Arithmetic Progression (AP), Geometric (GP) and Harmonic Progression (HP)

Arithmetic Progression (AP) Geometric (GP) and Harmonic Progression (HP):

Arithmetic Progression (AP)

The progression of the form: a, a + d, a + 2d, a + 3d … is known as an AP with *first term* = a, and *common difference* = d.

Arithmetic Progression (A.P.) *(or Arithmetic Sequence)*

Arithmetic progression (AP) is a sequence of numbers in which each term after the first is obtained by adding a constant, d to the preceding term. The constant d is called common difference.

An arithmetic progression is given by a, (a + d), (a + 2d), (a + 3d), ...
where a = the first term , d = the common difference

Examples for Arithmetic Progressions

1, 3, 5, 7, ... is an arithmetic progression (AP) with a = 1 and d = 2

7, 13, 19, 25, ... is an arithmetic progression (AP) with a = 7 and d= 6

Number of terms of an arithmetic progression

$$n = \frac{(l-a)}{d} + 1$$

where n = number of terms, a= the first term , l = last term, d= common difference

Sum of first n terms in an arithmetic progression

$$S_n = \frac{n}{2}\left[\,2a + (n-1)d\,\right] = \frac{n}{2}\left[\,a + l\,\right]$$

where a = the first term, d= common difference, $l = t_n =$ nth term $= a + (n-1)d$

Arithmetic Progression (AP), Geometric (GP) and Harmonic Progression (HP)

Arithmetic Mean

If a, b, c are in AP, b is the Arithmetic Mean (A.M.) between a and c.

In this case, $b = \frac{1}{2}(a + c)$

The Arithmetic Mean (A.M.) between two numbers a and b = $\frac{1}{2}(a + b)$

If a, a_1, a_2 … a_n, b are in AP we can say that a_1, a_2 … a_n are the n Arithmetic Means between a and b.

If a, b, c are in AP, 2b = a + c

To solve most of the problems related to A.P., the terms can be conveiently taken as

3 terms : (a − d), a, (a +d)

4 terms : (a − 3d), (a − d), (a + d), (a +3d)

5 terms : (a − 2d), (a − d), a, (a + d), (a +2d)

Relationship Between Arithmetic Mean, Harmonic Mean, and Geometric Mean of Two Numbers

If GM, AM and HM are the Geometic Mean, Arithmetic Mean and Harmonic Mean of two positive numbers respectively, then

$GM^2 = AM \times HM$

Harmonic Progression (H.P.) *(or Harmonic Sequence)*

Non-zero numbers $a_1,\ a_2,\ a_3,\ \ldots\ a_n$ are in H.P. if $\frac{1}{a_1},\ \frac{1}{a_2},\ \frac{1}{a_3},\ \ldots\ \frac{1}{a_n}$ are in A.P.

Examples for Harmonic Progressions

$\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \cdots$ is a harmonic progression (H.P.)

Three non-zero numbers a, b, c will be in HP, if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

If a, (a+d), (a+2d), . . . are in A.P., n^{th}term of the A.P. = a + (n - 1)d

Hence, if $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots$ are in H.P., n^{th}term of the H.P. = $\frac{1}{a+(n-1)d}$

If a, b, c are in HP, b is the Harmonic Mean(H.M.) between a and c

In this case, $b = \frac{2ac}{a+c}$

The Harmonic Mean (H.M.) between two numbers a and b = $\frac{2ab}{a+b}$

If a, a_1, a_2 ... a_n, b are in H.P. we can say that a_1, a_2 ... a_n are the n Harmonic Means between a and b.

If a, b, c are in HP, $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

Arithmetic Progression (AP), Geometric (GP) and Harmonic Progression (HP)

Geometric Progression (G.P.) *(or Geometric Sequence)*

A sequence of non-zero numbers is a Geometric Progression (G.P.) if the ratio of any term and its preceding term is always constant.

A Geometric Progression (G.P.) is given by a, ar, ar^2, ar^3, ... where a = the first term , r = the common ratio

Examples for Geometric Progressions

1, 3, 9, 27, ... is a geometric progression (G.P.) with a = 1 and r = 3

2, 4, 8, 16, ... is a geometric progression (G.P.) with a = 2 and r = 2

n^{th} term of a geometric progression (G.P.)

$t_n = ar^{n-1}$ where t_n = n^{th} term, a= the first term , r = common ratio, n = number of terms

Example 1 : Find the 10^{th} term in the series 2, 4, 8, 16, ...

a = 2, r = $\frac{4}{2}$ = 2, n = 10

10^{th} term, t_{10} = $ar^{n-1} = 2 \times 2^{10-1} = 2 \times 2^9 = 2 \times 512 = 1024$

Some Important Properties:

Three numbers a, b and c are in AP if $b = \frac{a+c}{2}$
Three non-zero numbers a, b and c are in HP if $b = \frac{2ac}{a+c}$
Three non-zero numbers a, b and c are in HP if $\frac{a-b}{b-c} = \frac{a}{c}$

Sum of an infinite geometric progression (G.P.)

$$S_\infty = \frac{a}{1-r} \quad (\text{if} \quad 0 < r < 1)$$

where a= the first term , r = common ratio

Example : Find $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$

a = 1, r = $\frac{\left(\frac{1}{2}\right)}{1} = \frac{1}{2}$

Here 0 < r < 1. Hence, $S_\infty = \frac{a}{1-r} = \frac{1}{\left(1-\frac{1}{2}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$

Sum of first n terms in a geometric progression (G.P.)

$$S_n = \begin{cases} \frac{a(r^n-1)}{r-1} & (\text{if } r > 1) \\ \frac{a(1-r^n)}{1-r} & (\text{if } r < 1) \end{cases}$$

where a= the first term , r = common ratio, n = number of terms

Example 1 : Find 4 + 12 + 36 + ... up to 6 terms

a = 4, r = $\frac{12}{4}$ = 3, n = 6

Here r > 1. Hence,

$$S_6 = \frac{a(r^n-1)}{r-1} = \frac{4(3^6-1)}{3-1} = \frac{4(729-1)}{2} = \frac{4 \times 728}{2} = 2 \times 728 = 1456$$

Relationship Between the Means of AP, GP and HP

If AM, GM and HM be the arithmetic, geometric and harmonic means between a and b, then the following results hold:

$$\text{AM} = \frac{a+b}{2} \qquad \ldots(i)$$

$$\text{GM} = \sqrt{ab} \qquad \ldots(ii)$$

$$\text{HM} = \frac{2ab}{a+b} \qquad \ldots(iii)$$

Therefore, we can write:

$$\text{AM} \times \text{HM} = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = \text{GM}^2$$

Or GM^2 = AM x HM........(iv)

Also, we have:

$$AM - GM = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b+2\sqrt{ab}}{2}$$

$$= \left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{2}}\right)^2 \qquad (v)$$

which is +ve if a and b are +ve; therefore, the AM of any two +ve quantities is greater than their GM.

Also, from equation (iv) we have, GM^2 = AM xHM

Clearly then, GM is a value that would fall between AM and HM and from equation (v) it is known that AM > GM, therefore we can conclude that GM > HM.

In words, we can say that the arithmetic, geometric and harmonic means between any two +ve quantities are in descending order of magnitude.