A Probability Approach for Solving Counting Problems

There are some GMAT questions where solving is not the problem! The problem is time! Let me give a simple example:

How many times will the digit 7 be written when listing the integers from 1 to 1000?

A) 110
B) 111
C) 271
D) 300
E) 304

This is an easy question to solve. All you need in order to solve this question is a pen, a paper and time. On the GMAT you have a pen and a paper (actually you don't - you have a wipe off board and an erase marker, but that's another problem) but you don't have time. In order to solve this problem in the most basic possible way you simply need to write down the numbers from 1 to 1000 and just count the sevens – using an Excel spreadsheet it took me more then two minutes which is the allocated time per question on the quantitative section of the GMAT. (The detailed solution will be discussed later in this paper).

I have seen some very good shortcuts for solving this question (and others like it). But I would like to offer you a fast and foolproof method. My method is probability based and I call it "A probability approach for solving counting problems" method.

Now let Ψ be the ... don't worry! No Ψ is needed, just wanted to sound smart! (And as you know every good math/physics/finance paper starts with "let something be something")

So I would like to start with a simple problem, very simple problem:

How many integers from 1 to 100 including, has a units digit of 5?

A) 5
B) 9
C) 10
D) 11
E) 20

This is a 350 level question; all you need to do is to list the integers that has 5 as the units digit, I call this the basic method, meaning Count(5,15,25,35,45,55,65,75,85,95) = 10. The answer is (C).

Using the probability method we can see we have two places for digit in the integer XX and we need to fill in the tens digit and units digit (i.e. 33 or 51 or 03) each of them represent a number from 1 to 99 (for 01 = 1).

So after we know we have XX and we know that the total number of integers is 100 (from1 to 100 including). All we have to do is to find the probability that the second X in XX will be 5 and to multiply this probability in the number of integers given (i.e. N = 100).

For the second X we can only choose the number 5 from ten digits (0-9) so the probability is 1/10. The probability for forming a number that has a unit digit of 5 is P = 1/10.

And using the probability method

P*N = 100*1/10 = 10. The answer is (C).

As you can see the basic method is easier than the probability method but this is only true in simple problems. As the problems get more and more complex it will be harder to solve them in the time frame of two minutes in any method other than the probability method.

One footnote on the concept of XX: I said that you can choose out of ten digits for the first X and out of ten digits for the other X. This is not entirely true since 00 is not a valid integer (out of range) but 100 is also out of range (XXX) so these two integers cancel out each other (you will see that this will give us a headache in more complex problems).

How many integers between 324,700 and 458,600 have tens digit 1 and units digit 3?

A) 10,300
B) 10,030
C) 1,353
D) 1,352
E) 1,339

Using the probability method we need to find the number of integers (N) so here N = 458,600 – 324,700 = 133,900 and now we need to figure out the probability (P). Since we have two digits XX for the tens digit and units digit and the probability for the first X is 1/10 and the probability for unit digit of 3 is 1/10 then P = 1/10*1/10 = 1/100.

P*N = 1/100*133,900 = 1,339

And the answer is (E)

How many odd integers between 200 and 600 are there such that their tens digit is even?

A) 20
B) 25
C) 100
D) 150
E) 200

Using the probability method

N = 600-200 = 400

P = 5/10*5/10 = 25/100

5/10 is for the even tens and 5/10 is for the last digit being odd (thus) making the integer odd.

N*P = 25/100*400 = 100

The answer is (C)

How many 4-digit positive integers are there in which all 4 digits are even?

A) 625
B) 600
C) 500
D) 400
E) 256

Using the probability method

N = 10,000-1,000 = 9,000

P = 4/9*5/10*5/10*5/10 = 500/9000 = 5/90

N*P = 9,000*5/90 = 500

The answer is (C)

Since the first digit cannot be zero (i.e. 01111 is not a four digit number) thn the probability that the first digit will be even is 4/9. For the other three digits we can choose from 0 to 9 so 5/10.

How many 7 digit numbers are even and have a "3" in the hundreds place?

Using the probability method

N = 10,000,000 – 1,000,000 = 9,000,000

P = 1/10*5/10 = 5/100

N*P = 9,000,000*1/20 = 450,000

Since the number is even it has to end with an even digit, so 5/10 for the last digit, the hundreds place has to be 3 so 1/10 is the probability for that.

Of the three-digit integers greater than 700, how many have two digits that are equal to each other and the remaining digit different from the other two?

A) 90
B) 82
C) 80
D) 45
E) 36

Using the probability method

N = 1,000 – 700 = 300

P1 = 1/10*9/10 = 9/100
P2 = 9/10*1/10 = 9/100
P3 = 9/10*1/10 = 9/100

ΣP = 9/100+9/100+9/100 = 27/100

N*P = 300*27/100 = 81

As you can see in the given answers, there isn't such an answer 81. This is due to the fact that the answer asks us for numbers that are greater then 700 and less then 1,000. If we let N = 300 this means 700 and 1,000 are in the range. If you recall I warned you this will sometimes give us a headache. We need to remove 700 from N*P so the answer is N*P-1 = 81-1 = 80.

For calculating P we need to understand that there are three possible P's. In the first one (P1) the first digit is not important, let's say we choose 7. The second digit has to be also 7 so the probability is 1/10 and the last digit has to be different than 7 so 9/10.

In P2 the first digit is not important (let's say we choose 8) but the second digit can't be 8 so 9/10 and the last digit has to be 8 so 1/10.

In P3 the same logic is used. The first digit is not important – in the second digit we can choose nine out of ten digits (the only one we can't choose is the first digit) so – 9/10 and in the last digit we need to choose exactly the second digit so 1/10. We can think about it like this:

For P1 = 77x or 88x or 99x
For P2 = 7x7 or 8x8 or 9x9
For P3 = 7xx or 8xx or 9xx

The answer is (C).

How many times will the digit 7 be written when listing the integers from 1 to 1000?

A) 110
B) 111
C) 271
D) 300
E) 304

This is the problem we started with (see above) - Using the probability method to solve:

N = 1,000

When 7 appears three times (i.e. 777)

$P1_1$ = 1/10*1/10*1/10 = 1/1000

P1 = 1/1000*3 = 3/1000 (since we are counting the appearances of sevens!)

When 7 appears twice (i.e. X77 or 77X or 7X7)

$P2_1$ = 1/10*1/10*9/10 = 9/1000
$P2_2$ = 1/10*1/10*9/10 = 9/1000
$P2_3$ = 1/10*1/10*9/10 = 9/1000

1/10 is for picking 7 out of 10 digits and the 9/10 is for picking any other digit except 7.

P2 = 27/1000*2 = 54/1000 (since we are counting the appearances of sevens!)

$P3_1$ = 1/10*9/10*9/10 = 81/1000
$P3_2$ = 9/10*1/10*1/10 = 81/1000
$P3_3$ = 9/10*9/10*1/10 = 81/1000

P3 = 243/1000*1 = 243/1000 (since we are counting the appearances of sevens!)

When 7 appears once (i.e. XX7 or 7XX or X7X)

ΣP = 3/1000+54/1000+243/1000 = 300/1000

N*P = 1000*300/1000 = 300

The answer is (D)

Of the three-digit positive integers that have no digits equal to zero, how many have two digits that are equal to each other and the remaining digit different from the other two?

A) 24
B) 36
C) 72
D) 144
E) 216

Using the probability method

N = 1,000 – 100 = 900

P1 = 1/10*8/10 = 8/100
P2 = 8/10*1/10 = 8/100
P3 = 8/10*1/10 = 8/100

ΣP = 8/100+8/100+8/100 = 24/100

N*P = 900*24/1,000 = 216

The answer is (E)

For calculating P we need this time to understand that again there are three possible P's - but with a twist. In P1 the first digit is not important, let's say we choose 1. The second digit has to be also 1 but we have 10 digits to choose from (0-9) so the probability is 1/10 and the last digit has to be different than 1 but we cannot choose zero so 8/10.

In P2 the first digit is not important (let's say we choose 8) but the second digit can't be 8 and can't be zero so 8/10 and the last digit has to be 8 so 1/10.

In P3 the same logic is used. The first digit is not important (let's say we choose 5) – in the second digit we can choose 8 out of 10 digits (we can't choose 5 which we've already chosen, nor can we choose 0) so – 8/10 and in the last digit we need to choose the second digit exactly so 1/10.

Written by KillerSquirrel – 2007