle. We will use what we already know about triangles to arrive at the required relations.

Look at the figure given below. AB, BC and AC are sides (of length 'a') of the equilateral triangle. OA, OB and OC are radii (of length 'r') of the circle.

Quarter Wit_Quarter Wisdom- Part-2



The interior angles of an equilateral triangle are 60 degrees each. Therefore, angle OBD is 30 degrees (since ABC is an equilateral triangle, BO will bisect angle ABD). So, triangle BOD is a 30-60-90 triangle.

As discussed in your geometry book, the ratio of sides in a 30-60-90 triangle is 1:?3:2 therefore, a/2 : r = ?3:2 or a:r = ?3:1

Side of the triangle = ?3 * Radius of the circle

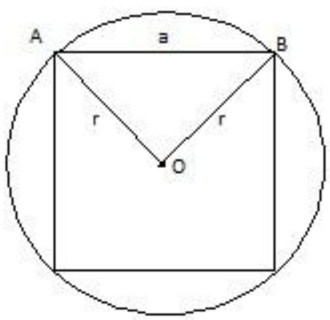
You don't have to learn up this result. You can derive it if needed. Note that you can derive it using many other methods.

Another method that easily comes to mind is using the altitude AD. Altitude AD of an equilateral triangle is given by (?3/2)*a.

The circum center is at a distance 2/3rd of the altitude so AO (radius) = (2/3)* (?3/2)*a = a/?3

Or side of the triangle = ?3 * radius of the circle

Let's look at a square now.



AB is the side of the square and AO and BO are the radii of the circle. Each interior angle of a square is 90 degrees so half of that angle will be 45 degrees. Therefore, ABO is a 45-45-90 triangle. We know that the ratio of sides in a 45-45-90 triangle is 1:1:?2.

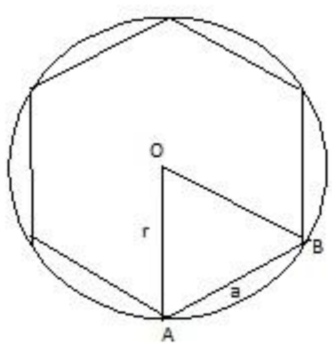
r:a = 1: ?2

Side of the square = ?2*Radius of the circle

Again, no need to learn up the result. Also, there are many methods of arriving at the relation. Another one is using the diagonal of the square. The diagonal of a square is ?2 times the side of the square. The radius of the circle is half the diagonal. So the side is the square is ?2*radius of the circle.

The case of a pentagon is more complicated since it needs the working knowledge of trigonometry which is beyond GMAT scope so we will not delve into it.

We will look at a hexagon though.



Notice that the interior angle of a regular hexagon is 120 degrees so half of that will be 60 degrees. Therefore, both angles OAB and OBA will be 60 degrees each. This means that triangle OAB is an equilateral triangle with all angles 60 degrees and all sides equal. Hence,

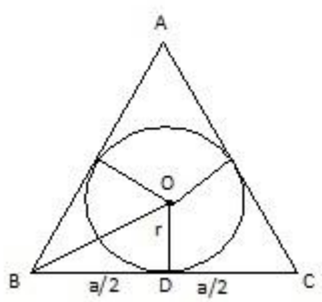
Side of the regular hexagon = Radius of the circle.

The higher order regular polygons and more complicated and we will not take them up. We will discuss a circle inscribed in a polygon next week.

74. <u>And now, other way.</u>

Today we will work with circles inscribed in regular polygons.

We begin by considering an equilateral triangle whose each side is of length ‘a’. Recall that every triangle has an incircle i.e. a circle can be inscribed in every triangle. The diagram given below shows the circle of radius ‘r’ inscribed in an equilateral triangle.



How can we find the relation between ‘r’ and ‘a’? Every angle of an equilateral triangle is 60 degrees. Since it is an equilateral triangle, due to the symmetry, angle OBD = angle OBA = 30 degrees. So we see that triangle BOD is a 30-60-90 triangle. So the ratio of the sides OD:BD:OB = 1: ?3 = r : a/2.

Therefore, a = 2?3 * r

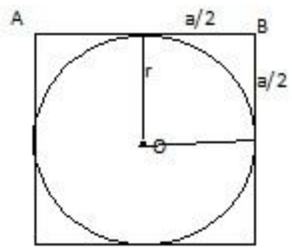
Side of the triangle = 2?3 * Radius of the circle

As discussed last week, there are many other methods of getting this result. We can use the altitude method.

Altitude of an equilateral triangle is given by (?3/2)*a. The incenter is at a distance 2/3rd of the altitude so OD (radius) = (1/3)* (?3/2)*a = a/2?3

Or Side of the triangle = 2?3 * Radius of the circle

Now we will look at a square.

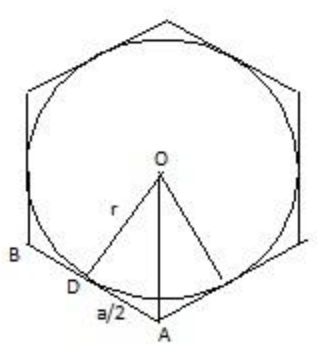


The figure itself shows us that r = a/2

Side of the square = 2 * Radius of the circle

There is no need to delve deeper into it. Though, here is something for you to think about: Can you have a circle inscribed in a rectangle?

Now let's consider a circle inscribed in a regular hexagon.



We know that the interior angle of a regular hexagon is 120 degrees. OA will bisect that angle making angle OAD = 60 degrees. Since AB is tangent to the circle, OD will be perpendicular to AB. Hence OAD is a 30-60-90 triangle. Therefore, a/2 : r = 1: ?3

Hence, a = 2r/?3

Side of the hexagon = (2/?3) * Radius of the circle

Again, remember, you are not expected to 'know' these results so don't try to learn them up. You can always derive any relation you want once you know some basic tricks. The intent of these posts is to familiarize you with those tricks.

Next week, we will look at some interesting Geometry questions based on these concepts!

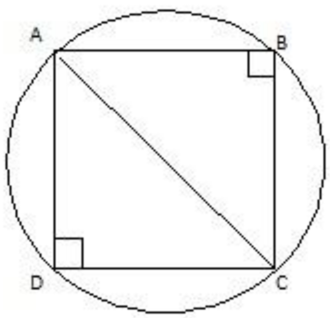
75. Questions on Polygons inscribed in Circles

For today's post, I have two questions for you – both on polygons inscribed in a circle. You must go through the previous post based on this topic before trying these questions.

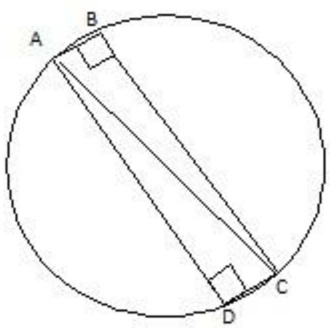
Question 1: Four points that form a polygon lie on the circumference of the circle. What is the area of the polygon ABCD?

Statement I: The radius of the circle is 3 cm.

Statement II: ABCD is square.

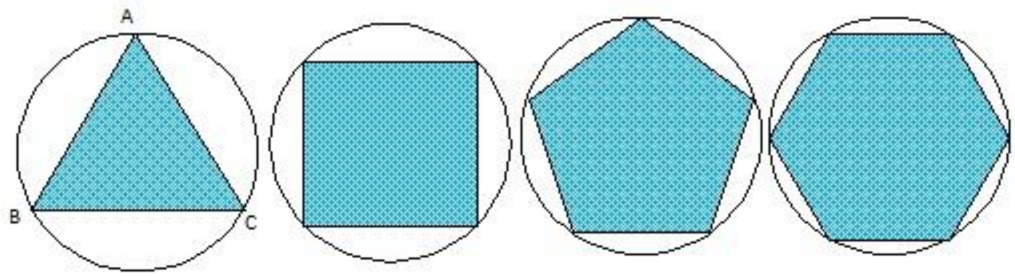


Solution:
Notice that you have been given that angles B and D are right angles. Does that imply that the polygon is a square? No. You haven't been given that the polygon is a regular polygon. The diagonal AC is the diameter since arc ADC subtends a right angle ABC. Hence arc ADC and arc ABC are semi-circles. But the sides of the polygon (AB, BC, CD, DA) may not be equal. Look at the diagram given below:



Statement I: The radius of the circle is 3 cm.
This statement alone is not sufficient. Look at the two figures given above. The area in the two cases will be different depending on the length of the sides. Just knowing the diagonal AC is not enough. Hence this statement alone is not sufficient.
Statement II: ABCD is square.
This tells us that the first figure is valid i.e. the polygon is actually a square. But this statement alone doesn't give us the measure of any side/diagonal. Hence this statement alone is not sufficient.
Using both statements together, we know that ABCD is a square with a diagonal of length 6 cm. This means that the side of the square is 6/?2 cm giving us an area of (6/?2)^2 = 18 cm^2.
Answer (C)
Let's look at a more complicated question now.
Question 2: A regular polygon is inscribed in a circle. How many sides does the polygon have?
Statement I: The length of the diagonal of the polygon is equal to the length of the diameter of the circle.
Statement II: The ratio of area of the polygon to the area of the circle is less than 2:3.
Solution:
In this question, we know that the polygon is a regular polygon i.e. all sides are equal in length. As the number of sides keeps increasing, the area of the circle enclosed in the regular polygon keeps increasing till the number of sides is infinite (i.e. we get a circle) and it overlaps with the original circle. The diagram given below will make this clearer.

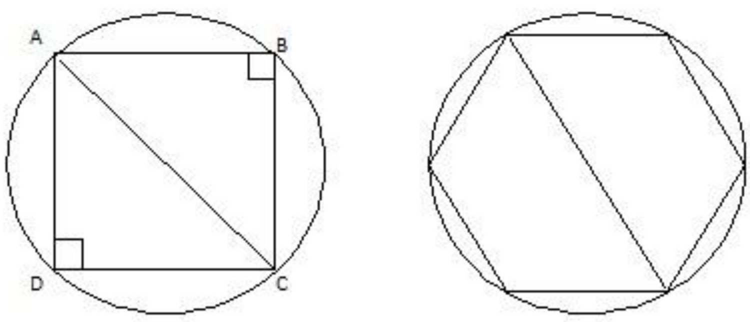
Quarter Wit_Quarter Wisdom- Part-2



Let's look at each statement:

Statement I: The length of one of the diagonals of the polygon is equal to the length of the diameter of the circle.

Do we get the number of sides of the polygon using this statement? No. The diagram below tells you why.



Regular polygons with even number of sides will be symmetrical around their middle diagonal and hence the diagonal will be the diameter. Hence the polygon could have 4/6/8/10 etc sides. Hence this statement alone is not sufficient.

Statement II: The ratio of area of the polygon to the area of the circle is less than 2:3.

Let's find the fraction of area enclosed by a square.

In the previous post we saw that

Side of the square = ?2 * Radius of the circle

Area of the square = Side^2 = 2*Radius^2

Area of the circle = ?*Radius^2 = 3.14*Radius^2

Ratio of area of the square to area of the circle is 2/3.14 i.e. slightly less than 2/3.

So a square encloses less than 2/3 of the area of the circle. This means a triangle will enclose even less area. Hence, we see that already the number of sides of the regular polygon could be 3 or 4. Hence this statement alone is not sufficient.

Using both statements together, we see that the polygon has 4/6/8 etc sides but the area enclosed should be less than 2/3 of the area of the circle. Hence the regular polygon must have 4 sides. Since the area of a square is a little less than 2/3rd the area of the circle, we can say with fair amount of certainty that the area of a regular hexagon will be more than 2/3rd the area of the circle. But just to be sure, you can do this:

Side of the regular hexagon = Radius of the circle

Area of a regular hexagon = 6*Area of each of the 6 equilateral triangles = 6*(?3/4)*Radius^2 = 2.6*Radius^2

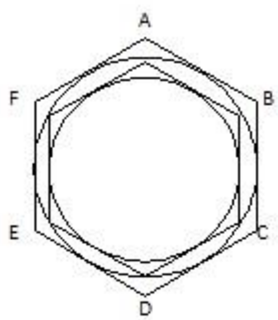
2.6/3.14 is certainly more than 2/3 so the regular polygon cannot be a hexagon. The regular polygon must have 4 sides only.

Answer (C)

76. Circles inscribed in polygons.

Last week we looked at questions on polygons inscribed in a circle. This week, let's look at questions on circles inscribed in regular polygons. As noted earlier, it's important to keep in mind that regular polygons are symmetrical figures. You need very little information to solve for anything in a symmetrical figure.

Question 1: A circle is inscribed in a regular hexagon. A regular hexagon is inscribed in this circle. Another circle is inscribed in the inner regular hexagon and so on. What is the area of the tenth such circle?



Statement I: The length of the side of the outermost regular hexagon is 6 cm.

Statement II: The length of a diagonal of the outermost regular hexagon is 12 cm.

Solution: Thankfully, in DS questions, we don't need to calculate the answer. We just need to establish the sufficiency of the given data. Note that we have found that there is a defined relation between the sides of a regular hexagon and the radius of an inscribed circle and there is also a defined relation between the radius of a circle and the side of an inscribed regular hexagon.

When the circle is inscribed in a regular hexagon,

Radius of the inscribed circle = (?3/2)* Side of the hexagon

When a regular hexagon is inscribed in a circle,

Side of the inscribed regular hexagon = Radius of the circle

So all we need is the side of any one regular hexagon or the radius of any one circle and we will know the length of the sides of all hexagons and the radii of all circles.

Statement I: The length of the side of the outermost regular hexagon is 6 cm.

If length of the side of the outermost regular hexagon is 6 cm, the radius of the inscribed circle is (?3/2)*6 = 3?3 cm

In that case, the side of the regular hexagon inscribed in this circle is also 3?3 cm. Now we can get the radius of the circle inscribed in this second hexagon and go on the same lines till we reach the tenth circle. This statement alone is sufficient.

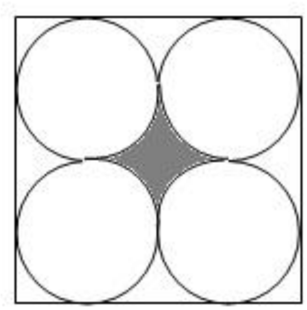
Statement II: The length of a diagonal of the outermost regular hexagon is 12 cm.

Note that a hexagon has diagonals of two different lengths. The diagonals that connect vertices with one vertex between them are smaller than the diagonals that connect vertices with two vertices between them. Length of AC will be shorter than length of AD. Given the length of a diagonal, we do not know which diagonal it is. Is AC = 12 or is AD = 12? The length of the side will be different in the two cases. So this statement alone is not sufficient.

Answer (A)

Keep in mind that you don't actually need to solve for an answer is DS; in fact, in some questions you will not be able to solve for the answer under the given time constraints. All you need to do is ensure that given unlimited time, you will get a unique answer.

Question 2: Four identical circles are drawn in a square such that each circle touches two sides of the square and two other circles (as shown in the figure below). If the side of the square is of length 20 cm, what is the area of the shaded region?



(A) 400 – 100?

(B) 200 – 50?

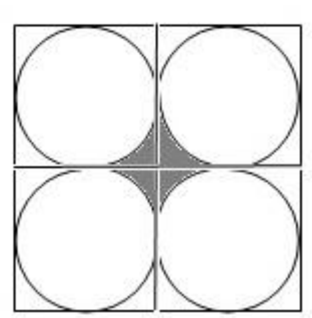
(C) 100 – 25?

(D) 8?

(E) 4?

Solution: First let's recall that squares and circles are symmetrical figures. The given figure is symmetrical.

We don't know any formula that will help us get the area of the curved shaded grey shape in the center. In such cases, very often what you need is to find the area of one region and subtract the area of another out of it. Here, if we subtract the area of the four circles out of the area of the square, the leftover area includes the shaded region but it includes other regions (around the corners etc) too. This is where symmetry helps us.



Notice that we can split the figure into four equal regions to get four smaller squares. Now focus on the diagram give below which shows you one such smaller square. The area around the four corners of the smaller squares is equal i.e. the area of the red region = area of the blue region = area of the yellow region = area of the green region.

Our shaded grey region has four such equal areas so

Area of the shaded grey region = Area of the smaller square – Area of one circle

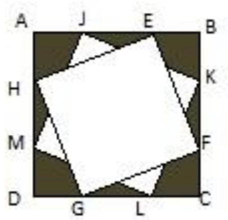
Area of the shaded grey region = (10)^2 – ?(5)^2 = 100 – 25?

Answer (C)

77. The matters of Squares

Let's look at a question today which encompasses most of what we have discussed in this topic. This will be the last post on this topic for a while now. We assume that after going through these posts thoroughly, if you come across any question on 'this inscribed in that', you should be able to handle it. Just a reminder, keep in mind the symmetry of the figures you are handling.

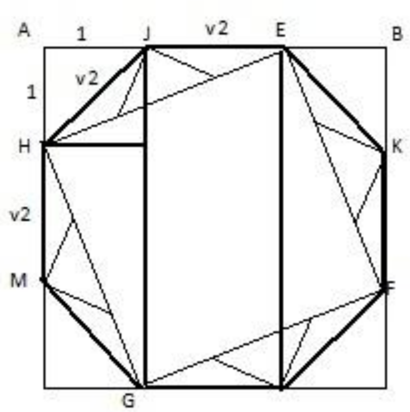
Question: Two identical squares EFGH and JKLM are inscribed in a square ABCD such that AJ:JE:EB = 1:?2:1. What is the area of the octagon obtained by joining points E, K, F, L, G, M, H and J if AB = (2 + ?2) cm?



(A) 8 cm^2
(B) 4 cm^2
(C) 4?2 + 2 cm^2
(D) 4(?2+1) cm^2
(E) 2(?2 + 1) cm^2

Solution:

We are given the length of side AB = (2 + ?2) cm
Also AJ:JE:EB = 1: ?2:1
AJ + JE + EB = (2 + ?2) = a + ?2a + a
a = 1 cm
AJ = 1 cm
JE = ?2 cm
EB = 1 cm
Now let's make the octagon as required.



Since AJ = 1 cm and AH = 1 cm, JH = ?(1^2 + 1^2) = ?2 cm

Notice that the octagon is a regular octagon: JE = KF = LG = MH = ?2 cm. Also, HJ = EK = FL = GM = ?2 cm

The area of the octagon = Area of trapezoid MHJG + area of rectangle JELG + Area of trapezoid KFLE

Area of trapezoid MHJG = (1/2) *(Sum of parallel sides)*Altitude = (1/2)*(?2 + 2 + ?2)*(1) = (?2+1) cm^2

Area of trapezoid KFLE = (?2 +1) cm^2 (by symmetry)

Area of rectangle JELG = ?2*(2 + ?2) = 2(?2 + 1) cm^2

Area of the octagon = (?2 + 1) + 2(?2 + 1) + (?2 + 1) = 4(?2 + 1) cm^2

Answer (D)

Hope you see that it doesn't matter how the question setter twists the concepts, they are still easy to apply if you understand them well!

78.Plugging in Numbers without using Transition Points

A few months back, one of our posts talked about knowing which numbers to plug-in in case you want to use the number–plugging method. To be more exact, we discussed that you need to find the transition points i.e. the points where the two sides of the inequality become equal. The transition points tend to reverse the relation between the two sides. For a detailed discussion of this concept, revisit this post.

A question that arises here is what if the transition points are not apparent? What do we do in that case? First of all, let us say that the use of logic is preferable in every question. There are few questions (but there sure are!) where the number plugging method is the only decent option. But that's beside the point. Let's take up a question and see what to do in case the transition point is hard to see.

Question: If x is an integer, is 4^x < 3^(x+1)?

Statement I: x is positive

Statement II: |x – 1| < 2

Solution:

Again, our moral duty is to first give you the logical solution since we would like you to think in those terms as far as possible. (Though in this question, plugging in numbers might seem easier.) We will discuss how to get the answer by plugging in appropriate numbers in this case.

Method 1: Logical Solution

The question stem only tells us that x can take any integral value. We need to find whether 4^x is less than 3^(x+1).

Consider statement I: x is positive. Given that x is positive, is 4^x less than 3^(x+1)? If we put x = 1, it is easy to see that 4^x is less than 3^(x+1). So the inequality holds in this case. The point is how do we prove that this will be true for all positive values of x? It's tough to prove that something holds for a lot of numbers. It's easier to show that it doesn't hold for at least one number since we need only one suitable value in that case.

Question Stem: Is 4^x < 3^(x+1)?

Reframe it as: Is 4^x < 3^x * 3?

Or Is (4/3)^x < 3?

Note that 4/3 (= 1.33) is greater than 1. When you raise it to a very high power, it will take a very large value. There is no reason it should stay less than 3. Hence the inequality will not hold for large values of x. Hence this statement alone is not sufficient.

Some properties to note:

- When you raise a positive number greater than 1 to a large positive integer power, it takes a large value.
- When you raise 1 to a large positive integer power, it stays 1.
- When you raise a positive number less than 1 to a large positive integer power, the number becomes even smaller than the original value.

These are some number properties you need to work through and be comfortable with.

Consider statement 2: $|x - 1| < 2$

Hopefully, you understand modulus well now. We can say that this inequality implies that x is a point at a distance less than 2 from the point 1 on the number line i.e. $-1 < x < 3$.

Is $(4/3)^x < 3$?

For small values of x e.g. x = 0, we know the inequality holds. Let's check for only the largest value x can take i.e. 2 since x must be an integer. Even if x were 2, $(4/3)^x = 16/9$ i.e. less than 2. $(4/3)^x$ would still be less than 3. Hence the inequality will hold in this case. This statement alone is sufficient to answer the question.

Note that we did use some basic number plugging here too but that number plugging helps us get a clear picture and makes us ask the right questions. There is nothing wrong with plugging in some numbers here and there to understand the logic. If you know why you are plugging in a particular number, it means you are on the right track. Blindly plugging in is the problem.

2. Putting in Numbers

Let's look at this method too now.

Is $4^x < 3^{(x+1)}$?

Here it's not easy to find the transition point. We would have to plug in numbers again to find where the two sides of the inequality are equal! So let's ignore the transition points and directly start plugging in numbers.

Statement 1: x is positive.

Put x = 1, you get 4 < 9 (Holds)

Put x = 2, you get 16 < 27 (Holds)

Put x = 3, you get 64 < 81 (Holds)

What do we do now? How long are we supposed to keep putting in numbers? We cannot do it for all positive integers. How do we decide when to stop? Note that the relative difference between the left hand side and the right hand side is reducing. 9 is more than twice of 4. 27 is more than 16 but not quite twice. It is more than 1.5 times of 16. 81 is somewhat more than 64 but not more than 1.5 times of 64. What this means is that soon enough, the difference will go to 0 and left hand side will become more than the right hand side. If you want to check and you are comfortable with higher powers of numbers,

Put x = 4, you get 256 < 243 (Does not hold)

What you need to do in case the transition point is not apparent is focus on the pattern of the numbers. Is the difference between them narrowing or widening?

This statement alone is not sufficient to answer the question.

Statement II: $|x - 1| < 2$

As above, we get $-1 < x < 3$ so this is simply a matter of putting in x = 0, 1 and 2 to see that the inequality holds in each case. Sufficient.

Answer (B)

79. Using the Number Line

By now, you know that we like to discuss visual approaches to problems. A visual tool that we have used before for solving inequality and modulus questions is the number line. The number line is also useful in helping us solve many number properties questions.

A few things to keep in mind when dealing with number line:

1. $x < y$ (in other words, x is less than y) implies x is to the left of y on the number line. x and y could be in any region i.e. negative or positive but x must be to the left of y in any case.
2. '$x - y > 0$' (in other words, $x - y$ is positive) implies x is to the right of y on the number line. Again, x and y could be in any region of the number line but x will be to the right of y i.e. x will be greater than y in any case.

The importance of these points is not apparent without a couple of questions.

Question 1: If a, b, and c are positive integers, is b between a and c?

Quarter Wit_Quarter Wisdom- Part-2

Statement 1: b is 3 greater than a, and b is 5 less than c.

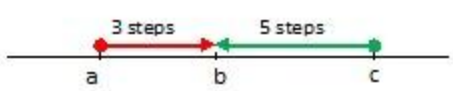
Statement 2: c is 5 greater than b, and c is 8 greater than a.

Solution: You might be tempted to use algebra with equations such as b = a + 3, b = c – 5 etc. But the question 'is b between a and c' should remind you of the number line. If we can figure out the relative position of 'a', 'b' and 'c' on the number line, we can say whether 'b' is between 'a' and 'c'. Many of these 'is this number less than that number' questions can be easily done using the number line.

The question 'Is b between a and c?' essentially means 'does b lay between a and c on the number line?'

Statement 1: b is 3 greater than a, and b is 5 less than c.

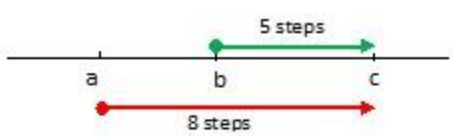
This means 'b' is 3 steps to the right of 'a' but 5 steps to the left of 'c' on the number line. It must lay between 'a' and 'c'.



This statement alone is sufficient to answer the question.

Statement 2: c is 5 greater than b, and c is 8 greater than a.

'c' is 5 steps to the right of 'b' which means 'b' is 5 steps to the left of 'c'. 'c' is 8 steps to the right of 'a' which means 'a' is 8 steps to the left of 'c'. 'a' is further to the left of 'c' than 'b'. So 'b' must be between 'a' and 'c'.



This statement alone is sufficient to answer the question too.

Hence the **answer is (D)**.

Working with equations would have been far too cumbersome. Don't take my word for it; try it on your own.

Let's look at another question based on the same concepts.

Question 2: The points A, B, C and D are on a number line, not necessarily in this order. If the distance between A and B is 18 and the distance between C and D is 8, what is the distance between B and D?

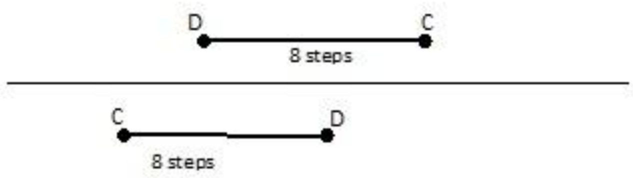
Statement 1: The distance between C and A is the same as the distance between C and B.

Statement 2: A is to the left of D on the number line.

Solution: This question specifically mentions number line.

We are given that distance between A and B is 18. We don't know how to place A and B on the number line yet:

We don't know in which region they lay. We can make a similar diagram for C and D. Note that we don't know how to place these points. All we know is the relative distance between them. We also don't know which one lays to the left and which one lays to the right.



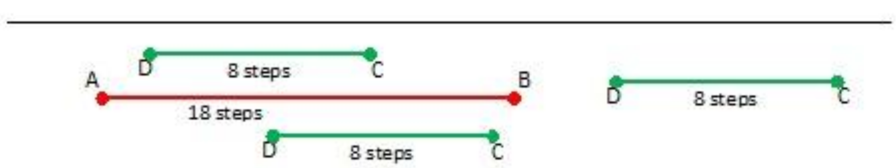
Statement 1: The distance between C and A is the same as the distance between C and B.

Since distance between C and A is the same as distance between C and B, C must lay in the center of A and B. There are still many different ways of placing B and D so the distance between B and D is not known yet.

This statement alone is not sufficient.

Statement 2: A is to the left of D on the number line.

If the only constraint is that A is to the left of D, there are many different ways of placing A relative to D.



The distance between B and D will be different in different cases. This statement alone is not sufficient.

Let's consider both the statements together. C is in the middle of A and B and A is to the left of D. There are still two different cases possible.

The distance between B and D will be different in the two cases. Hence, we still cannot say what the distance between the two points is.

Answer (E)

Number line is a very simple yet powerful visualization tool. Try to use it in various questions for a simpler, more intuitive solution.

80. When Does Order Matters

I have to admit that probability is confusing. The problem is not so much that students find it hard to understand as that teachers find it hard to explain. There are subtle points in a probability question that make all the difference in the world and it takes a ton of ingenuity to explain them in a manner that others understand. You either get the point right away, or you don't.

Today, I will try to explain a probability concept I have always found very difficult to explain in person so the fact that I am attempting to explain it in a post is making me queasy. Nevertheless, the concept is important and I think it deserves a post.

Let me give you two questions:

Question 1: First 15 positive integers are written on a board. If two numbers are selected one by one from the board at random (the numbers are not necessarily different), what is the probability that the sum of these numbers is odd?

Question 2: There is a group of people consisting of 10 men and 6 women. Among these 16 people, there are 4 married couples (man-woman couples). If one man and one woman are selected at random, what is the probability that a married couple gets selected?

Now, let me give you the solutions to these questions. Note that the two solutions are different. We will discuss the reasons behind the difference today.

Solution 1:

Numbers: 1, 2, 3, 4, …, 13, 14, 15

When will the sum of two of these numbers be odd? When one number is odd and the other is even.

P(Sum is Odd) = P(First number is Odd)*P(Second number is even) + P(First number is Even)*P(Second number is Odd)

P(Sum is Odd) = (8/15)*(7/15) + (7/15)*(8/15) = 112/225

Solution 2:

P(Selecting a Married Man) = 4/10

P(Selecting the Wife of that Man) = 1/6

P(Married Couple is Selected) = (4/10)*(1/6) = 4/60

The question I come across here is this: **Why is the second question not solved the way we solved the first question?** After all, selecting two things together is the same as selecting them one after another (explained in your Combinatorics book) i.e. why don't we solve the second question in this way:

P(Selecting a Married Couple) = P(Selecting a Married Man)*P(Selecting the Wife of that Man) + P(Selecting a Married Woman)*P(Selecting the Husband of the Woman)

= (4/10)*(1/6) + (4/6)*(1/10)

Other than the fact that it gives the wrong answer, why can't we solve it like this? Because the order doesn't matter here. It doesn't matter whether we pick the husband first or the wife first. The end result is the same. After we pick either one, the probability of picking the other one stays the same. The two selections have to be made from two different groups. They cannot be made from the same group (contrary to the first question). It doesn't matter whether you catch hold of the man first or the woman first.

In the first question, the probability of picking the correct second number depends on what you picked in the first selection. Hence we consider the order. I will explain this by trying to solve the first question the way we solved the second question:

On first selection, we can pick any number so the probability is 1. The second selection depends on what you selected in the first pick. If you selected an odd number in the first pick, the probability of selecting an even number is 7/15. If you selected an even number in the first pick, the probability of selecting an odd number is 8/15. So what do you do? Do you use 7/15 or 8/15 with 1? You cannot say so you must take individual cases.

Case 1: Select an odd number and then an even number: (8/15) * (7/15)

Case 2: Select an even number and then an odd number: (7/15) * (8/15)

The total cases considered here are 15*15 (select first number in 15 ways and select the second number in 15 ways since the second number can be the same as the first number). In 8*7 ways, you will select an odd number and then an even number. In 7*8 ways, you will select an even number and then an odd number. In both the cases, the sum will be odd. This gives us a probability of (56+56)/225 = 112/225

The total probability of 1 is obtained as follows:

1 = P(first number odd, second number even) + P(first number even, second number odd) + P(first number odd, second number odd) + P(first number even, second number even)

= 56/225 + 56/225 + 64/225 + 49/225 = 1

We are only interested in the 56/225 + 56/225 part.

In the second question, we need to select a couple. Here, it doesn't matter whether you select the man first or the woman; the two member types are different and there is only one way in which you can select the corresponding partner. You cannot select two members of the same type e.g. two men or two women. Hence we don't need to bother with calculating the different cases of selecting the man first or the woman first.

Of course, even if we do it, we will get the correct answer. Let me show you the calculation.

The total number of ways of selecting a man and a woman are 'select a man in 10 ways' and 'a woman in 6 ways'. Then 'select a woman in 6 ways' and 'then a man in 10 ways' i.e. total 120 ways. To select a couple, you can select a man in 4 ways and the woman in 1 way. You can select a woman in 4 ways and the man in 1 way. So total 4 + 4 = 8 ways.

Probability of selecting a couple = 8/120 = 4/60 (same as before).

To sum it, the two questions are quite different.

In the first question, you have two groups of numbers: Even Numbers and Odd Numbers

You can select the two numbers from different groups or from the same group. Hence the total number of cases is 15*15. Also, you can select the same number again.

In the second question, you have two groups of members: Men and Women

You must select the two members from different groups. You cannot select two men or two women. Hence the total number of cases is only 10*6 (and not 16*15). You cannot select the same member again.

In case of confusion, just use the combinations approach rather than probability. You will invariably get the correct answer.

81. When is the question harder than the solution

Last week we looked at a question whose solution was quite hard to explain. This week we will look at a question in which the question itself is hard to explain (so no point worrying about the difficulty in explaining the solution as of now!) So why are we discussing such a question? Because it is certainly not out of GMAT-scope. It uses the concepts of relative speed and GMAT could give you some pretty intimidating questions at higher levels. So what should be your strategy when you come across a question which takes a minute or more to sink in? After you understand the question, first of all you should congratulate yourself that the toughest part is already over. If the question is hard to understand, the solution would be cake walk (well, at least it will feel like it).

Of course, another approach is to skip such a question within 20 secs and move on but in the interest of this post, we will assume that you will not do that. Also, if you get such a question, chances are that you are 'good' at quant and that you would have performed quite well in the test till then. In that case, you would have plenty of extra time to challenge your intellect with such a question.

Let's look at this question now:

Question: On a straight road, a biker noticed that every 12 minutes a bus overtakes him and every 4 minutes he meets an oncoming bus. If all buses leave the station at the same fixed time intervals and run at the same constant speed and the biker moves at a constant speed, what is the time interval between consecutive buses?

(A) 5 minutes

(B) 6 minutes

(C) 8 minutes

(D) 9 minutes

(E) 10 minutes

Solution:

First let's review the information given in the question:

- All buses run at the same constant speed.
- They leave the station at fixed time intervals, say every t mins. (We have to find the value of t) Had the biker been stationary, he would have met a bus every t mins from both directions.
- The biker is moving at a constant speed which is less than the speed of the bus. We can infer that the biker's speed must be less than the speed of the bus because buses overtake him from behind every 12 mins. Had his speed been equal to or more than the speed of the buses, the buses could not have overtaken him.
- Since the biker is moving too, (say going due east) he meets buses coming from one direction (say going from east to west) more frequently and buses coming from the other direction less frequently.

Imagine a scenario where the biker is stationary:

Bus –> Bus –> Bus –> Bus –> Bus –> Bus –>

Biker

Bus <– Bus <– Bus <– Bus <– Bus <– Bus <–

He will meet a bus coming from either direction every t mins. Note that the distance between consecutive buses will stay the same. Why? Let me explain this using an example:

Assume that starting from a bus station, all buses run at the same speed of 50 mph.

Say a bus starts at 12:00 noon. Another starts at 1:00 pm i.e. exactly one hr later on the same route. Can we say that the previous bus is 50 miles away at 1:00 pm? Yes, so the distance between the two buses initially will be 50 miles. The 1 o clock bus also runs at 50 mph. Will the distance between these two buses always stay the same i.e. the initial 50 miles? Since both buses are moving at the same speed of 50 mph, relative to each other, they are not moving at all and the distance between them remains constant. The exact same concept is used in this question.

Now imagine what happens when the biker starts moving too. Say, he is traveling due east.

Bus –> *Bus* –> Bus –> Bus –> Bus –> Bus –>

…………………………..Biker ->

Bus <– Bus <– Bus <– **Bus** <– Bus <– Bus <–

Say, he just met two buses, one from each direction. Now the **Bus** (in bold) is a fixed distance away from him. The biker and the **Bus** are traveling toward each other so they will cover the distance between them faster. Their relative speed is the sum of the speed of the bus and the speed of the biker. They take only 4 mins to meet up. t must be more than 4.

On the other hand, the biker is moving away from the *Bus* (in italics) so the effective speed of *Bus* is only the difference between the speed of the bus and the speed of the biker. So *Bus* takes 12 mins to catch up with the biker. t must be less than 12.

Now that we have understood the question, solving it is relatively easy.

Say the speed of the biker is K and the speed of the bus is B. The ratio of the relative speeds in the two cases will be the inverse of the ratio of time taken (Ratio of speed and time is covered in a previous post).

When the relative speed is $(B + K)$, the time taken is 4 mins.

When the relative speed is $(B - K)$, the time taken is 12 mins.

$(B + K)/(B - K) = 12/4$ (inverse of 4/12)

This gives us $K = (1/2)B$

This means that the bus travelling at a relative speed which is half its usual speed ($B-K = B/2$) takes 12 minutes to meet the man. If it were travelling at its usual speed (i.e. if the biker were stationary), it would have taken half the time i.e. 12/2

= 6 mins to meet the biker. Had the biker been stationary, the time taken by the bus to cover the distance between them would be the same as the time interval between consecutive buses i.e. t mins.
Hence the value of t is 6 mins.
Answer (B)
Hope you see that once you understand the question well, solving it becomes quite easy.

82. Dealing with the 3rd Degree

One of the basic things you need to know before you start your GMAT preparation is how to solve quadratic equations i.e. factorize the quadratic and equate each factor to 0 to get the possible values that x can take. Today we will discuss how you can solve a third degree equation.
Say an equation such as $x^3 – 6x^2 + 11x – 6 = 0$.
How do we get the values of x which satisfy this equation?
If you do get a third degree equation, it will have one very easy root such as 0 or 1 or -1 or 2 or -2 etc. Try a few of these values to get the first root. Here $x = 1$ works. It is easy to see since you have two 6s, a 1 and an 11 as the coefficients.
Putting $x = 1$: $(1)^3 – 6(1)^2 + 11(1) – 6 = 0$
So you know that $(x – 1)$ is a factor. Now figure out the quadratic which when multiplied by $(x – 1)$ gives the third degree expression
$(x – 1)(ax^2 + bx + c) = x^3 – 6x^2 + 11x – 6$
How do we figure out the values of a, b and c? Let's see.
Coefficient of x^3 on right hand side is 1. So you know that all you need is x^2 so that it multiplies with x to give x^3 on left hand side too. So 'a' must be 1.
$(x – 1)(x^2 + bx + c) = x^3 – 6x^2 + 11x – 6$
The constant term, 'c', is easy to figure out too. It should multiply with -1 to give -6 on right hand side. Hence c must be 6.
$(x – 1)(x^2 + bx + 6) = x^3 – 6x^2 + 11x – 6$
Getting the middle term is slightly more complicated. bx multiplies with x to give x^2 term and you also get the x^2 term by multiplying -1 with x^2. You have $-6x^2$ on right hand side so you need the same on the left hand side too. You already have $-x^2$ (by multiplying -1 with x^2) so you need another $-5x^2$ from bx^2. So b must be -5.
$(x – 1)(x^2 – 5x + 6) = x^3 – 6x^2 + 11x – 6$
Now you just factorize the quadratic in the usual way. Let's see how exactly we would do it using a question.
Question: Is $x^3 + 2x^2 – 5x – 6 < 0$
Statement 1: $-3 < x \le -1$
Statement 2: $-1 \le x < 2$
Solution:
We know how to deal with inequalities with multiple factors (discussed here). But the inequality is not split into factors here so we will have to do it on our own.
Let's first try to find the simple root that this expression must have. Try x = 1, -1 etc. We see that when we put x = -1, the expression becomes 0.
$x^3 + 2x^2 – 5x – 6 = (-1)^3 + 2(-1)^2 – 5(-1) – 6 = 0$
So the first factor we get is $(x + 1)$.
$(x + 1)(ax^2 + bx + c) = x^3 + 2x^2 – 5x – 6$
a must be 1 since we have x^3 on the right hand side.
c must be -6 since we have -6 on the right hand side.
$(x + 1)(x^2 + bx – 6) = x^3 + 2x^2 – 5x – 6$
$bx^2 + x^2 = 2x^2$

So b must be 1.

We get: $(x + 1)(x^2 + x - 6)$ which is equal to $(x + 1)(x + 3)(x - 2)$ (after splitting the quadratic)

So the question becomes:

Is $(x + 1)(x + 3)(x - 2) < 0$?

We already know how to deal with inequalities with multiple factors. The transition points here will be -3, -1 and 2. The expression will be negative in the ranges $-1 < x < 2$ and $x < -3$

Statement 1: $-3 < x <= -1$

In this region, the expression is positive (when $-3 < x < -1$) or 0 (when $x = -1$). Hence it will certainly not be negative. This is sufficient to answer the question with 'No'. Hence statement 1 is sufficient to answer the question.

Statement 2: $-1 <= x < 2$

In this region, the expression is negative (when $-1 < x < 2$) or 0 (when $x = -1$). We cannot say for certain whether it will be negative or not. Hence statement 2 alone is not sufficient to answer the question.

Answer (A)

It's not hard to deal with third degree equations. All you have to do is bring it down to second degree by figuring out one root and then the problem is in a format you already know.

83. The Curious Case of Incorrect answers.

Many of us are hooked on to using algebra in Quant questions. The thought probably is that how can it be a Quant question if one did not need to take a couple of variables and make a couple of equations/inequalities. We, at Veritas Prep, love to harp on about how algebra is time consuming and unnecessary in most cases. But today we will go one step further and discuss how indiscriminate use of algebra can actually result in incorrect answers. Surprised, eh?

Of course if you make correct equations and solve them correctly, there is no reason you shouldn't get the correct answer. The problem that arises is in making correct equations/inequalities. Now I am sure you are thinking that you know how to make equations and so probably this post is a waste of your time. Hold on to that thought – Let me give you a statement:

The total number of apples is more than 20.

How will you convert it in terms of algebra? Will you write it as 'Na > 20'? If this is what you did, please do go through the post. I am sure there will be a couple of things you will find interesting.

Let me take an example to show you why something like this may not be enough.

Question: Harry bought some red books that cost \$8 each and some blue books that cost \$25 each. If Harry bought more than 10 red books, how many blue books did he buy?

Statement 1: The total cost of blue books that Harry bought was at least \$150.

Statement 2: The total cost of all books that Harry bought was less than \$260.

Solution: The first thing we will do is look at the most common algebra solution.

Let the number of blue books be B and red books be R.

He bought more than 10 red books so $R > 10$.

Statement 1: The total cost of blue books that Harry bought was at least \$150

$25B >= 150$

So $B >= 6$

This statement alone is not sufficient to give the actual value of B.

Statement 2**:** The total cost of all books that Harry bought was less than \$260

$8R + 25B < 260$ …………. (I)

Also, $10 < R$ (from above) which gives $80 < 8R$ …………..(II)

Adding (I) and (II), we get (note that the two inequalities have the same sign '<' so they can be added)
8R + 25B + 80 < 260 + 8R
B < 7.2
This statement alone is not sufficient to give the actual value of B.
Using both statements together, we get that B >= 6 and B < 7.2
So B could be 6 or 7 (since number of blue books must be an integer value)
Answer (E)
This is incorrect and actually, answer is (C). The question is how? There is no calculation mistake in the above given solution. Then why do we get the incorrect answer? Let me give you the logical solution and prove that answer is actually (C). Then we will discuss why algebra fails us here.
Logical Solution:
Red Books – $8 each
Blue Books – $25 each
No of Red books is more than 10.
Statement 1: The total cost of blue books that Harry bought was at least $150
Blue books cost $25 each so he bought at least 6 books. He could have bought more too. This statement alone is not sufficient.
Statement 2**:** The total cost of all books that Harry bought was less than $260
He bought more than 10 red books so he bought at least 11 red books. He spent at least $8*11 = $88 on the red books. Out of the total 260, he is left with 260 – 88 = $172 for the blue books. Since each blue book costs $25, he could have bought at most 6 blue books. He could have bought fewer too. This statement alone is not sufficient.
Using both statements together: He bought at least 6 and at most 6 blue books. So he must have bought 6 blue books.
Answer (C)
I think you would have figured out the problem with the algebra solution by now. In the algebra solution, the inequality 10 < R does not include all the information you have available. You know that R cannot be 10.5 or 10.8 i.e. a decimal. It must be an integer since it represents the number of red books. So you might want to use 11 <= R to get a tighter value for B. Mind you, it is true that R is greater than 10. Important is that it is equal to or greater than 11 too.
Hence the analysis of statement 2 changes a little:
8R + 25B < 260 ………(I)
11 <= R which gives us 88 <= 8R …………. (II)
When you add (I) and (II) now, you get 25B < 172 i.e. B < 6.9. So B must be 6 or less.
This gives us enough information such that when considering both statements together, we get B = 6.
So when you use algebra, but be mindful of the hidden constraints.

84. The Play of words

Some days back I came across a question which was a slight twist on a regular question type. The usual active voice of the sentence had been changed to passive but in such a way that the meaning had been altered. It was a lesson in DS as well as SC – read every word carefully. One word could change a 600 level to a 750 level one, a mundane everyday question to a smart question. We often see this interesting transformation in P&C questions but for that to happen in algebra was quite a delight. Let's discuss that particular question today.
First let's look at the mundane version.
Question: If 9 notebooks and 3 pencils cost 20 Swiss Francs, do 12 notebooks and 12 pencils cost 40 Swiss Francs?
Statement 1: 7 notebooks and 5 pencils cost 20 Swiss Francs.
Statement 2: 4 notebooks and 8 pencils cost 20 Swiss Francs.

Solution: Both statements give very similar information. It looks like the answer will be (D). That is, if one statement is enough to answer the question alone, the other will probably be enough to answer alone too. Also it seems that we will have two simultaneous equations in two variables so we will be able to solve for the variables.

Let's quickly review how we actually solve this

Given: If 9 notebooks and 3 pencils cost 20 Swiss Francs –> $9N + 3P = 20$ (assuming N is the cost of each notebook and P is the cost of each pencil)……..(I)

Question: do 12 notebooks and 12 pencils cost 40 Swiss Francs – > Is $12N + 12P = 40$? OR Is $6N + 6P = 20$?

Statement 1: 7 notebooks and 5 pencils cost 20 Swiss Francs.

$7N + 5P = 20$ ……… (II)

Equating (I) and (II), we get $N = P = 20/12$. This is sufficient to answer whether $6N + 6P$ is equal to 20.

Statement 2: 4 notebooks and 8 pencils cost 20 Swiss Francs.

$4N + 8P = 20$ ……… (III)

Equating (I) and (III), we get $N = P = 20/12$. This is sufficient to answer whether $6N + 6P$ is equal to 20.

So as expected, **answer is (D)** in this case.

The problem arises when the question is changed a bit.

Question 2: If 20 Swiss Francs is enough to buy 9 notebooks and 3 pencils, is 40 Swiss Francs enough to buy 12 notebooks and 12 pencils?

Statement 1: 20 Swiss Francs is enough to buy 7 notebooks and 5 pencils.

Statement 2: 20 Swiss Francs is enough to buy 4 notebooks and 8 pencils.

Solution: Note that the numbers are unchanged. Does the changed wording convey the same meaning? At first glance, you may think so but that is not true. Now, 9 notebooks and 3 pencils may cost less than 20 SF too. All that the statement tells us is that 20 SF is enough – whether it is just enough or comfortably enough, we don't know. So we don't have the actual cost of 9N and 3 pencils. We just know that $9N + 3P \le 20$. So here we will have to solve inequalities. But it still seems that both statements give very similar information and so if one alone is sufficient, the other alone should be sufficient too.

Given: If 20 Swiss Francs is enough to buy 9 notebooks and 3 pencils –> $9N + 3P \le 20$ ……. (I)

Question: is 40 Swiss Francs enough to buy 12 notebooks and 12 pencils – > "Is $12N + 12P \le 40$?" OR "Is $6N + 6P \le 20$?" OR "Is $N + P < 10/3$?"

Statement 1: 20 Swiss Francs is enough to buy 7 notebooks and 5 pencils.

$7N + 5P \le 20$ ……..(II)

Adding (I) and (II), we get $2N + P \le 5$. We want the coefficients of N and P to be the same in the resulting inequality. Since coefficient of N is greater in both inequalities, we cannot have the same coefficient of N and P in the resultant inequality. So we cannot say whether $N + P < 10/3$ so this statement alone is not sufficient.

Statement 2: 20 Swiss Francs is enough to buy 4 notebooks and 8 pencils.

$4N + 8P \le 20$ …………(III)

Again, we need the coefficient of N and P to be the same in the resultant inequality. You can get this as

$2N + 4P \le 10$ (from III)

$3N + P \le 20/3$ (from I)

Adding them, we get $5N + 5P \le 50/3$ OR $N + P \le 10/3$

This statement alone is sufficient to answer the question.

Answer (B)

As opposed to our instinct, we find that the second statement alone is sufficient while the first is not.

Let's try to understand why this is so using a logical solution.

Given: $9N + 3P \le 20$

Question: Is $12N + 12P \le 40$? OR Is $6N + 6P \le 20$?

We know that 9 notebooks and 3 pencils cost 20 SF or less. We want to find whether 6 notebooks and 6 pencils will cost 20 SF or less i.e. if you drop 3 notebooks but take another 3 pencils, will your total cost still not exceed 20 SF? That depends on the relative cost of notebooks and pencils. If pencils are cheaper than (or have same cost as) notebooks, then obviously the total cost will stay less than or equal to 20. If pencils are more expensive than notebooks, we need to know how much more expensive they are to be able to judge whether the cost of 6 notebooks and 6 pencils will exceed 20 SF.

Statement 1: *20 Swiss Francs is enough to buy 7 notebooks and 5 pencils.*

Now we know that we can substitute 2 notebooks with 2 pencils i.e. pencils may be cheaper than notebooks, may have the same cost or may be a little more expensive but there is enough leeway in our total cost for us to bear the extra cost of 2 pencils in place of 2 notebooks. But do we have enough leeway in our total cost to replace 3 notebooks with 3 pencils, we don't know. Let me explain this using an example:

Say 1 notebook costs 1 SF and 1 pencil costs 1 SF too. So 9 notebooks and 3 pencils costs 12 SF. 7 notebooks and 5 pencils cost 12 SF. 6 notebooks and 6 pencils will cost 12 SF.

Take a different case – say 1 notebook costs 1.5 SF and 1 pencil costs 1.85 SF

Then 9 notebooks and 3 pencils cost 19.05 SF (which is less than 20 SF). 7 notebooks and 5 pencils cost 19.75 SF. (So even though 1 pencil costs more than 1 notebook, 2 pencils can substitute 2 notebooks because total cost is less than 20 SF. Obviously 1 pencil can substitute 1 notebook since there was enough leeway for even 2 pencils in place of 2 notebooks)

But 6 notebooks and 6 pencils cost 20.1 SF. (Now we see that 3 pencils cannot substitute 3 notebooks because there isn't enough leeway. This time it crossed 20 SF)

Hence this statement alone is not sufficient to answer the question.

Statement 2: 20 Swiss Francs is enough to buy 4 notebooks and 8 pencils.

Now we know that we can drop 5 notebooks and buy 5 extra pencils in their place and the total cost will still stay below 20 SF. Hence there is enough leeway for 5 replacements. This obviously means that there is enough leeway for 3 replacements and hence the cost of 6 notebooks and 6 pencils will stay below 20 SF. This statement alone is sufficient to answer the question.

Answer (B)

We hope all this made sense. If you are reeling after reading all these numbers, give it another try. The question is a good 750 level question and certainly not easy.

85. <u>Facing too much Knowledge Problem</u>

Continuing our scrutiny of interesting standalone questions with important takeaways, let's discuss today how too much knowledge can actually let you down. We often come across people wondering whether they should learn up the many formulas in permutation/combination, co-ordinate geometry etc. Our take on the question is a flat 'No'. Formulas won't take you far in GMAT, perhaps up to 600 but certainly not further. In fact, until and unless you have an eidetic memory or a Math PHD, chances are that knowing too many formulas will be a disadvantage. Let me show you why:

Say you see this question: What is the area of a triangle with vertices at (1, 4), (7, 1) and (4, 7)?

What is the first thing that comes to your mind? I am fairly certain that my fellow engineering grads will think of either Matrices or Heron's formula. With Matrices, the confusion will be: was it (y2 – y1) or (y1 – y2)? Or was it –x2 or +x2? With Heron's formula, the problem will be too many calculations. So we need to get the length of the sides first, then find s, then plug it all in the formula which was ummm… ?s(s-a)… or s?(s-a)… Hope you get my point. Until and unless you spend a fair bit of time everyday with all the formulas you intend to remember and the exact cases in which they can be used, it's a waste of time and effort. In fact, if you are too attuned to the use of formulas, you will find it hard to think of a non-formula method to solve the problem. It will be hard for you to think beyond the formula and you will spend the two minutes you get in recalling the exact formula to be used in this particular situation, that is assuming there are formulas for most situations.

Now, try to think of a non-formula method and in this, I am sure the non Math background people will do better because they are used to figuring out innovative methods of solving problems. They do not come to the floor with pre-conceived notions on 'methods to be used while solving particular question types' and hence can keep their minds open. I can think of and have come across at least 3 different methods of solving this problem without using any exotic formulas. We just have to think in terms of right angles since we know how to find the area of figures with right angles. Let's discuss each of those methods:

Question: What is the area of a triangle with vertices at (1, 4), (7, 1) and (4, 7)?

(A) 9/2

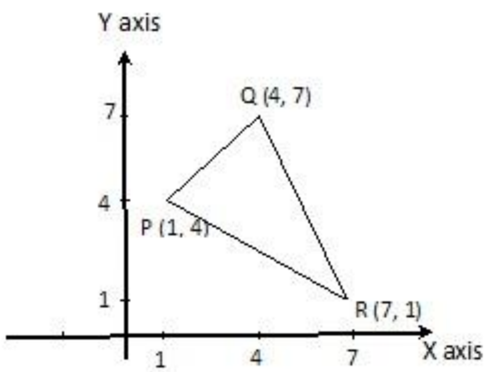
(B) 9

(C) 27/2

(D) 18

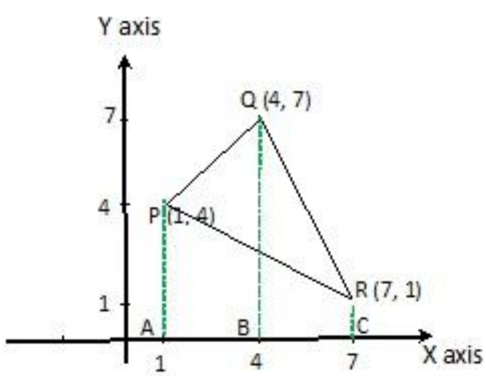
(E) 27

Solution: First of all, follow the golden rule of coordinate geometry – draw the triangle.



We don't see any right triangles so let's make some. We know how to find the area when we have right angles around.

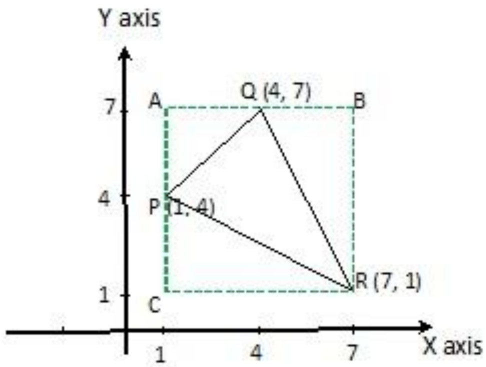
Method 1: Use Trapezoids



Area of PQR = Area of APQB + Area of QBCR – Area of APRC

Area of PQR = (1/2)*3*(4 + 7) + (1/2)*3*(7 + 1) – (1/2)*6*(4+1) = 27/2

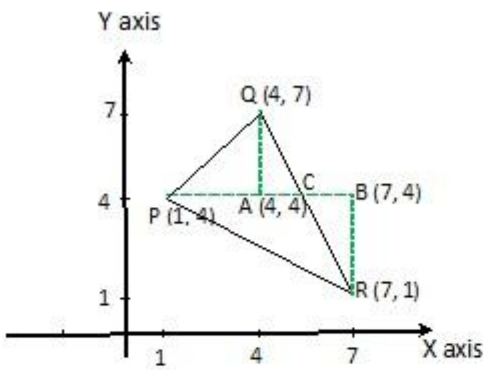
Method 2: Use a Rectangle



Area of PQR = Area of ABRC – Area of AQP – Area of BQR – Area of PCR

Area of PQR = 6*6 – (1/2)*3*3 – (1/2)*3*6 – (1/2)*3*6 = 36 – (1/2)*45 = 27/2

Method 3: Use right triangles



Area of PQR = Area of PAQ + Area of QAC + Area of PRC

There is one complication here – we don't know the coordinates of point C. It is easy to figure out. Just find the equation of line QR and find the value of x when y = 4.

Equation of a line is given by: y – y1 = (y2 – y1)/(x2 – x1) * (x – x1)

Equation of QR: y – 1 = (7 – 1)/(4 – 7) * (x – 7)

2x + y = 15

When y = 4, x = 11/2. So C (11/2, 4)

Area of PQR = Area of PAQ + Area of QAC + Area of PRC

Area of PQR = (1/2)*3*3 + (1/2) *(11/2 – 4)*3 + (1/2) *(11/2 -1)*3 = 27/2

Answer (C)

I am sure you can come up with some other methods if you try!

86. Hard Quadratic Equations

When faced with an unusual quadratic equation, some people waste a lot of time while trying to 'split the middle term'. The common refrain is 'I am just not good at it.' Actually it has little to do with intuition and a lot to do with understanding how numbers work. If I am looking at a quadratic equation and am unable to find the required factors, I will go back to check my quadratic to see if it is correct rather than try to use the esoteric quadratic formula.

Quarter Wit_Quarter Wisdom- Part-2

To solve a quadratic, you need to find the two factors that add up to give the middle term and multiply to give the product of the constant term and co-efficient of x^2. What you may have problems with is 'splitting the middle term'. Even though solving a quadratic is a basic skill one must possess to crack the GMAT, we have seen people struggle with it especially if the coefficient of x^2 is something other than 1. Let's discuss how we can split the middle term quickly in such cases.

Question 1: Solve for x: 5x^2 – 34x + 24 = 0

To factorize, we need to find two numbers a and b such that:

a + b = -34

a*b = 5*24

Step 1: Prime factorize the product.

a*b = 5*24 = 2*2*2*3*5

Step 2: Check the signs and decide what you need. Here the sum of a and b is negative (-34) while product is positive. This means a and b both are negative. Both will add to give a negative number and multiply to give a positive number (more on this at the end*). It also means that both a and b are smaller than 34 (since both are negative, their absolute values will be added to give 34).

Step 3: Try to split the prime numbers into two groups such that their sum is 34. Try the most obvious group first i.e. 2*2*2 and 3*5 -> 8 and 15

If a = 8 and b = 15, we get a + b = 23

But the sum needs to be 34, i.e. a number greater than 23.

Before we discuss the next step, let's talk about how adding numbers works:

Let's say the prime factorization we have is 2*2*5*5.

We split it into 2 groups -> 2*5 and 2*5 (10 and 10). The sum of 10 and 10 is 20.

We split it in another way -> 2*2 and 5*5 (4 and 25). Their sum is 29. The sum increased.

We split it in yet another way -> 2 and 2*5*5 (2 and 50). Their sum is 52. The sum increased further.

Notice that further apart the numbers are, the greater is their sum. We get the least sum (20) when the numbers are equal. If we need a higher sum, we increase the distance between the numbers.

Going back to the original question, the prime factorization is 2*2*2*3*5 and we split it as 2*2*2 and 3*5. This gave us a sum of 8 + 15 = 23. We need 34 so we need to get the numbers farther from each other but not too far either. Let's say, we pick a 2 from 8 and give it to 15. We get two numbers 4 and 30. They are farther apart and their sum is 34. So the numbers we are looking for are -4 and -30 (to get -34 as sum).

Now the quadratic is simply: 5*(x - 4/5)*(x – 30/5) = 0 (a shortcut to the usual '5x^2 – 4x – 30x + 24 = 0 and proceed' method)

x = 4/5 or 6

Question 2: Solve for x: 8x^2 – 47x – 63 = 0

To factorize, we need to find two numbers a and b such that:

a + b = -47

a*b = 8*(-63) = – 2*2*2*3*3*7

The product of a and b is negative which means one of a and b is negative. The sum of a and b is also negative therefore, the number with higher absolute value is negative and the other is positive. When these two numbers will be added, the difference of their absolute values will be the sum and the sign of the sum will be negative. So at least one of a and b is greater than 47. The greater one will be negative and the smaller one will be positive.

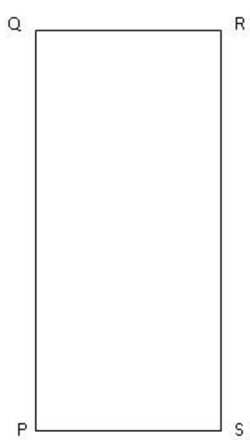
Let's try to split the factors now. To start, we split the primes into two easy groups: 2*2*2 and 3*3*7 to get 8 and 63 but -63 + 8 = -55. We need the sum to have lower absolute value than 55 so we need to get the numbers closer together. Take off a 3 from 63 and give it to 8 to get 2*2*2*3 and 3*7. Now a and b are -24 and 21; the numbers are too close. Instead, take off 7 from 63 and give it to 8 to get 2*2*2*7 and 3*3. Now a and b are -56 and 9.

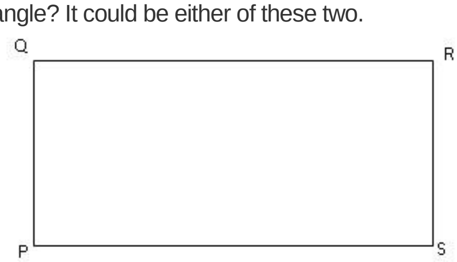
-56 + 9 = -47 -> the required sum.

Now the quadratic is 8*(x + 9/8)*(x – 56/8) = 0
x = -9/8 or 7
With a little bit of practice, the hardest questions can be quickly solved.
*How do you decide the sign of a and b:
Product is positive – This means both a and b have the same sign. If sum is negative, both a and b are negative; if sum is positive, they both are positive.
Product is negative – This means a and b have opposite signs; one is negative, the other is positive. If sum is positive, the number which is positive has a higher absolute value. If sum is negative, the number which is negative has a higher absolute value.

87. Elementary, My Dear Watson

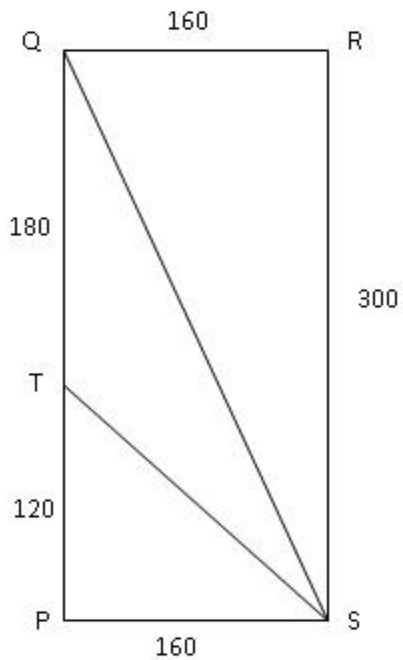
hile eagerly awaiting the kick off of season 3 of BBC's Sherlock, let's put our time to good use. Though we have already spent a lot of it speculating over what really happened to Sherlock (HOW did he come back?!), perhaps we can take a leaf out of his book and learn to notice little things in whatever is leftover. There is a good reason to do that – there are little clues in some questions that the test maker unwittingly leaves to bring clarity to the question. If we understand those clues, a seemingly mysterious problem could be easily unraveled. Let us show you with an example.
Question: Peter and Jacob are at the northwest corner of a field, which is a rectangle 300 ft long and 160 ft wide. Peter walks in a straight line directly to the southeast corner of the field. If Jacob walks 180 ft down the west side of the field and then walks in a straight line directly to the southeast corner of the field, what is the difference in the distance traveled by the two?
(A) 20
(B) 40
(C) 80
(D) 120
(E) 140
Solution: The first thing we do in these "direction" questions is draw the diagram. But there is a problem here: how do we decide the orientation of the rectangle? It could be either of these two.





A few things help us decide this. There are two definitions of length:
1. Length is the longest side of the rectangle.
2. Width is from side to side and length is whatever width isn't (i.e. the side from up to down in a rectangle) (this definition is less embraced than the first one)

If the side from up to down is the longest side, then there is no conflict.
Keeping this in mind, when drawing the figure, given that length is the longer of the two, one could make the rectangle on the left and there will be no conflict. But the question maker may not want to take for granted that you know this.
So he/she leaves a clue – the question mentions that 'Jacob walks 180 ft down the west side of the field'. There needs to be at least 180 ft on the west side of the field for him to travel that much. So the orientation on the left makes sense. This is something the question maker would have put to try to give you a hint of the orientation. Now that we know what our diagram should look like, we can proceed to solve this question.



If you just remember some of your pythagorean triplets, this question can be solved in moments (and that's why we suggest you to remember them!) If not, it would involve some calculations.
QR = 160, RS = 300
So QR:RS = 8:15
Remember 8-15-17 pythagorean triplet? (the third triplet after 3-4-5 and 5-12-13)
Since the two sides are in the ratio 8:15, the hypotenuse must be 17. The common multiplier is 20 so QS should be 17*20 = 340
Therefore, Peter traveled 340 feet.
TP = 120, PS = 160
TP:PS = 3:4
Does it remind you of 3-4-5 triplet?
120 is 3*40 and 160 is 4*40 so TS will be 5*40 = 200
So Jacob traveled a total distance of 180 + 200 = 380 feet.
Difference between the distance traveled = 380 – 340 = 40 feet
Note: The following triplets come in handy: (3, 4, 5) (5, 12, 13) (8, 15, 17) (7, 24, 25) (20, 21, 29) and (9, 40, 41)

88. Can you find out correct answers to tricky GMAT questions

This is hard to confess publicly but I must because it is a prime example of how GMAT takes advantage of our weaknesses – A couple of days back, I answered a 650 level question of weighted averages incorrectly. Those of you who have been following my blog would understand that it was an unpleasant surprise – to say the least. I know my weighted averages quite well, thank you! For this comedown, I blame the treachery of GMAT because it knows how to get you when you become too complacent. The takeaway here is – no matter how easy and conventional the question seems, you MUST read it carefully.

Let me share that particular question with you. I will also share two solutions which give you two different answers. It is an exercise for you to figure out which one is the correct solution (that is, if one of them is the correct solution). Needless to say, the error in the solution(s) is conceptual and very easy to see (not some sly calculation mistake). It's just that in your haste, it's very easy to miss this important point. I hope to see some comments with some good explanations.

Question: The price of each hair clip is ¢ 40 and the price of each hair band is ¢ 60. Rashi selects a total of 10 clips and bands from the store, and the average (arithmetic mean) price of the 10 items is ¢ 56. How many bands must Rashi put back so that the average price of the items that she keeps is ¢ 52?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Solution 1:

Price of each clip (Pc) = 40

Price of each band (Pb) = 60

Average price of each item (Pavg) = 56

Wc/Wb = (Pb – Pavg)/(Pavg – Pc) = (60 – 56)/(56 – 40) = 1/4 (our weighted average formula)

Since the total number of items is 10, number of clips = 1*2 = 2 and number of bands = 4*2 = 8

If the average price is changed to 52,

Wc/Wb = (Pb – Pavg)/(Pavg – Pc) = (60 – 52)/(52 – 40) = 2/3

Now the ratio has changed to 2:3. This gives us number of clips as 4 and number of bands as 6.

Since previously she had 8 bands and now she has 6 bands, she must have put back 2 bands.

Answer (B)

Solution 2:

Say the number of hair clips is C and the number of hair bands is 10 – C.

(40C + 60(10 – C))/10 = 56 (Using the formula: Average = Sum/Number of items)

On solving, you get C = 2

Number of clips is 2 and number of bands is (C – 2) = 8.

Now, let's consider the scenario when she puts back some bands, say x.

(2*40 + (8 – x)*60)/(10 – x) = 52

On solving, you get x = 5

So she puts back 5 bands so that the average price is 52.

Answer (E)

Obviously, there is only one correct answer. It's your job to figure out whether it is (B) or (E) or some third option. Also what's wrong with one or both of these solutions?

89. How to identify terminating decimals

We know the basics of decimals and rational numbers.

- Decimals can be rational or irrational.

Quarter Wit_Quarter Wisdom- Part-2

- Decimals which terminate and those which are non-terminating but repeating are rational. They can be written in the form a/b.
- Decimals which are non-terminating and non-repeating are irrational such as ?2, ?3 etc.

The problem comes when we get a question based on these basics. That's when we realize that our basics are not as strong as we assumed them to be. For example, look at this question:

Question: Which of the following fractions has a decimal equivalent that is a terminating decimal?

(A) 10/189

(B) 15/196

(C) 16/225

(D) 25/144

(E) 39/128

If your first thought is that we will simply divide the numerator by the denominator in each case and figure out which terminates and which doesn't, you must realize that that is a very time consuming process. There has to be another logical approach to this problem. Well, here it is:

A fraction in its lowest term can be expressed as a terminating decimal if and only if the denominator has powers of only 2 and/or 5. Let's try to understand the logic behind it.

Say, a and b are two integers.

a/b = a * 1/b

For a/b to be terminating, 1/b must be a terminating decimal. What happens when you start dividing 1 by b? You add a decimal point and start adding 0s. You will get 1 followed by as many 0s as you require in the numerator.

10/100/1000/10000 etc have only two prime divisors: 2 and 5. If the denominator has 2s or 5s or both, we will be able to terminate the decimal by choosing the required multiple of 10. If there are any other primes, we will never be able to divide a multiple of 10 completely and hence the decimal will not terminate. It is obvious, isn't it?

1/3 = .333333333333333333…

1/7 = .142857142857142857…

1/11 = .09090909090909090…

Now the question we posed above is quite simple. Let's look at it again.

Question 1: Which of the following fractions has a decimal equivalent that is a terminating decimal?

(A) 10/189

(B) 15/196

(C) 16/225

(D) 25/144

(E) 39/128

Only option (E) has a denominator of the form 2^a*5^b.

128 = 2^7

Therefore, 39/128 will terminate. All the other denominators have other prime numbers as well and hence will not terminate.

Using the same concepts, let's look at another question.

Question 2: If 1/(2^11 * 5^17) is expressed as a terminating decimal, how many non-zero digits will the decimal have?

(A) 1

(B) 2

(C) 4

(D) 6

(E) 11

Solution:

First realize that $2^{11} * 5^{17} = 2^{11} * 5^{11} * 5^6 = 10^{11} * 5^6$

So $1/(10^{11} * 5^6)$ is just $0.00…001/5^6$.

Now let's try to figure out the answer intuitively:

What do you get when you divide .01 by 5? You get .002. You write 0s till you get 10 and then you get a non-zero digit.

What do you get when you divide .01 by 125 (which is 5^3)? You get .00008.

Do you notice something? The non zero term is $8 = 2^3$

The reason is this: You have 1 followed by as many 0s as you require in the dividend. $125 = 5^3$ so you will need 2^3 i.e. you will need 10^3 as the dividend and then 125 will be able to divide it completely (i.e. the decimal will terminate).

Now, using the same logic, what will be the non zero digits if you are dividing .00001 by 625?

$625 = 5^4$. You will need $2^4 = 16$ to get 10^4 and that will end the terminating decimal. So you will have two non 0 digits: 16

What will you get when you divide .000…0001 by 5^6? Your non zero digits will be $2^6 = 64$ i.e. you will have 2 non-zero digits.

Another way to look at the problem is this:

$1/(10^{11} * 5^6) = 2^6/(10^{17})$ (multiply and divide by 2^6)

$= 64/(10^{17})$

Since the denominator is a power of 10, it will just move the decimal 17 places to the left. The non-zero digits will remain 64 only i.e. 2 digits.

Answer (B)

We will look at some DS questions on terminating and non terminating decimals next week.

90. Terminating decimals in DS

Last week, we discussed the basics of terminating decimals. Let me review the important points here:

- To figure out whether the fraction is terminating, bring it down to its lowest form.
- Focus on the denominator – if it is of the form $2^a * 5^b$, the fraction is terminating, else it is not.

Keeping this in mind, let's look at a couple of DS questions on terminating decimals.

Question 1: If a, b, c, d and e are integers and $m = 2^a*3^b$ and $n = 2^c*3^d*5^e$, is m/n a terminating decimal?

Statement 1: $a > c$

Statement 2: $b > d$

Solution:

Given: a, b, c, d and e are integers

Question: Is m/n a terminating decimal?

Or Is $(2^a*3^b)/(2^c*3^d*5^e)$?

We know that powers of 2 and 5 in the denominator are acceptable for the decimal to be terminating. If there is a power of 3 in the denominator after reducing the fraction, then the decimal in non- terminating. So our question is basically whether the power of 3 in the denominator gets canceled by the power of 3 in the numerator. If b is greater than (or equal to) d, after reducing the fraction to lowest terms, it will have no 3 in the denominator which will make it a terminating decimal. If b is less than d, even after reducing the fraction to its lowest terms, it will have some powers of 3 in the dominator which will make it a non-terminating decimal.

Question: Is $b >= d$?

Statement 1: $a > c$

This statement doesn't tell us anything about the relation between b and d. Hence this statement alone is not sufficient.

Statement 2: $b > d$

This statement tells us that b is greater than d. This means that after we reduce the fraction to its lowest form, there will be no 3 in the denominator and it will be of the form $2^c * 5^e$ only. Hence it will be a terminating decimal. This statement alone is sufficient.

Answer (B)

Now onto another DS question.

Question 2: If $0 < x < 1$, is it possible to write x as a terminating decimal?

Statement 1: 24x is an integer.

Statement 2: 28x is an integer.

Solution:

Given: $0 < x < 1$

Question: Is x a terminating decimal?

Again, x will be a terminating decimal if it is of the form $m/(2^a * 5^b)$

Statement 1: 24x is an integer.

$24x = 2^3 * 3 * x = m$ (an integer)

$x = m/(2^3 * 3)$

Is x a terminating decimal? We don't know. If m has 3 as a factor, x will be a terminating decimal. Else it will not be. This statement alone is not sufficient.

Statement 2: 28x is an integer.

$28x = 2^2 * 7 * x = n$ (an integer)

$x = n/(2^2 * 7)$

Is x a terminating decimal? We don't know. If n has 7 as a factor, x will be a terminating decimal. Else it will not be. This statement alone is not sufficient.

Taking both together,

$m/24 = n/28$

$m/n = 6/7$

Since m and n are integers, m will be a multiple of 6 (and thereby of 3 too) and n will be a multiple of 7. So x will be a terminating decimal.

Answer (C)

91. Converting non terminating repeating decimals to fractions

Last week we discussed the properties of terminating decimals. We also discussed that non-terminating but repeating decimals are rational numbers.

For GMAT, we must know how to convert these non-terminating repeating decimals into rational numbers. We know how to do vice versa i.e. given a rational number, we can divide the numerator by the denominator to find its decimal equivalent.

For example:

1/3 = 0.333333333… (infinite number of 3s)

But given 0.555555555…, how will you convert it to its exact fraction equivalent?

For GMAT, we must know how to convert non-terminating repeating decimals into rational numbers. We know how to do vice versa i.e. given a rational number, we can divide the numerator by the denominator to find its decimal equivalent. For example,

1/3 = 0.333333333… (infinite number of 3s)

We write this as $0.\overline{3}$.

Similarly, 1/6 = .16666666666... (infinite number of 6s)

We write this as $0.1\overline{6}$.

1/7 = 0.142857142857...

We write this as $0.\overline{142857}$.

The problem arises when we are given a decimal which we need to convert to a fraction. We know that every non-terminating repeating decimal can be written cleanly and then used in calculations by converting it to fraction. But given, say $1.\overline{8}$, how do we know which fraction it represents? We can approximate and say that 1.88888... is almost 1.9 but approximation may not be suitable in every question or you might be asked for the actual value of the fraction.

So how do you convert $1.\overline{8}$ to a fraction? A terminating decimal is easy to handle. Say, a decimal such as 1.8. We get rid of the decimal sign by dividing by 10 i.e. 1.8 = 18/10.

When we have a decimal such as $1.\overline{8}$, the problem is that we have infinite 8s so we will need infinite 0s in the denominator.

1.888 = 1888/1000

1.888888 = 1888888/1000000

But what do we do when we have infinite 8s? It is very hard for us to fathom infinite numbers and harder still to work with them. We need to get rid of the infinite sequence in some way. The good thing about the infinite sequence is that even if we pull away one 8 out of it, the sequence still remains infinite.

$X = 1.\overline{8}$

$10X = 18.\overline{8}$ (When you multiply by 10, the decimal moves one place to the right but you still have infinite 8s leftover)

Subtract the first equation from the second to get,

$9X = 18.\overline{8} - 1.\overline{8} = 17$ (Infinite 8s subtract out the infinite 8s and you are left with 18 – 1 = 17)

So X = 17/9

So the exact value of $1.\overline{8}$ is 17/9.

To sum it, the manipulation required is this – we need the infinite sequence (and only the infinite sequence) to the right of decimal in both equations. We subtract out the equations to get rid of the infinite sequence. We are left with a clean fraction.

We use the exact same logic to convert any number with a non terminating repeating part into a fraction. Let's take another example. Say,

$X = 2.13\overline{46}$

Note that multiplying by 10 or 100 here doesn't help us. We need the part after the decimal to be the same in both the equations so that we can cancel it off. Note how we work around this problem:

$100X = 213.\overline{46}$

$10000X = 21346.\overline{46}$

Now we have two equations where we have only the infinite sequence to the right of the decimal point. When we subtract these two equations, we get rid of the repetitive part and get,

$9900X = 21133$

$X = 21133/9900$

The point is to separate out the repetitive part and then cancel it. You can now convert any repeating non-terminating decimal into a fraction.

For practice, try out these numbers.

1. $0.08\overline{6}$
2. $34.02\overline{76}$
3. $1.\overline{452}$

92.How to Deal with Maximizing/Minimizing Strategies in GMAT

We haven't dealt with maximizing/minimizing strategies in our QWQW series yet (except in sets). The reason for this is that the strategy to be used varies from question to question. What works in one question may not work in another. You might have to think up on what to do in a question from scratch and you have only 2 mins to do it in. The saving grace is that once you know what you have to do, the actual work involved to arrive at the answer is very little.

Let's look at some maximizing minimizing strategies in the next few weeks. We start with an OG question today with a convoluted question stem.

Question: List T consists of 30 positive decimals, none of which is an integer, and the sum of the 30 decimals is S. The estimated sum of the 30 decimals, E, is defined as follows. Each decimal in T whose tenths digit is even is rounded up to the nearest integer, and each decimal in T whose tenths digits is odd is rounded down to the nearest integer. If 1/3 of the decimals in T have a tenths digit that is even, which of the following is a possible value of E – S?

I. -16

II. 6

III. 10

A. I only

B. I and II only

C. I and III only

D. II and III only

E. I, II, and III

Solution:

There is a lot of information in the question stem and a lot of variables are explained. Let's review the given data in our own words first.

T has 30 decimals. The sum of all the decimals is S.

10 decimals have even tenths digit. They will be rounded up.

20 decimals have odd tenths digit. They will be rounded down.

The sum of rounded numbers is E.

E – S can take many values so how do we figure which ones it cannot take? We need to find the minimum value E – S can take and the maximum value it can take. That will help us figure out the values that E – S cannot take. Note that E could be greater than S and it could be less than S. So E – S could be positive or negative.

Step 1: Getting Minimum Value of E – S

Let's try to make E as small as possible. For that, we need to do two things:

1. When we round up the decimals (even tenths digit), the difference between actual and estimate should be very small. The estimate should add a very small number to round it up so that E is not much greater than S. Say the numbers are something similar to 3.8999999 (the tenths digit is the largest even digit) and they will be rounded up to 4 i.e. the estimate gains about 0.1 per number. Since there are 10 even tenths digit numbers, the estimate will be approximately .1*10 = <u>1 more than actual</u>.

2. When we round down the decimals (odd tenths digit), the difference between actual and estimate should be as large as possible. Say the numbers are something similar to 3.999999 (tenths digit is the largest odd digit) and they will be rounded down to 3 i.e. the estimate loses approximately 1 per number. Since there are 20 such numbers, the estimate is 1*20 = <u>20 less than actual</u>.

Overall, the estimate will be approximately 20 – 1 = 19 less than actual.

Minimum value of E – S = -19

Step 2: Getting Maximum Value of E – S

Now let's try to make E as large as possible. For that, we need to do two things:

1. When we round up the decimals (even tenths digit), the difference between actual and estimate should be very high. Say the numbers are something similar to 3.000001 (tenths digit is the smallest even digit) and they will be rounded up to 4 i.e. the estimate gains 1 per number. Since there are 10 even tenths digit numbers, the estimate will be approximately 1*10 = <u>10 more than actual</u>.

2. When we round down the decimals (odd tenths digit), the difference between actual and estimate should be very little. Say the numbers are something similar to 3.1 (tenths digit is the smallest odd digit). They will be rounded down to 3 i.e. the estimate loses approximately 0.1 per number. Since there are 20 such numbers, the estimate is approximately 0.1*20 = <u>2 less than actual</u>.

Overall, the estimate will be approximately 10 – 2 = 8 more than actual.

Maximum value of E – S = 8.

The minimum value of E – S is -19 and the maximum value of E – S is 8.

So E – S can take the values -16 and 6 but cannot take the value 10.

93. <u>Min- Max Strategies-Establishing Base Case</u>

Continuing our discussion on maximizing/minimizing strategies, let's look at another question today. Today we discuss the strategy of establishing a base case, a strategy which often comes in handy in DS questions. The base case gives us a starting point and direction to our thoughts. Otherwise, with the number of possible cases in any given scenario, we may find our mind wandering from one direction to another without reaching any conclusions. That is a huge waste of time, a precious commodity.

Question: Four friends go to Macy's for shopping and buy a top each. Three of them buy a pillow case each too. The prices of the seven items were all different integers, and every top cost more than every pillow case. What was the price, in dollars, of the most expensive pillow case if the total price of the seven items was $89?

Statement 1: The most expensive top cost $16.

Statement 2: The least expensive pillow case cost $9.

Solution: The first problem here is figuring out the starting point. There must be many ways in which you can price the seven items such that the total cost is $89. So we need to establish a base case (which conforms to all the conditions given in the question stem) first and then we will tweak it around according to the additional information obtained from our statements.

'Seven items for $89' means the average price for each item is approximately $12. But 12 is not the exact average. 12*7 = 84 which means another $5 were spent.

A sequence with an average of 12 and different integers is $9, $10, $11, $12, $13, $14, $15.

But actually another $5 were spent so the prices could be any one of the following variations (and many others):

$9, $10, $11, $12, $13, $14, $20 (Add $5 to the highest price)

$9, $10, $11, $12, $13, $16, $18 (Split $5 into two and add to the two highest prices)

$9, $10, $12, $13, $14, $15, $16 (Split $5 into five parts of $1 each and add to the top 5 prices)

$7, $9, $13, $14, $15, $16, $17 (Take away some dollars from the lower prices and add them to the higher prices along with the $5)

etc

Let's focus on another piece of information given in the question stem: "every top cost more than every pillow case." This means that when we arrange all the prices in the increasing order (as done above), the last four are the prices of the four tops and the first three are the prices of the three pillow cases. The most expensive pillow case is the third one. Now that we have accounted for all the information given in the question stem, let's focus on the statements.

Statement 1: The most expensive top cost $16.

We have already seen a case above where the maximum price was $16. Is this the only case possible? Let's look at our base case again:

$9, $10, $11, $12, $13, $14, $15

(a further $5 needs to be added to bring the total price up to $89)

Since the prices need to be all unique, if we add 1 to any one price, we also need to add at least $1 to each subsequent price. E.g. if we increase the price of the least expensive pillow case by $1 and make it $10, we will need to increase the price of every subsequent item by $1 too. But we have only $5 more to give.

If the maximum price is $16, it means the rightmost price can increase by only $1. So all prices before it can also only increase by $1 only and except the first two prices, they must increase by $1 to adjust the extra $5.

Hence the only possible case is $9, $10, $12, $13, $14, $15, $16.

So the cost of the most expensive pillow case must have been $12.

Statement 1 is sufficient alone.

Statement 2: The least expensive pillow case cost $9.

A restriction on the lowest price is much less restrictive. Starting from our base case

$9, $10, $11, $12, $13, $14, $15,

we can distribute the extra $5 in various ways. We can do what we did above in statement 1 i.e. give $1 to each of the 5 highest prices: $9, $10, $12, $13, $14, $15, $16

We can also give the entire $5 to the highest price: $9, $10, $11, $12, $13, $14, $20

So the price of the most expensive pillow case could take various values. Hence, statement 2 alone is not sufficient.

Answer (A)
Note that the answer is a little unexpected, isn't it? If we were to read the question and guess within 20 secs, we would probably guess that the answer is (C), (D) or (E). The two statements give similar but complementary information. It would be hard to guess that one will be sufficient alone while other will not be. This is what makes this question interesting and hard too.
Our strategy here was to establish a base case and tweak it according to the information given in the statements. This strategy is often useful in DS – not just in max-min questions but others too.

94. Min-Max Strategies- Focus on Extremes

In the last two weeks, we discussed some max min strategies. Today, let's look at another max-min question in which we apply the strategy of focusing on the extremes. The largest or the smallest values are often found at the extremes of a given range.
Question: If x and y are integers such that (x+1)^2 is less than or equal to 36 and (y-1)^2 is less than 64, what is the sum of the maximum possible value of xy and the minimum possible value of xy?
(A) -16
(B) -14
(C) 0
(D) 14
(E) 16
Solution: To get the sum of the maximum and minimum possible values of xy, we need to know the maximum and minimum values of xy. For those, we need to find the values that x and y can take. So first, we should review the information given:
(x + 1)^2 <= 36
(y – 1)^2 < 64
We need to find the values that x and y can take. There are many ways of doing that. We can solve the inequality using the wave method discussed in this post or using the concept of absolute values. Let's discuss both the methods.
Wave method to solve inequalities:
Solve for x: (x + 1)^2 <= 36
(x + 1)^2 – 6^2 <= 0
(x + 1 + 6)(x + 1 – 6) <= 0
(x + 7)(x – 5) <= 0
-7 <= x <= 5 (Using the wave method)
Solve for y: (y – 1)^2 < 64
(y – 1)^2 – 8^2 < 0
(y – 1 + 8)(y – 1 – 8) < 0
(y + 7)(y – 9) < 0
-7 < y < 9 (Using the wave method)
Or you can solve taking the square root on both sides
Solve for x: (x + 1)^2 <= 36
|x + 1| <= 6
-6 <= x + 1 <= 6 (discussed in your Veritas Algebra book)
-7 <= x <= 5
So x can take values: -7, -6, -5, -4, … 3, 4, 5
Solve for y: (y – 1)^2 < 64

|y – 1| < 8
-8 < y – 1 < 8 (discussed in your Veritas Algebra book)
-7 < y < 9
So y can take values: -6, -5, -4, -3, … 6, 7, 8.
Now that we have the values of x and y, we should try to find the minimum and maximum values of xy.
Note that the values of xy can be positive as well as negative. The minimum value will be the negative value with largest absolute value (largest negative) and the maximum value will be the positive value with the largest absolute value.
Minimum value – For the value to be negative, one and only one of x and y should be negative. Focus on the extreme values: if x is -7 and y is 8, we get xy = -56. This is the negative value with largest absolute value.
Maximum value – For the value to be positive, both x and y should have the same signs. If x = -7 and y = -6, we get xy = 42. This is the largest positive value.
The sum of the maximum value of xy and minimum value of xy is -56 + 42 = -14
Answer (B)
Try to think of it in terms of a number line. x lies in the range -7 to 5 and y lies in the range -6 to 8. The range is linear so the end points give us the maximum/minimum values. Think of what happens when you plot a quadratic – the minimum/maximum could lie anywhere.

95. How well you know your factors

In the last three weeks, we discussed a couple of strategies we can use to solve max-min questions: 'Establishing Base Case' and 'Focus on Extremes'. Now try to use those to solve this question:
Question: A carpenter has to build 71 wooden boxes in one week. He can build as many per day as he wants but he has decided that the number of boxes he builds on any one day should be within 4 off the number he builds on any other day.
(A) What is the least number of boxes that he could have build on Saturday?
(B) What is the greatest number of boxes that he could have build on Saturday?
Meanwhile, let's move on to something else today. What we will discuss today is a very simple concept but it seems odd to us when we first confront it even if we are very comfortable with factors and divisibility. If we tell you the concept right away, you will probably not believe us when we say that many people are unable to come up with it on their own. Hence, we will first give you a question which you need to answer in 30 seconds. If you are unable to do so, then we will discuss the concept with you!
Question: A, B, C and D are positive integers such that A/B = C/D. Is C divisible by 5?
Statement 1: A is divisible by 210
Statement 2: B = 7^x, where x is a positive integer
Solution: Let's discuss the solution till the point I assume you will be quite comfortable.
We need to find whether C is divisible by 5. So let's separate the C out of the variables.
C = AD/B
Since C is an integer, AD will be divisible by B but what we don't know is that after the division, is the quotient divisible by 5?
Statement 1: A is divisible by 210
We still have no idea what B is so this statement alone is not sufficient. Let's take an example of how the value of B could change our answer. Assume A is 210.
If B is 3, AD/B will be divisible by 5.
If B is 10, AD/B may not be divisible by 5 (depending on the value of D).
Statement 2: B = 7^x, where x is a positive integer

We have no idea what A and D are hence this statement alone is not sufficient.
Using both together: Now, this is where the trick comes in. Using both statements together, we see that C = (210*a*D)/(7^x)
Now we can say for sure that C will be divisible by 5. If you are not sure why, read on.
The Concept:
As you know, factors (also called divisors) of a number N are those positive integers which completely divide number N i.e. they do not leave a remainder on dividing N. If F is a factor of N, N/F leaves no remainder. This also means that N can be written as F*m where m is an integer. Sure you feel this is elementary but this concept is not as internalized in your conscience as you believe. To prove it, let me give you a question.
Example 1: Is 3^5 * 5^9 * 7 divisible by 18?
Did you take more than 2 seconds to say 'No' confidently?
For N to be divisible by F, you should be able to write N as F*m i.e. N must have F as a factor. F here is 18 (= 2*3^2) but we have no 2 in N (which is 3^5 * 5^9 * 7) though we do have a couple of 3s. Hence this huge product is not divisible by 18.
This helps us deduce that odd numbers are never divisible by even numbers.
Example 2: Is 3^5*7^6*11^3 divisible by 13?
The answer is simply 'No'.
For the numerator to be divisible by the denominator, the denominator MUST BE a factor of the numerator. In the entire numerator, there is no 13 so the numerator is not divisible by 13.
Example 3: On the other hand, is 3^5*7^6*11^3*13 divisible by 13?
Yes, it is. 13 gets cancelled and the quotient will be 3^5*7^6*11^3.
Example 4: Is 2^X divisible by 3?
No. No matter what X is, you will only have X number of 2s in the numerator and will never have a 3. So this will not be divisible by 3.
Let's come back to the original question now:
Given that C = (210*a*D)/(7^x)
Whatever x is, 7^x will get cancelled out by the numerator and we will be left with something. That something will include 5 (obtained from 210) since only 7s will be cancelled out from the numerator. Hence C is divisible by 5.
Answer (C)

96. Properties of Absolute Values on GMAT

We have talked about quite a few concepts involving absolute value of x in ourprevious posts. But some absolute value questions involve two variables. Then do we need to consider the positive and negative values of both x and y? Certainly! But there are some properties of absolute value that could come in handy in such questions. Let's take a look at them:
(I) For all real x and y, |x + y| <= |x| + |y|
(II) For all real x and y, |x – y| >= |x| – |y|
We don't need to learn them of course and there is no need to look at how to prove them either. All we need to do is understand them – why do they hold, when is the equality sign applicable and when can they be useful. Let's look at both the properties one by one.
(I) For all real x and y, |x + y| <= |x| + |y|
The result of both the left hand side and the right hand side will be positive or zero. On the right hand side, the absolute values of x and y will always get added irrespective of the signs of x and y. On the left hand side, the absolute values of

x and y might get added or subtracted depending on whether they have the same sign or different signs. Hence the result of the left hand side might be smaller than or equal to that of the right hand side.

For which values of x and y will the equality hold and for which values will the inequality hold? Let's think logically about it.

The absolute values of x and y get added on the right hand side. We want the absolute values of x and y to get added on the left hand side too for the equality to hold. This will happen when x and y have the same sign. So the equality should hold when they have the same signs.

For example, $x = 4$, $y = 8$:

$|4 + 8| = |4| + |8| = 12$

OR $x = -3$, $y = -4$:

$|-3 -4| = |-3| + |-4| = 7$

Also, when at least one of x and y is 0, the equality will hold.

For example, $x = 0$, $y = 8$:

$|0 + 8| = |0| + |8| = 8$

OR $x = -3$, $y = 0$:

$|-3 + 0| = |-3| + |0| = 3$

What happens when x and y have opposite signs? On the left hand side, the absolute values of x and y get subtracted hence the left hand side will be smaller than the right hand side (where they still get added). That is when the inequality holds i.e. $|x + y| < |x| + |y|$

For example, $x = -4$, $y = 8$:

$|-4 + 8| < |-4| + |8|$

$4 < 12$

OR $x = 3$, $y = -4$:

$|3 -4| < |3| + |-4|$

$1 < 7$

Let's look at our second property now:

(II) For all real x and y, $|x – y| >= |x| – |y|$

Thinking on similar lines as above, we see that the right hand side of the inequality will always lead to subtraction of the absolute values of x and y whereas the left hand side could lead to addition or subtraction depending on the signs of x and y. The left hand side will always be positive whereas the right hand side could be negative too. So in any case, the left hand side will be either greater than or equal to the right hand side.

When will the equality hold?

When x and y have the same sign and x has greater (or equal) absolute value than y, both sides will yield a positive result which will be the difference between their absolute values

For example, $x = 9$, $y = 2$;

$|9 – 2| = |9| – |2| = 7$

OR $x = -7$, $y = -3$

$|-7 – (-3)| = |-7| – |-3| = 4$

Also when y is 0, the equality will hold.

For example, $x = 8$, $y = 0$:

$|8 – 0| = |8| – |0| = 8$

OR $x = -3$, $y = 0$:

$|-3 – 0| = |-3| – |0| = 3$

What happens when x and y have the same sign but absolute value of y is greater than that of x?

It is easy to see that in that case both sides have the same absolute value but the right hand side becomes negative.

For example, $x = -4$, $y = -9$

$|x – y| = |-4 – (-9)| = 5$

$|x| – |y| = |-4| – |-9| = -5$

So even though the absolute values will be the same since we will get the difference of the absolute values of x and y on both sides, the right hand side will be negative. If we were to take further absolute value of the right hand side, the two will become equal i.e. the right hand side will become $|(|x| – |y|)| = |-5| = 5$ in our example above. In that case, the equality will hold again.

Similarly, what happens when only $x = 0$? The right hand side becomes negative again so taking further absolute value will make both sides equal.

For example, $x = 0$, $y = -5$

$|x – y| = |0 – (-5)| = 5$

$|x| – |y| = |0| – |5| = -5$

Taking further absolute value, $|(|x| – |y|)| = |-5| = 5$

So when we take further absolute value of the right hand side, this property becomes similar to property 1 above: $|x – y| = |(|x| – |y|)|$ when x and y have the same sign or at least one of x and y is 0.

Now let's look at the inequality part of property 2.

Whenever x and y have opposite signs, $|x – y| > |x| – |y|$

On the left hand side, the absolute values will get added while on the right hand side, the absolute values will get subtracted. So the absolute value of the left hand side will always be greater than the absolute value of the right hand side. The left hand side will always be positive while the right hand side could be negative too. Hence even if we take the further absolute value of the right hand side, the inequality will hold: $|x – y| > |(|x| – |y|)|$ when x and y have opposite signs

For example, $x = -4$, $y = 8$:

$|-4 – 8| > |-4| – |8|$

$12 > -4$

Taking further absolute value of the right hand side, we get $|(|x| – |y|)| = |-4| = 4$

Still, $12 > 4$ i.e. $|x – y| > |(|x| – |y|)|$

OR $x = 3$, $y = -4$:

$|3 –(-4)| > |3| – |-4|$

$7 > -1$

Taking further absolute value of the right hand side, we get $|(|x| – |y|)| = |-1| = 1$

Still, $7 > 1$ i.e. $|x – y| > |(|x| – |y|)|$

Note that the inequality of the original property 2 also holds when x and y have the same sign but absolute value of y is greater than the absolute value of x since the right hand side becomes negative. It also holds when x is 0 but y is not.

To sum it all neatly,

(I) For all real x and y, $|x + y| <= |x| + |y|$

$|x + y| = |x| + |y|$ when (1) x and y have the same sign (2) at least one of x and y is 0.

$|x + y| < |x| + |y|$ when (1) x and y have opposite signs

(II) For all real x and y, $|x – y| >= |x| – |y|$

$|x – y| = |x| – |y|$when (1) x and y have the same sign and x has greater (or equal) absolute value than y (2) y is 0

$|x – y| > |x| – |y|$ in all other cases

(III) For all real x and y, $|x – y| >= |(|x| – |y|)|$

$|x – y| = |(|x| – |y|)|$ when (1) x and y have the same sign (2) at least one of x and y is 0.

$|x – y| > |(|x| – |y|)|$ when (1) x and y have opposite signs

Note that property (III) matches property (I).

There is another property we would like to discuss but let's take it up next week along with some GMAT questions where we put these properties to use.

97. Properties of Absolute Values-II

We pick up this post from where we left the post of last week in which we looked at a few properties of absolute values in two variables. There is one more property that we would like to talk about today. Thereafter, we will look at a question based on some of these properties.

(III) |x – y| = 0 implies x = y

x and y could be positive/negative integer/fraction; if the absolute value of their difference is 0, it means x = y. They cannot have opposite signs while having the same absolute value. They must be equal. This also means that if and only if x = y, the absolute value of their difference will be 0.

Mind you, this is different from 'difference of their absolute values'

|x| – |y| = 0 implies that the absolute value of x is equal to the absolute value of y. So x and y could be equal or they could have opposite signs while having the same absolute value.

Let's now take up the question we were talking about.

Question: Is |x + y| < |x| + |y|?

Statement 1: | x | is not equal to | y |

Statement 2: | x – y | > | x + y |

Solution: One of the properties we discussed last week was

"For all real x and y, |x + y| <= |x| + |y|

|x + y| = |x| + |y| when (1) x and y have the same sign (2) at least one of x and y is 0.

|x + y| < |x| + |y| when (1) x and y have opposite signs"

We discussed in detail the reason absolute values behave this way.

So our question "Is |x + y| < |x| + |y|?" now becomes:

Question: Do x and y have opposite signs?

We do not care which one is greater – the one with the positive sign or the one with the negative sign. All we want to know is whether they have opposite signs (opposite sign also implies that neither one of x and y can be 0)? If we can answer this question definitively with a 'Yes' or a 'No', the statement will be sufficient to answer the question. Let's go on to the statements now.

Statement 1: | x | is not equal to | y |

This statement tells us that absolute value of x is not equal to absolute value of y. It doesn't tell us anything about the signs of x and y and whether they are same or opposite. So this statement alone is not sufficient.

Statement 2:| x – y | > | x + y |

Let's think along the same lines as last week – when will | x – y | be greater than | x + y |? When will the absolute value of subtraction of two numbers be greater than the absolute value of their addition? This will happen only when x and y have opposite signs. In that case, while subtracting, we would actually be adding the absolute values of the two and while adding, we would actually be subtracting the absolute values of the two. That is when the absolute value of the subtraction will be more than the absolute value of the addition.

For Example: x = 3, y = -2

| x – y | = |3 – (-2)| = 5

| x + y | = |3 – 2| = 1

or

x = -3, y = 2

| x – y | = |-3 – 2| = 5

| x + y | = |-3 + 2| = 1
If instead, x and y have the same sign, | x + y | will be greater than| x – y |.
If at least one of x and y is 0, | x + y | will be equal to| x – y |.
Since this statement tells us that | x – y | > | x + y |, it implies that x and y have opposite signs. So this statement alone is sufficient to answer the question with a 'Yes'.
Answer (B)
Takeaway from this question:
If x and y have the same signs, | x + y | >| x – y |.
If x and y have opposite signs, | x + y | <| x – y |.
If at least one of x and y is 0, | x + y | =| x – y |.
You don't need to 'learn this up'. Understand the logic here. You can easily recreate it in the exam if need be.

98. An official Question on Absolute Value

Now that we have discussed some important absolute value properties, let's look at how they can help us in solving official questions.
Knowing these basic properties can help us quickly analyze the question and arrive at the answer without getting stuck in analyzing different ranges, a cumbersome procedure.
First we will look at a GMAT Prep question.

Question 1: Is |m – n| > |m| – |n|?
Statement 1: n < m
Statement 2: mn < 0
Solution 1:
Recall the property number 2
Property 2: For all real x and y, |x – y| >= |x| – |y|
|x – y| = |x| – |y| when (1) x and y have the same sign and x has greater (or equal) absolute value than y (2) y is 0
|x – y| > |x| – |y| in all other cases
So if m and n have the same sign with |m| >= |n|, equality will hold.
Also, if n is 0, equality will hold.
If we can prove that both these conditions are not met, then we can say that |m – n| is definitely greater than |m| – |n|.
Statement 1: n < m
We have no idea about the signs of m and n. Are they same? Are they opposite? We don't know. Also n may or may not be 0. Hence we don't know whether the equality will hold or the inequality. Statement 1 alone is not sufficient to answer the question.
Statement 2: mn < 0
Since mn is negative, it means one of m and n is positive and the other is negative. This also implies that n is definitely not 0. So we know that m and n do not have the same sign and n is not 0. So under no condition will the equality hold.
Hence |m – n| is definitely greater than |m| – |n|. Statement 2 alone is sufficient to answer the question.
Answer (B)
Let's look at one more question now.
Question 2: If xyz ? 0, is x(y + z) >= 0?
Statement 1: |y + z| = |y| + |z|
Statement 2: |x + y| = |x| + |y|
Solution 2: xyz ? 0 implies that all, x, y and z, are non zero numbers.

Question: Is x(y + z) >= 0?

If we can prove that x(y + z) is not negative that is x and (y+z) do not have opposite signs, we can say that x(y + z) is positive or 0.

Looking at the statements given, let's review our property number 1:

Property 1: For all real x and y, |x + y| <= |x| + |y|

|x + y| = |x| + |y| when (1) x and y have the same sign (2) at least one of x and y is 0.

|x + y| < |x| + |y| when (1) x and y have opposite signs

The two statements give us equalities which means that the relevant part of the property is this:

|x + y| = |x| + |y| when (1) x and y have the same sign (2) at least one of x and y is 0.

We are also given in the question stem that x, y and z are not 0. Hence, given |x + y| = |x| + |y|, we can infer that x and y have the same sign.

Statement 1: |y + z| = |y| + |z|

This implies that y and z have the same signs. But we have no information about the sign of x hence this statement alone is not sufficient.

Statement 2: |x + y| = |x| + |y|

This implies that x and y have the same signs. But we have no information about the sign of z hence this statement alone is not sufficient.

Using both statements together, we know that x, y and z have the same sign. Whatever is the sign of y and z, the same will be the sign of (y+z). Hence x and (y+z) have the same sign. This implies that x(y + z) cannot be negative.

Hence we can answer our question with a definite 'yes'.

Answer (C).

Mind you, both these questions can get time consuming (even though they aren't really tough) if you don't understand these properties well. You can certainly start your thinking from the scratch, arrive at the properties and then proceed or resort to more desperate measures such as number plugging but that is best avoided in DS questions.

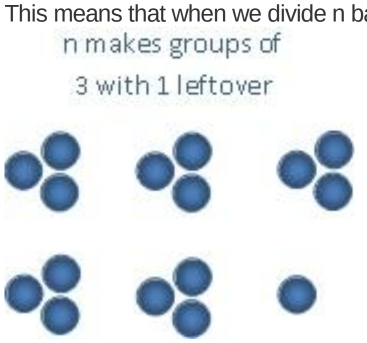
99. All about negative remainders

I could have sworn that I had discussed negative remainders on my blog but the other day I was looking for a post discussing it and much as I would try, I could not find one. I am a little surprised since this concept is quite useful and I should have taken it in detail while discussing divisibility. Though we did have a fleeting discussion of it here.

Since we did miss it, we will discuss it in detail today but you must review the link given above before we proceed.

Consider this: When n is divided by 3, it leaves a remainder 1.

This means that when we divide n balls in groups of 3 balls each, we are left with 1 ball.



This also means that n is 1 MORE than a multiple of 3. Or, it also means that n is 2 less than the next multiple of 3, doesn't it?

Say, n is 16. When you split 16 balls into groups of 3 balls each, you get 5 groups of 3 balls each and there is one ball leftover. n is 1 more than a multiple of 3 (the multiple being 15). But we can also say that it is 2 LESS than the next multiple of 3 (which is 18). Hence, the negative remainder in this case is -2 which is equivalent to a positive remainder of 1.

Generally speaking, if n is divided by m and it leaves a remainder r, the negative remainder in this case is -(m – r).

When n is divided by 7, it leaves a remainder of 4. This is equivalent to a remainder of -3.

n is 3 more than a multiple of m. It is also 2 less than the next multiple of m. This means m is 5.

This concept is very useful to us sometimes, especially when the divisor and the remainder are big numbers.

Let's take a question to see how.

Question 1: What is the remainder when 1555 * 1557 * 1559 is divided by 13?

(A) 0

(B) 2

(C) 4

(D) 9

(E) 11

Solution: Since it is a GMAT question (a question for which we will have no calculator), multiplying the 3 numbers and then dividing by 13 is absolutely out of question! There has to be another method.

Say n = 1555 * 1557 * 1559

When we divide 1555 by 13, we get a quotient of 119 (irrelevant to our question) and remainder of 8. So the remainder when we divide 1557 by 13 will be 8+2 = 10 (since 1557 is 2 more than 1555) and when we divide 1559 by 13, the remainder will be 10+2 = 12 (since 1559 is 2 more than 1557).

So n = (13*119 + 8)*(13*119 + 10)*(13*119 + 12) (you can choose to ignore the quotient and just write it as 'a' since it is irrelevant to our discussion)

So we need to find the remainder when n is divided by 13.

Note that when we multiply these factors, all terms we obtain will have 13 in them except the last term which is obtained by multiplying the constants together i.e. 8*10*12.

Since all other terms are multiples of 13, we can say that n is 8*10*12 (= 960) more than a multiple of 13. There are many more groups of 13 balls that we can form out of 960.

960 divided by 13 gives a remainder of 11.

Hence n is actually 11 more than a multiple of 13.

We did not use the negative remainders concept here. Let's see how using negative remainders makes our calculations easier here. The remainder of 8, 10 and 12 imply that the negative remainders are -5, -3 and -1 respectively.

Now n = (13a – 5) * (13a – 3) * (13a – 1)

The last term in this case is -5*-3*-1 = -15

This means that n is 15 less than a multiple of 13 i.e. actually 2 less than a multiple of 13 because when you go back 13 steps, you get another multiple of 13. This gives us a negative remainder of -2 which means the positive remainder in this case will be 11.

Here we avoided some big calculations.

I will leave you now with a question which you should try to solve using negative remainders.

Question 2: What is the remainder when 3^(7^11) is divided by 5? (here, 3 is raised to the power (7^11))

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Hint: I solved this question orally in a few secs using cyclicity and negative remainders. Don't get lost in calculations!

100. A tricky questions on Negative Remainders

Today, we will discuss the question we left you with last week. It involves a lot of different concepts – remainder on division by 5, cyclicity and negative remainders. Since we did not get any replies with the solution, we are assuming that it turned out to be a little hard.

It actually is a little harder than your standard GMAT questions but the point is that it can be easily solved using all concepts relevant to GMAT. Hence it certainly makes sense to understand how to solve it.

Question: What is the remainder when 3^(7^11) is divided by 5? (here, 3 is raised to the power (7^11))

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Solution: As we said last week, this question can easily be solved using cyclicity and negative remainders. What is the remainder when a number is divided by 5? Say, what is the remainder when 2387646 is divided by 5? Are you going to do this division to find the remainder? No! Note that every number ending in 5 or 0 is divisible by 5.

2387646 = 2387645 + 1

i.e. the given number is 1 more than a multiple of 5. Obviously then, when the number is divided by 5, the remainder will be 1. Hence the last digit of a number decides what the remainder is when the number is divided by 5.

On the same lines,

What is the remainder when 36793 is divided by 5? It is 3 (since it is 3 more than 36790 – a multiple of 5).

What is the remainder when 46^8 is divided by 5? It is 1. Why? Because 46 to any power will always end with 6 so it will always be 1 more than a multiple of 5.

On the same lines, if we can find the last digit of 3^(7^11), we will be able to find the remainder when it is divided by 5.

Recall from the discussion in your books, 3 has a cyclicity of 4 i.e. the last digit of 3 to any power takes one of 4 values in succession.

3^1 = <u>3</u>

3^2 = <u>9</u>

3^3 = 2<u>7</u>

3^4 = 8<u>1</u>

3^5 = 24<u>3</u>

3^6 = 72<u>9</u>

and so on… The last digits of powers of 3 are 3, 9, 7, 1, 3, 9, 7, 1 … Every time the power is a multiple of 4, the last digit is 1. If it is 1 more than a multiple of 4, the last digit is 3. If it is 2 more than a multiple of 4, the last digit is 9 and if it 3 more than a multiple of 4, the last digit is 7.

What about the power here 7^(11)? Is it a multiple of 4, 1 more than a multiple of 4, 2 more than a multiple of 4 or 3 more than a multiple of 4? We need to find the remainder when 7^(11) is divided by 4 to know that.

Do you remember the binomial theorem concept we discussed many weeks back? If no, check it out here.

7^(11) = (8 – 1)^(11)

When this is divided by 4, the remainder will be the last term of this expansion which will be (-1)^11. A remainder of -1 means a positive remainder of 3 (if you are not sure why this is so, check last week's post here). Mind you, you are not to mark the answer as (D) here and move on! The solution is not complete yet. 3 is just the remainder when 7^(11) is divided by 4.

So 7^(11) is 3 more than a multiple of 4.

Review what we just discussed above: If the power of 3 is 3 more than a multiple of 4, the last digit of 3^(power) will be 7.

So the last digit of $3^{(7^{11})}$ is 7.

If the last digit of a number is 7, when it is divided by 5, the remainder will be 2. Now we got the answer!

Answer (C)

Interesting question, isn't it?