In this post, I would like to focus on a particular type of remainder questions and how to solve them in a particular way. For the type of questions I am going to discuss today, I like to use "Binomial Theorem." You might be tempted to run away right now and save yourself some precious time if you are not a Math geek but wait! We will just use an application of Binomial which I will explain in as simple a language as I can think of. I am quite certain that you will be comfortable with the method if you just give it a chance.

Question 1: What is the remainder when $(3^{84})/26$

(A) 0
(B) 1
(C) 2
(D) 24
(E) 25

First up, GMAT questions don't involve any painful calculations. So my thought is that there has to be an obvious link between 3 and 26. 26 is 1 less than the cube of 3. (It helps one to know the squares of first 20 numbers and cubes of first 10 numbers.)
So, $3^3 = 27$

But how is it going to help us? Now we come to binomial theorem. Let me start with something you already know.

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3ba^2 + 3ab^2 + b^3$

What about $(a + b)^4$ or $(a + b)^5$ or higher powers? Binomial theorem just tells us how to expand these expressions. It gives you a general formula:

$(a + b)^n = a^n + n*a^{(n-1)}*b + n(n-1)/2*a^{(n-2)}*b^2 +\ldots\ldots+ n*a*b^{(n-1)} + b^n$

I know the above looks intimidating but our concern is limited to the last term of the expression. Notice that every term above is divisible by 'a' except for the last term b^n. Every term but the last has 'a' as a factor. That is all you need to understand about Binomial Theorem.

Now for some quick applications:

What is the last term when you expand $(8 + 1)^{20}$? It is 1^{20} (which is just '1').

When you expand $(8 + 1)^{20}$, is every term divisible by 8? Yes, except for the last term, 1, because every term has 8 as a factor except for the last term.

If I divide $(8 + 1)^{20}$ by 8, what will be the remainder? Since every term (except for the last one) in the expansion of $(8 + 1)^{20}$ is divisible by 8, we can say that $(8 + 1)^{20}$ is 1 more than a multiple of 8. Hence the remainder when we divide it by 8 will be 1.

Or I can say that when I divide 9^{20} (which is just $(8 + 1)^{20}$) by 8, the remainder is 1.

Now let's look at our original question.

$(3^{84}) = (3^3)^{28} = 27^{28} = (26 + 1)^{28}$

Every term of $(26 + 1)^{28}$ will be divisible by 26 except for the last one. The last term will be $1^{28} = 1$. Hence, when you divide 27^{28} by 26, the remainder will be 1.

Answer (B).

All you had to do was to look for a power of 3 which is 1 more or 1 less than 26. We found that the third power of 3 is 1 more than 26. We adjusted the power to make 27 the base and split it into (26 + 1). We got the remainder as 1. Why do we necessarily look for a power 1 more or 1 less? We do that because 1^n is always 1. If we are left with 2^{28}, we again have a problem since we don't know what 2^{28} is. Let's use this concept in another problem now:

Question 2: What is the remainder of 2^{86} is divided by 9?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 8

I have added a few complications in this question. Let's tackle them one by one. We start by looking for a power of 2 which is 1 more or 1 less than 9. We know $2^3 = 8$ which is 1 less than 9.

Next, let's adjust the power to make the base 8.

$2^{86} = 8^?$

86 is not divisible by 3. The closest integer less than 86 that is divisible by 3 is 84. So, separate out two 2s and work with the rest of the 84 2s as of now.

$2^{86} = (2^2) * (2^{84}) = (4) * (2^3)^{28} = 4 * (8^{28})$

I am going to forget about the 4 for the time being.

$8^{28} = (9 – 1)^{28} = [9 + (-1)]^{28}$

Every term of this expression will be divisible by 9 except for the last term $(-1)^{28}$ which is again equal to 1.

Hence, 8^{28} will give a remainder 1 when divided by 9.

I can say that $8^{28} = 9m + 1$ where m is some positive integer. Now, we need to consider the 4 that we left out in the previous step. Our actual expression is

$4 * 8^{28} = 4 * (9m + 1) = 4*9m + 4$

When I divide this by 9, $4*9m$ is divisible by 9. So, $4*9m + 4$ is 4 more than a multiple of 9. Hence the remainder will be 4.

A question to ponder on: How will you solve this question if I change it to "What is the remainder of 2^{83} is divided by 9?"