Chapter 1: Divisibility & Primes

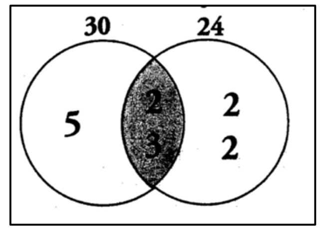
An integer is divisible by:

2	If the integer is even.	3	If the SUM of the integer's DIGITS is divisible by 3.	4	If the LAST TWO digits are divisible by 4.
5	if the integer ends in 0 or 5.	6	if the integer is divisible by BOTH 2 and 3.	7	Perform Long Division
8	if the integer is divisible by 2 THREE TIMFS, or if the LAST THREE digits are divisible by 8.	9	if the SUM of the integer's DIGITS is divisible by 9.	10	if the integer ends in 0.

- Fewer Factors, more multiples: Any integer only has a limited number of factors. By contrast, there is an infinite number of multiples of an integer.
- An easy way to find all the factors of a SMALL number is to use factor pairs
- One very helpful way to analyze a number is to break it down into its prime factors.
  - (1) Determining whether one number is divisible by another number
  - (2) Determining the greatest common factor of two numbers
  - (3) Reducing fractions
  - (4) Finding the least common multiple of two (or more) numbers
  - (5) Simplifying square roots
  - (6) Determining the exponent on one side of an equation with integer constraints

Small	Large
1	72
2	36
3	24
4	18
6	12
8	9

- **Greatest Common Factor (GCF)**: the largest divisor of two or more integers
- **Least Common Multiple (LCM):** the smallest multiple of two or more integers.
- **If *j* is divisible by 12 and 10, is *j* divisible by 24?** → Do not double count.



Chapter 2: Odds & Evens

- If there are X even integers in a set of integers being multiplied together, the result will be divisible by 2^X.
- The sum of any two primes will be even, unless one of those primes is the number 2.

To summarize so far:

Odd ± Even = ODD	Odd × Odd = ODD
Odd ± Odd = EVEN	Even × Even = EVEN (and divisible by 4)
Even ± Even = EVEN	Odd × Even = EVEN

Chapter 3: Positives & Negatives Strategies

If $ab > 0$, which of the following must be negative?

(A) $a + b$ (B) $|a| + b$ (C) $b - a$ (D) $\frac{a}{b}$ (E) $-\frac{a}{b}$

In these situations, you should set up a table listing all the possible positive/negative combinations of the variables.

Chapter 4: Consecutive Integers

- {12, 16,20, 24} is a set of **consecutive multiples** **Average = Sum/ Number**
- **Properties of Evenly Spaced Sets**
  - The arithmetic mean (average) and median are equal to each other.
  - The mean and median of the set are equal to the average of the FIRST and LAST terms.
- Remember that if both extremes should be counted, you need to add one before you are done.
  - Ex: How many integers are there from 14 to 765, inclusive?
  - Consecutive Multiples: ((Last – First) / Increment) + 1
    - Ex: All of the even integers between 12 and 24
- **The Sum of Consecutive Integers**:
  - Use the methods above: Find the median/Average and multiply by the number of integers.
    - Ex: What is the sum of all the integers from 20 to 100, inclusive?

- For any odd number of consecutive integers, the sum of those integers is divisible by the number ofintegers.
  - This means that the average is an integer. Only odd consecutive numbers have an integer as average. Thus, k is odd.
- **Products of Consecutive Integers and Divisibility**
  - The product of k consecutive integers is always divisible by k factorial (k!).
- **Sums of Consecutive Integers and Divisibility**
  - For any set of consecutive integers with an ODD number of items, the sum of all the integers is ALWAYS a multiple of the number of items. (Ex: 4 + 5 + 6 + 7 + 8 =30)
  - For any set of consecutive integers with an EVEN number of items, the sum of all the items is NEVER a multiple of the number of items. (Ex: 1+2 +3+4 =10)
- **Consecutive Integers and Divisibility:**
  - If x is an even integer, is x(x + l)(x + 2) divisible by 4?
    Use a prime box to keep track of factors of consecutive integers.
    (x is even, so it is divisible by 2). (x+2 Is even, so it is divisible by 2). The product is divisible by 2 x 2 = 4

Chapter 5: Exponents

- You can only simplify exponential expressions that are linked by multiplication or division if if they have either a base or an exponent in common. You cannot simplify expressions linked by addition or subtraction (although in some cases, you can factor them and otherwise manipulate them).
- Few problems:

$$\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^{3} = \frac{64}{27}$$

$$7^4+7^6 = 7^4(1+7^2)$$

If $x = 4^{20} + 4^{21} + 4^{22}$, what is the largest prime factor of x?

All three terms contain 4^{20}, so we can factor the expression: $x = 4^{20}(4^0 + 4^1 + 4^2)$. Therefore, $x = 4^{20}(1 + 4 + 16) = 4^{20}(21) = 4^{20}(3 \times 7)$. The largest prime factor of x is 7.

$25^{3/2} = \sqrt{25^3} = \sqrt{(5^2)^3} = 5^3 = 125$. We can also write $25^{3/2} = (5^2)^{3/2} = 5^{2\times\frac{3}{2}} = 5^3 = 125$.

INCORRECT	CORRECT
$(x+y)^2 = x^2 + y^2$? $(3+2)^2 = 3^2 + 2^2 = 13$?	$(x+y)^2 = x^2 + 2xy + y^2$ $(3+2)^2 = 3^2 + 2(3)(2) + 2^2 = 25$
$a^x \cdot b^y = (ab)^{x+y}$? $2^4 \cdot 3^5 = (2 \cdot 3)^{4+5}$?	Cannot be simplified further (different bases **and** different exponents)
$a^x \cdot a^y = a^{xy}$? $5^4 \cdot 5^3 = 5^{12}$?	$a^x \cdot a^y = a^{x+y}$ $5^4 \cdot 5^3 = 5^7$
$(a^x)^y = a^{(x+y)}$? $(7^4)^3 = 7^7$?	$(a^x)^y = a^{xy}$ $(7^4)^3 = 7^{12}$
$a^x + a^y = a^{x+y}$? $x^3 + x^2 = x^5$?	Cannot be simplified further (addition **and** different exponents)
$a^x + a^x = a^{2x}$? $2^x + 2^x = 2^{2x}$?	$a^x + a^x = 2a^x$ $2^x + 2^x = 2(2^x) = 2^{x+1}$
$a \cdot a^x = a^{2x}$? $5 \cdot 5^z = 25^z$?	$a \cdot a^x = a^{x+1}$ $5 \cdot 5^z = 5^{z+1}$
$-x^2 = x^2$? $-4^2 = 16$?	$-x^2$ cannot be simplified further $-4^2 = -16$ **Compare:** $(-x)^2 = x^2$ **and** $(-4)^2 = 16$
$-b^x = (-b)^x$?	

Exponent Rule	Examples
$x^a \cdot x^b = x^{a+b}$	$c^3 \cdot c^5 = c^8$ $3^5 \cdot 3^8 = 3^{13}$ $5(5^n) = 5^1(5^n) = 5^{n+1}$
$a^x \cdot b^x = (ab)^x$	$2^4 \cdot 3^4 = 6^4$ $12^5 = 2^{10} \cdot 3^5$
$\frac{x^a}{x^b} = x^{(a-b)}$	$\frac{2^5}{2^{11}} = \frac{1}{2^6} = 2^{-6}$ $\frac{x^{10}}{x^3} = x^7$
$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$	$\left(\frac{10}{2}\right)^6 = \frac{10^6}{2^6} = 5^6$ $\frac{3^5}{9^5} = \left(\frac{3}{9}\right)^5 = \left(\frac{1}{3}\right)^5$
$(a^x)^y = a^{xy} = (a^y)^x$	$(3^2)^4 = 3^{2\cdot4} = 3^8 = 3^{4\cdot2} = (3^4)^2$
$x^{-a} = \frac{1}{x^a}$	$\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ $2x^{-4} = \frac{2}{x^4}$
$x^{a/b} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$	$27^{4/3} = \sqrt[3]{27^4} = \left(\sqrt[3]{27}\right)^4 = 3^4 = 81$ $\sqrt[5]{x^{15}} = x^{15/5} = x^3$
$a^x + a^x + a^x = 3a^x$	$3^4 + 3^4 + 3^4 = 3 \cdot 3^4 = 3^5$ $3^x + 3^x + 3^x = 3 \cdot 3^x = 3^{x+1}$

Chapter 6: Roots (also called radicals)

$1^2 = 1$	$\sqrt{1} = 1$
$1.4^2 \approx 2$	$\sqrt{2} \approx 1.4$
$1.7^2 \approx 3$	$\sqrt{3} \approx 1.7$
$2.25^2 \approx 5$	$\sqrt{5} \approx 2.25$
$2^2 = 4$	$\sqrt{4} = 2$
$3^2 = 9$	$\sqrt{9} = 3$
$4^2 = 16$	$\sqrt{16} = 4$
$5^2 = 25$	$\sqrt{25} = 5$
$6^2 = 36$	$\sqrt{36} = 6$
$7^2 = 49$	$\sqrt{49} = 7$
$8^2 = 64$	$\sqrt{64} = 8$
$9^2 = 81$	$\sqrt{81} = 9$
$10^2 = 100$	$\sqrt{100} = 10$
$11^2 = 121$	$\sqrt{121} = 11$
$12^2 = 144$	$\sqrt{144} = 12$
$13^2 = 169$	$\sqrt{169} = 13$
$14^2 = 196$	$\sqrt{196} = 14$
$15^2 = 225$	$\sqrt{225} = 15$
$16^2 = 256$	$\sqrt{256} = 16$
$20^2 = 400$	$\sqrt{400} = 20$
$25^2 = 625$	$\sqrt{625} = 25$
$30^2 = 900$	$\sqrt{900} = 30$

- **Rule**: Even roots only have a positive value. SQRT (4) = 2, NOT ±2.
- Within the exponent fraction, the numerator tells us what power to raise the base to, and the denominator tells us which root to take.

$$\left(\frac{1}{8}\right)^{-4/3} = 8^{4/3} = \sqrt[3]{8^4} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16$$

- 216 = 2 x 2 x 2 x 3 x 3 x 3 = 6^3
- **Rule**: You can only simplify roots by combining or separating them in multiplication and division. You cannot combine or separate roots in addition or subtraction.

$$\sqrt{25\times16} = \sqrt{25}\times\sqrt{16} = 5\times4 = 20$$
$$\sqrt{50}\times\sqrt{18} = \sqrt{50\times18} = \sqrt{900} = 30$$

$$\sqrt{144 \div 16} = \sqrt{144} \div \sqrt{16} = 12 \div 4 = 3$$
$$\sqrt{72} \div \sqrt{8} = \sqrt{72 \div 8} = \sqrt{9} = 3$$

- **Imperfect Squares**: We can rewrite imperfect squares as a product of primes under the radical.

$$\sqrt{52} = \sqrt{2\times2\times13} = 2\times\sqrt{13}$$

Chapter 7: PEMDAS

- The correct order of operations is: Parentheses-Exponents-(Multiplication- Division)-(Addition-Subtraction).
- Please Excuse My Dear Aunt Sally.
- Pay attention to x – (y – z). Distribute.

Simplify: $\frac{x-1}{2} - \frac{2x-1}{3}$

Chapter 8: Strategies for Data Sufficiency

- Your first task in solving a data sufficiency problem is to rephrase the question and/or the statements whenever possible. After rephrasing the question, you should also try to rephrase each of the two statements, if possible.

If p is an integer, is $\frac{p}{18}$ an integer?

(1) $\frac{5p}{18}$ is an integer.

(2) $\frac{6p}{18}$ is an integer.

→ Are there two 3's and a 2 in the prime box of p

If x is a positive integer, is $x^3 - 3x^2 + 2x$ divisible by 4?

(1) $x = 4y + 4$, where y is an integer
(2) $x = 2z + 2$, where z is an integer

→ is x even?

- Types of Data Sufficiency Problems: Value vs. Yes/No

If n is an integer and n^3 is between 1 and 100, inclusive, what is the value of n?

(1) $n = 2k + 1$, where k is an integer.
(2) n is a prime number.

→ **Test Numbers**: n = 1,2,3, or 4

- **Test Smart Numbers**: try your best to find numbers that yield multiple answers for a Value question, or a MAYBE answer for a YES/NO question.

Is $x^2 \leq 2x$?

(1) $x > 0$
(2) $x < 3$

- Whenever you find that your two statements contradict each other, it means that you have made a mistake.

If y and n are positive integers, is yn divisible by 7?

(1) $n^2 - 14n + 49 = 0$
(2) $n + 2$ is the first of three consecutive integers whose product is 990.

Chapter 10: DIVISIBILITY & PRIMES: ADVANCED

- All the primes up to 100 (2, 3, 5,7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97)
- GCF and LCM for 3 numbers (or more):

Number:	2		5		7	
100	2^2	×	5^2		–	
140	2^2	×	5^1	×	7^1	
250	2^1	×	5^3		–	
GCF:	2^1	×	5^1			$= 2^1 \times 5^1 = 10$
LCM:	2^2	×	5^3	×	7^1	$= 2^2 \times 5^3 \times 7^1 = 3{,}500$

- Three general properties of the GCF and LCM are worth noting:
  - (GCF of m and n) x (LCM of m and n) =m x n
  - The GCF of m and n cannot be larger than the difference between m and n.
  - Consecutive multiples of n have a GCF of n.

Is the integer z divisible by 6?

(1) The greatest common factor of z and 12 is 3.
(2) The greatest common factor of z and 15 is 15.

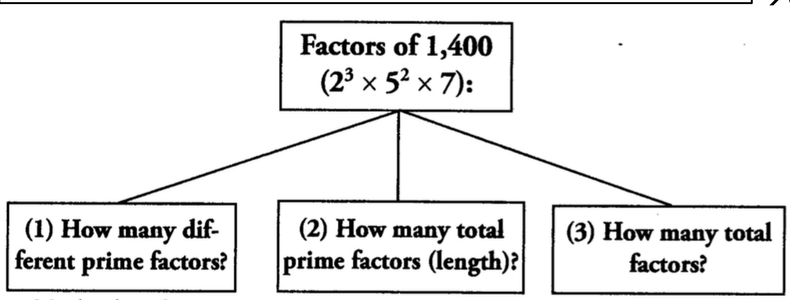
→Since z is not divisible by 2, it cannot be divisible by 6.

If j is divisible by 12 and 10, is j divisible by 24?

→ Do NOT double count

If the LCM of a and 12 is 36, what are the possible values of a?

→Use the table above

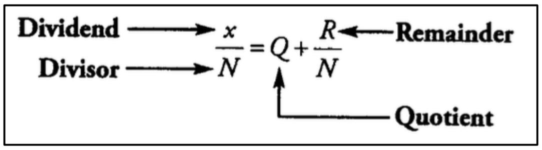


2	18
3	12
4	9
6	6

- Perfect squares always have an odd number of factors; Other integers always have an even number of factors
- Perfect squares are formed from the product of two copies of the same prime factors. Therefore, the prime factorization of a perfect square contains only even powers of primes.
- Prime factors of perfect squares MUST come in pairs; likewise, prime factors of perfect cubes MUST come in groups of 3.

> If k^3 is divisible by 240, what is the least possible value of integer k?
> (A) 12 (B) 30 (C) 60 (D) 90 (E) 120

→ Do the prime factorization of 240 and k^3

- N! is the product of all positive integers smaller than or equal to N. Therefore, N! must be divisible by all integers from 1 to N.
- 

> If $a \div b$ yields a remainder of 5, $c \div d$ yields a remainder of 8, and a, b, c and d are all integers, what is the smallest possible value for $b + d$?

→ b>=6, d>= 9

- Two useful tips for arithmetic with remainders, if you have the same divisor throughout:
  (1) You can add and subtract remainders directly, as long as you correct excess or negative remainders. "Excess remainders" are remainders larger than or equal to the divisor. To correct excess or negative remainders, just add or subtract the divisor.
  (2) You can multiply remainders, as long as you correct excess remainders at the end
- 17 /5 = 3.4. This quotient has an integer portion (3) and a decimal portion (0.4). The decimal portion represents the remainder 2 divided by 5.

> When positive integer A is divided by positive integer B, the result is 4.35. Which of the following could be the remainder when A is divided by B?
> (A) 13 (B) 14 (C) 15 (D) 16 (E) 17

→R/B = 7/20, 7B=2*2*5*R

- **Counting Total Factors**:
  - How many different factors does 2,000 have?
    - First, factor 2,000 into primes: $2{,}000 = 2^4 \times 5^3$.
      So the total number of factors of 2,000 must be (4 + 1)(3 + 1) = 5 x 4 = 20 different factors.
  - If a number has prime factorization $a^x \times b^y \times c^z$ (where a, b, and c are all prime), then the number has (x + 1)(y + 1)(z + 1) different factors.
- Interesting Problems:

> a, b and c are positive integers greater than 1. If $a < b < c$ and $abc = 286$, what is $c - b$?

> Is the sum of integers a and b divisible by 7?
> (1) a is not divisible by 7.
> (2) $a - b$ is divisible by 7.

> Statements (1) and (2) combined tell us that a is not divisible by 7, but $a - b$ is divisible by 7. This tells us that a and b have the SAME REMAINDER when divided by 7: if $a - b$ is divisible by 7, then the remainder of $a - b$ is zero. Therefore, the remainders of a and b must be equal.

Chapter 11: ODDS & EVENS / POSITIVES & NEGATIVES / CONSECUTIVE INTEGERS: ADVANCED

- **Special Case of Divisibility (Odds & Evens)**

> If x, y and z are integers and xyz is divisible by 8, is x even?
>
> (1) yz is divisible by 4
> (2) x, y, and z are all NOT divisible by 4

→ Use Prime Box

- Divisibility by 2 has a special property that divisibility by other numbers does not have. Recall from Chapter 10 that in general when we add or subtract two numbers, neither of which is divisible by x, we cannot tell whether the result will be divisible by x. However, when adding or subtracting two integers, neither of which is divisible by 2**, the result will always be divisible by 2**.
- <u>Remainder Rules to Remember:</u>
  - Odd integers are those integers that leave a remainder of 1 after division by 2.
  - Even integers are those integers that leave a remainder of 0 after division by 2.

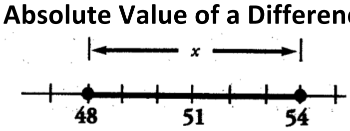
> If w, x, y and z are integers and $w + x = y$, is y divisible by z?
>
> (1) w and x each have a remainder of 1 when divided by z
> (2) $z = 2$

- **Representing Even & Odd algebraically**: Even Integer: 2n Odd Integer: 2n+1 or 2n-1

> What is the remainder when a is divided by 4 ?
> (1) a is the square of an odd integer.
> (2) a is a multiple of 3.

→ $4n^2 + 4n + 1$. Multiples of 4 have remainder 0

- **Absolute Value of a Difference**: 48<x<54 can be rewritten |x-51|<3



- **DisguisedPositive, &. Negative Questions:**

If $\frac{a-b}{c} < 0$, is $a > b$?

(1) $c < 0$
(2) $a + b < 0$ → (2) does not tell us if a>b; (2) is insuficient

Generally speaking, whenever you see inequalities with zero on either side of the inequality, you should consider testing positive/negative cases to help solve the ptoblem.

- **Complex Absolute Value Equations**: With an absolute value equation that contains <u>more than one variable</u> and <u>NO constants</u>, it is usually easiest to test positive/negative numbers to solve the problem.

If $|x| - |y| = |x + y|$ and xy does not equal 0, which of the following must be true?

(A) $x - y > 0$
(B) $x - y < 0$
(C) $x + y > 0$
(D) $xy > 0$
(E) $xy < 0$

Note that |x| has to be bigger than or equal to |y|, since |x|-|y| is equal to an absolute value and |x|-|y|>=0

x	y	$\lvert x\rvert - \lvert y\rvert$	$\lvert x + y\rvert$	Valid?
3	2	$\lvert 3\rvert - \lvert 2\rvert = 1$	$\lvert 3 + 2\rvert = 5$	NO
3	−2	$\lvert 3\rvert - \lvert -2\rvert = 1$	$\lvert 3 + (-2)\rvert = 1$	YES
−3	2	$\lvert -3\rvert - \lvert 2\rvert = 1$	$\lvert (-3) + 2\rvert = 1$	YES
−3	−2	$\lvert -3\rvert - \lvert -2\rvert = 1$	$\lvert (-3) + (-2)\rvert = 5$	NO

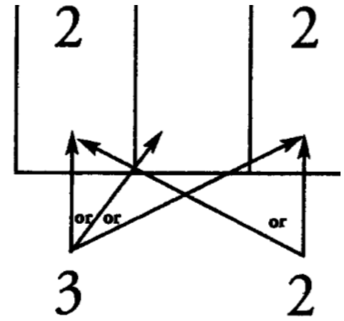
→ criterion: different sign

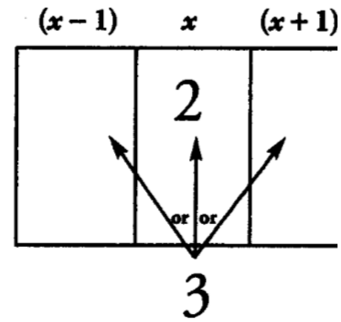
- **Consecutive Integers and Divisibility**:

  If $x^3 - x = p$, and x is odd, is p divisible by 24?

  → p = (x-1)(x)(x+1) and c is odd

  (x - 1) and (x + 1) are consecutive multiples of 2. So either (x - 1) or (x + 1) must have another 2 and be divisible by 4. Therefore, P is divisible by 8.
  In addition, one of the numbers – (x - 1), x, or (x + 1) – is divisible by 3, because in any set of 3 consecutive inte- gers, one of the integers will be a multiple of 3. We can therefore conclude that if x is odd, P will be divisible by at least 2 x 2 x 2 x 3 =24.





Chapter 12: EXPONENTS & ROOTS: ADVANCED

- **Simplifying Exponential Expressions:**

  Simplify the following expression: $\dfrac{(7^5+7^7)4^5}{8^3 \cdot 25}$

  4 Steps Process:
  (1) Simplify of factor any additive or substractive terms
  (2) Break every non-prime base into prime factors
  (3) Distribute the exponents to every prime factor
  (4) Combine the exponents for each prime factor and simplify

  A general rule of thumb is that when you encounter any exponential expression in which two or more terms include something common in the base, you should comider factoring. Similarly, when an expression is given in factored form, conslder distributing it.

DISTRIBUTED FORM		FACTORED FORM
$x^2 - x$	⟷	$x(x-1)$
$x^4 - x^2$	⟷	$x^2(x^2-1) = x^2(x+1)(x-1)$
$7^5 - 7^3$	⟷	$7^3(7^2-1) = 48 \cdot 7^3$
$5^8 + 5^9 + 5^{10}$	⟷	$5^8(1+5+5^2) = 31 \cdot 5^8$
$z^3 - z$	⟷	$z(z^2-1) = z(z+1)(z-1)$
$10^{(b+1)}$	⟷	$10(10^b)$
$10^{(b-1)}$	⟷	$\dfrac{(10^b)}{10}$
$3^5 + 3^5 + 3^5$	⟷	$3(3^5) = 3^6$
$a^b - a^{b-1}$	⟷	$a^b(1-a^{-1}) = a^{b-1}(a-1)$
$pq + pr + qs + rs$	⟷	$p(q+r)+s(q+r) = (p+s)(q+r)$

- **Simplifying Roots with Prime Factorization:**

  Ex: SQRT(180) = SQRT(3*3*2*2*5)

  To simplify a root, follow this procedure:
  (1) Factor the number under the radical sign into primes.
  (2) Pull out any pair of matching primes from under the radical sign, and place one of those primes outside the root.
  (3) Consolidate the expression.

- **Adding and Subtracting Roots**:

  Roots act like variables in addition and subtraction: you can only combine them if they are "like terms" or similar terms.

  $\sqrt{80} - \sqrt{45} = \sqrt{2\cdot2\cdot2\cdot2\cdot5} - \sqrt{3\cdot3\cdot5}$ →=SQRT(5)

  Therefore, you must simplify roots before you add or subtract them to see whether the final number under the radical is the same. Sometime roots that do not appear at first to be similar can in fact be combined.

- **Using Conjugates to Rationalize Denominators**

  However, simplifying a denominator that contains the sum or difference of a square root AND another term is more difficult:

  Simplify $\dfrac{4}{3-\sqrt{2}}$.

  To simplify this type of problem, we need to use **the conjugate** of the denominator. The conjugate for any square root expression involving addition or subtraction is defined as follows:

  For $a+\sqrt{b}$, the conjugate is given by $a-\sqrt{b}$.