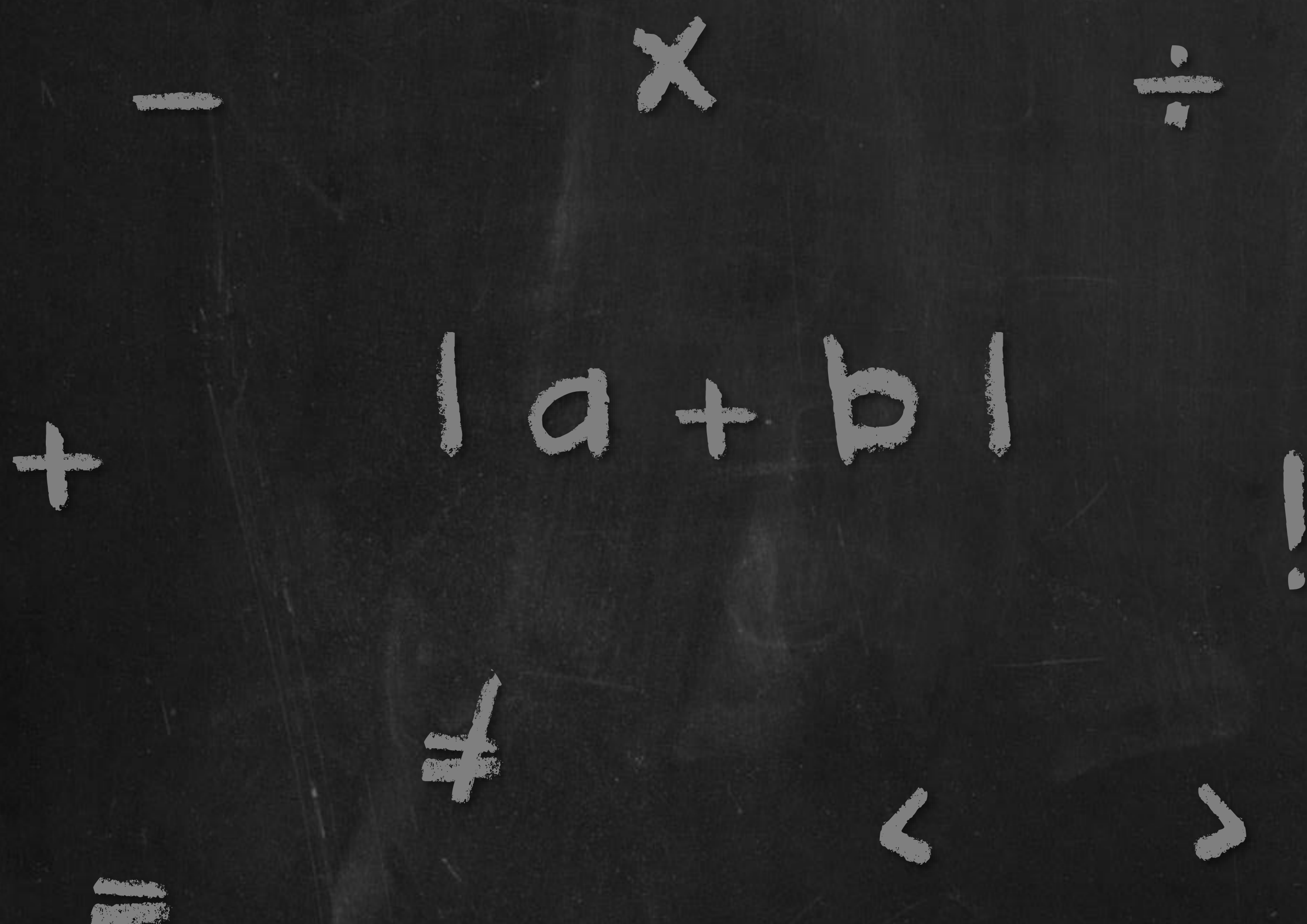
Different methods to solve absolute value equations and inequalities



By e-gmat.com

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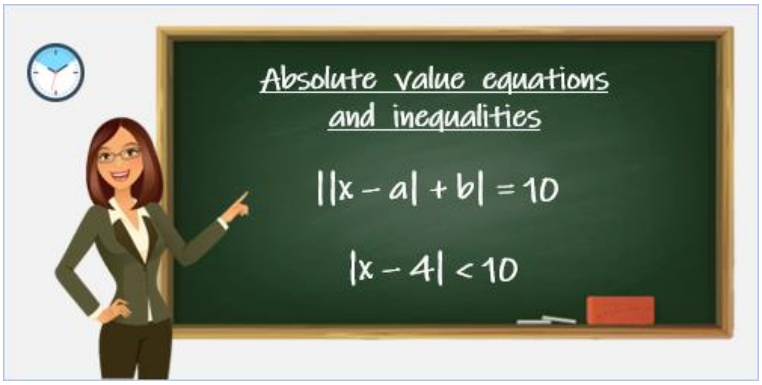

Objective of this article

- The primary objective of this article is to explain the application of absolute value while solving absolute value equations and inequalities.
- This article is helpful for those who have a basic understanding of absolute values and inequalities.
- After going through the article, one will be able to ace all levels of questions on absolute values that GMAT throws at you.

Agenda of the Article

- In this article, we will solve 3 examples to learn how to solve absolute value equations easily.
- And, we will also solve 3 more examples to understand how we can solve the questions that are related to the absolute value inequalities.

So, let us look at the first example.

Example 1

e-GMAT example with explanation

If x is an integer such that |x - 3| = 7. What is/are the values of x?

Solution:

We are given that x is an integer and the value of |x - 3| is 7, and we are asked to find out all the possible values of x.

Now, from the definition of absolute value, we know that:

- The value of |x| = x, if x ≥ 0, and
- The value of |x| = -x, if x < 0.

Note: *Many students may wonder, why |x| is written as -x, when absolute value represents the positive value of the number.*

- However, if you pay attention to the constraint that "*if x is negative*", then |x| is equal to -x.
- And, we all know that negative of a negative number is always positive.
  - Hence, |x| is always positive.

Therefore, it is utmost important for you to pay attention to the constraints.

For example: |3| is 3, since 3 is positive.

- And, |-3| is also 3.
  - Because -3 is negative. Hence |-3| will be - (-3) which is equal to 3.

Therefore, if we apply the definition of absolute value in the equation |x - 3| = 7, then we'll get two cases.

Case-1: *When x-3 ≥ 0.*

- Then, |x - 3| = x - 3, if x – 3 ≥ 0 or x ≥ 3,
  - Which also implies that x – 3 = 7 for x ≥ 3.
    - Thus, x = 10

Case-2: *When x-3 < 0.*

- Then, |x - 3| = - (x - 3), if x – 3 < 0 or x < 3,
  - Which implies, - (x – 3) = 7 for x < 3.
    - On multiplying by a negative sign in the above equation, we get: (x – 3) = -7
  - Thus, x = -4
  - Hence, the possible values of x are -4 and 7.

Now, if you carefully observe the two cases, then:

- In the first case, we have the equation as $x - 3 = 7$
- And, in the second case, we got the equation as $x – 3 = -7$.
    - So, generally, we can say that an equation of the form $|x-a|=b$ will give us two cases:
        - $x – a = b$ and $x - a = -b$
        - And, then we can solve it easily to find the value of x.

Let us quickly write down all our learnings from this question so that we can also apply them directly in further question.

Key Takeaways

1. We can see that the process to solve an absolute value equation is very easy. All we need to do is apply the basic concept of absolute values to solve this question.
2. We observed that if we have an equation in the form $|x-a|=b$ then we will get two cases:
    a. $x - a = b$ when $(x - a) \geq 0$.
    b. And, $x - a = -b$ when $(x - a) < 0$.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$|x - a| = b$$

- $x - a = b$ | $(x - a) \geq 0$
- $x - a = -b$ | $(x - a) < 0$

With these learnings in mind, let us now increase the difficulty a bit and solve another question.

Example 2

e-GMAT example with explanation

If x is an integer, then find the values of x that satisfy the equation $||x + 9| - 15| = 10$?

Solution:

As we can see that we are given one modulus inside another modulus:

- That is, $|x + 9|$ is inside $||x + 9| - 15|$.
- And, we are asked to find out all the possible values of x such that $||x + 9| - 15| = 10$.

So, how do we solve this question?

- Well, we will simply apply the basic properties of absolute values.
    1. We will first solve the outermost modulus.
        a. And, while solving it, we can substitute the inner modulus with an appropriate variable such as P or Q.
    2. Once we have solved the outer modulus, we will get the values for the assumed variable (P or Q) and then we can simplify the inner modulus, by substituting back the modulus.
    3. Now, this process will continue even if we had more than 2 moduli in the question.
        a. We will first solve the outermost and then we move from outermost modulus to the inner moduli.

So, let's apply the same process here.

- We will start from the outer modulus.
    - Therefore, let us substitute the inner modulus, $|x + 9| = y$.
        - Hence, we get $|y - 15| = 10$.
    - Now, we will get two cases:
        - First, $y – 15 = 10$ or $y = 25$, when $y – 15 \geq 0$ or $y \geq 15$.
        - Second, $y – 15 = -10$ or $y = 5$, when $y – 15 < 0$ or $y < 15$.
    - Then, we can substitute back y as $|x + 9|$, in $y = 5$ and $y = 25$ as shown below.
        - $|x + 9| = 5$.......... ①,
        - $|x + 9| = 25$......... ②
            - Thus, we have to solve these two equations and find the value of x.
        - Therefore, on solving the first equation, $|x + 9| = 5$, we get:
            - $x + 9 = 5$ or $x = -4$, when $x + 9 \geq 0$ or $x \geq -9$.
            - $x + 9 = -5$ or $x = -14$, when $x + 9 < 0$ or $x < -9$.
        - Similarly, on solving the second equation, $|x + 9| = 25$, we will get he two values of x as -34 and 16.

Therefore, the values of x are {-34, -14, -4, 16}

Key Takeaways

1. In this question, we learnt how to solve an absolute value equation involving more than 1 modulus.
    a. And, to do that, we always start simplifying the equation from the outer most modulus and then we move to inner moduli.
2. We also learnt that substituting the inner moduli as a variable helps us to solve the question easily.

With these learnings in mind, let us now increase the difficulty a bit and solve another question.

Example 3

e-GMAT example with explanation

Find the values of x, if |x - 13| =|2x + 6|, where x is an integer?

Solution:

Now, in this question, we have one modulus in the LHS and one modulus in the RHS.

- But, we have only learnt to solve the questions in which we have modulus and constant term on either of the side.
    - So, how should we go about to solve this question?
    - Well, do not panic. We will try to convert |x - 13| = |2x + 6| in to |x - a| = b form.
        - And, how should we do that?
        - We can substitute |2x + 6|= b and that will give us the equation as: |x - 13| = b.
            - Now, we can simply solve this.
            - Therefore, on solving the equation, we will get two cases:
                - x-13=b, when x ≥ 13 -------- ①
                - And, x-13= -b, when x <13 -------- ②
            - Now, we can substitute back b = |2x + 6| in both the equation.

So, let us first substitute in the equation, x - 13 = b.

Case-1

- Hence, we get x – 13 = |2x + 6| for $x \geq 13$
    - Let us assume that x - 13 is a constant.
    - Therefore, we will get two cases:
        - First, in which $(2x + 6) \geq 0$ or $x \geq -3$.
            - Second, in which $(2x + 6) < 0$ or $x < -3$.
        - However, don't forget to consider $x \geq 13$.
            - So, when **x** ≥ **-3** and $x \geq 13$, we get $2x + 6 = x - 13$
                - Now, if $x \geq -3$ and $x \geq 13$ then can we directly say that $x \geq 13$?
                    - Yes, we can.
                        - All the values that are greater or equal to 13 are also greater or equal -3.
                    - Hence, for $x \geq 13$, $2x + 6 = x - 13$
                        - Therefore, the value of x is -19.
                        - Now, tell me, is this possible that for $x \geq 13$, we can get x = 19?
                            - No, right?
                            - Hence, we can discard this case.
            - Now, when **x < -3** and $x \geq 13$, we get $2x + 6 = -(x - 13)$
                - But, is it possible that x can simultaneously be less than -3 and greater than 13?
                    - No, right?
                    - Hence, we will discard this equation.

Case-2

Let us now substitute b = |2x + 6| in the equation, x – 13 = -b.

- Hence, $x - 13 = -|2x + 6|$ or $2x + 6 = -(x - 13)$, when $x < 13$ and **x** ≥ **-3**
    - Or, $2x + 6 = -(x - 13)$, for $-3 \leq x < 13$
        - Therefore, $x = \frac{7}{3}$ and this also lies in the range.
            - But, can we take $x = \frac{7}{3}$?
                - No, we cannot.
                - Because, x is given as integer in the question.
                - Therefore, we can discard this case.
- Now, when $x < 13$ and **x < -3** then $2x + 6 = -(-(x - 13))$
    - Since x is less than -3 as well as 13, we can directly say that $x < -3$.
    - Thus, we get $2x + 6 = (x - 13)$, for $x < -3$
    - Therefore, x = -19 and we can select this value of x as it is less than -3 and also an integer.

Hence, we will get only one value of x and that is -19.

We can also solve this equation by an alternate method.

Alternate method:

Let's try to solve this question in a different way by applying the definition of $|x| = \sqrt{x^2}$

- We are given $|x - 13| = |2x + 6|$,
- Hence, we can write this as $\sqrt{(x-13)^2} = \sqrt{(2x+6)^2}$, since $|x| = \sqrt{x^2}$
- So, first, let us remove the square root by squaring the equation on both sides.
    - Therefore, $(x - 13)^2 = (2x + 6)^2$
    - And, expanding this, we get: $x^2 - 26x + 169 = 4x^2 + 24x + 36$
        - Which implies that $3x^2 + 50x - 133 = 0$
    - Or, we can express this quadratic equation as: $(x + 19)(3x - 7) = 0$
- Therefore, the values of x are -19 and $\frac{7}{3}$.
    - Since x is given as integer, we can only take x= -19.

Therefore, the value of x is -19.

Key Takeaways

- We solved this question by 2 methods:
    - By applying the basic concepts of absolute value
    - And, by applying the definition of $|x|$ that it is equal to $\sqrt{x^2}$.
- In method-1, we saw that we can substitute any one modulus to be a variable and then solve it to get the equations of the $|x - a| = b$.
    - So, if we are given an equation of the form $|x - a| = |x - b|$
        - Then, we can substitute $|x - b| = c$ and get $|x - a| = c$
            - Which will give us:
                - $x - a = c$ for $x \geq a$.
                - And, $x - a = -c$ for $x < a$.
            - Then we can substitute back $|x - a| = c$ and get the values of x.
- The second method involved a simple application of the definition of $|x| = \sqrt{x^2}$.
- You can apply any method to solve this type of questions.
    - However, make sure that the value of x you get, always lies within the considered range and also satisfies the constraint given in the question as we saw in case-1.

Now that we have seen different methods of solving absolute value equations by simple application of its definition, let's see how to solve absolute value inequalities.

Absolute Value Inequalities:

Before solving the questions on absolute value inequalities, let's brush up a few basics of absolute value inequalities:

Consider, $|x| > a$, where a is a positive number then what is the range of x?

- Well, we can solve this by applying the definition of $|x|$.
- The inequality $|x| > a$ can be written as:
    - $x > a$, if $x \geq 0$ .... ①
    - $-x > a$, if $x < 0$
        - Which implies that: $x < -a$ .... ②
        - Now, on combining ① and ②, we can say that if $|x|$ is greater than a, then $x > a$ and $x < -a$.

Now, consider, $|x| < a$, where a is a positive number, what will be the range of x?

- Applying the definition of $|x|$,
- The inequality, $|x| < a$ can be written as,
    - $x < a$, if $x \geq 0$............ ①
    - $-x < a$, if $x < 0$
        - Or, $x > -a$. .... ②
        - Now, on combining ① and ②, we can say that if $|x|$ is less than a, then $x > -a$ or $x < a$ or $-a < x < a$.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$|x - a| = b$

- $x - a = b$ $\quad (x - a) \geq 0$
- $x - a = -b$ $\quad (x - a) < 0$

Let's apply these inferences to solve a few questions.

Example 4

e-GMAT example with explanation

If |x - 8| > 20, what is the range of x?

Solution:

This is in the form of |x| > a

Therefore, the range of x is x > a and x < -a.

- Applying the same here, we get that, x – 8 > 20 or x - 8 < -20
  - Which implies that x > 28 or x < -12.

Thus, the range of x = (-∞, -12) U (28, ∞)

Example 5

e-GMAT example with explanation

If ||x - 3| - 8| < 6, what is the range of x?

Solution:

We solved a question of double modulus in absolute value equations.

So, we will follow the similar approach.

First, substitute |x - 3| as y, and write the inequality as,

- |y - 8| < 6
- Now, this is in the form of |x| < a
- Therefore, the range of x is -a < x < a.
  - Similarly, we can find the range of y if we substitute x by y-8 and a by 6.

    Thus, -6 < y-8< 6

    Adding 8 on all the sides of the inequality, we get: 2< y< 14.

Now, we can substitute back the value of y as |x - 3| and get:

- 2 < |x - 3| < 14
- Here, we need to consider two cases, when x – 3 is positive and when x - 3 is negative
  - Considering first case, x – 3 is positive, we have:
    - 2 < x - 3 < 14
    - Simplifying this inequality, we get
      - 5 < x < 17 .... ①

- Now, considering second case, x – 3 is negative, we have:
    - $2 < -(x-3) < 14$
    - So, let us now multiply by -1 on all the sides, we get
        - $-2 > x-3 > -14$ or $-14 < x-3 < -2$

Note: Did you notice that, the sign of the inequality is changed when you multiply the whole expression by -1?

So, always keep in mind to change the signs when you multiply an inequality by a negative number.

- Simplifying this inequality, we get
    - $-11 < x < 1$ .... ②

Therefore, the range of x is ① U ②, which is equal to (-11, 1) U (5, 17).

Key Takeaways

1. In this question, we learnt how to solve an inequality that involves more than 1 modulus.
2. We first substituted the value of inner modulus as y and found the range of y.
    - Then, we substituted back the inner modulus at the place of y and considered every possible case that is:
        - x – 3 as positive and negative to find the range of x.

With these learnings in the mind, let us discuss our final question of this article:

Key Takeaways from the article

- Then, we learnt to solve an equation of the $|x - a| = |x - b|$
  - We solved this question by two methods.
    - In first method, we substituted one of the modulus as a variable, let say C and then we solved it by using the concept of absolute value.
    - In the second method, we used the concept that $|x| = \sqrt{x^2}$ and it helped us to find the answer easily.
- We also saw the range of a few absolute inequalities.
  - For example: If $|x| > a$, then we get the range as: $x > a$ and $x < -a$.
  - And, if $|x| < a$, then we get the range as: $-a \le x \le a$
- And, at the end, we saw three questions to understand how we can solve absolute inequality questions.

Now, keeping all these learnings in mind, solve the practice questions.
Their link is given in the first comment of this article.

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