10C Combinatorics & Probability

Combinatorics
area of mathematics that deals with arrangements of different items

Basic counting principles - number of spots • number of possibilites for that spot
Permutations - arrangements where the ***sequence*** of elements ***is important*** (seating, passwords, etc.)
Combinations - arrangements where the ***order does not matter*** (teams, items chosen, etc.)

Factorial	*Value*		*Factorial*	*Value*	
0!	1		4!	24	(4 · 3 · 2 · 1)
1!	1		5!	120	(5 · 4 · 3 · 2 · 1)
2!	2	(2 · 1)	6!	720	(6 · 5· 4 · 3 · 2 · 1)
3!	6	(3 · 2 · 1)			

Permutation
arrangement of N distinct elements where the ***order matters*** ("slot" for each item)

Number of arrangements of N distinct elements (placement of 1st item matters) **N !**

Number of arrangements of N distinct elements in a ***circle*** (placement of 1st item *arbitrary*) **(N - 1) !**

If there are repeating elements, must divide by number of arrangements of repeated elements. If A are identical ***and*** B are the identical: $\dfrac{N!}{A! \cdot B!}$

No. of arrangements of K distinct elements selected from a pool of N items $\quad {}_{N}P_{K} = \dfrac{N!}{(N-K)!}$

Note: usually easier to use basic counting principles than to use this formula for permutations.

Combination
a set of elements where the ***order is irrelevant***

Number of ***unordered*** arrangements consisting of K items selected from a pool of N items

Combination formula: $\quad {}_{N}C_{K} = \dfrac{N!}{K!\,(N-K)!} = \dfrac{{}_{N}P_{K}}{K!}$

Note: ${}_{N}C_{N} = 1$ and ${}_{N}C_{1} = {}_{N}P_{1} = N$

If choosing 1 from a set of **m** objects, another from **n** objects, different combinations of the two: **m · n**

General Approach to Combinatorics

0. Sketch it! Laying out specific arrangement(s) can help you to understand the problem.
1. Does **changing the order** of elements create a new/distinct arrangement?
   If *yes* (RGB ≠ GRB), then permutations (**P**). If *no* (RGB ≈ GRB), then combinations (**C**).
2. What is the structure of the problem? If number of items is small, consider writing out the possibilities.
   May items be repeated? If yes, use basic counting principles.
   Are elements selected from a larger pool (K < N)? If so, may need to use formula.
   Are any subgroups selected or arranged separately? If so, subdivide the problem.
3. Are any **elements identical?** If yes, remember the formula for permutations with repeating elements.
4. Is there a range of values of K? If yes, compute the no. of ways for each allowable number of elements, then sum.
   Look for trigger words ***at least***, ***at most***, and ***or*** (e.g. if ***at least*** K, calculate for K, K+1, etc.)
5. Are there any special conditions or constraints? Constraints - possible arrangements that are excluded
   Circular arrangement

Restrictions and Constraints

1. Solve the problem ***without the constraint***, ***then exclude cases*** that do not satisfy the constraint ***or***
2. Consider ***only*** the scenarios that ***meet the constraints*** and ***add up the total no.*** of possibilities

Probability
likelihood of the occurrence of a certain event. $0 \le P(A) \le 1$

Probability of 0 means no chance. Probability of 1 (100%) means the event is certain to occur.

Probability of an event A $\quad P(A) = \dfrac{\textbf{number of outcomes when A occurs}}{\textbf{total number of possible outcomes}}$

Mutually exclusive events ***can never occur together***. If events A and B are mutually exclusive, P(A **and** B) = 0

Events are ***complementary*** if one and ***only*** one of them ***must*** occur (such as a coin flip). P(A) + P(B) = 1

If an action results in ***only two possible results***, A or B, then P(A) = 1 – P(B)

Construct complementary events to simplify the problem (e.g. if *"at least one"*) P(A) = 1 – P(B)

Look for complementary answer choices: answer the right question; make educated guess

Binomial probabilities: successive events with 2 possible outcomes. For ≤3, write out; for >3, use counting principles.

General case: probability of either one of two events A **or** B occurring: P(A **or** B) = P(A) + P(B) – P(A **and** B)

For mutually exclusive events A and B, P(A and B) = 0, so the last term drops out.

Events A and B are ***independent*** if the occurrence of one event does ***not*** affect the probability of another.

For independent events A and B, the joint probability is the product of the probabilities: P(A and B) = P(A) • P(B)

Dependent events and ***conditional probability*** - events A and B occurring in sequence P(A and B) = P(A) • $\mathbf{P_A(B)}$

Events A and B are ***dependent*** if the occurrence of one event affects the probability of another.

If dealing with dependent probability, remember to change nos. for both specified outcomes and total outcomes.

Pairs probability a tricky type of dependent probability. First pick may not matter. "Any pair"≠ "Pair of twos"

If asked for probability of two events simultaneously, consider as "one at a time, without replacement."