There are not many mathematical concepts on the GMAT that cause the savviest of test-takers to cringe. In fact, difficulty on the GMAT quantitative section is typically achieved not through the testing of extremely difficulty concepts, but through the packaging of intermediate level concepts in challenging disguises. One exception to this rule: The concept of Absolute Value. For even the savviest of test-takers, Absolute Value is just a plain-old-hard concept.

In this month's strategy series we will take an up-close look at absolute value expressions. We will start with the basics - the definition of an absolute value expression, both conceptually and algebraically. We will then apply the definition to the solution of equations and inequalities involving absolute value expressions. Finally we will analyze how to deal with a more conceptual application of absolute value on the GMAT.

Absolutely positive. That is what everyone knows about an absolute value expression. /3/ = 3 and /-3/ = 3. But why do absolute value bars around a number neutralize, or do away with, the sign of the number? The answer is that absolute value bars are conceptually a measure of the distance of the expression inside the bars from zero. -3 and 3 are both three units away from zero. We will use these two ways of thinking absolute values-- (1) "the guarantee of positive" and (2) the measure of distance from zero--as the basis for delving deeper into absolute values.

Absolute value expressions start to become difficult when variable expressions are placed inside the bars. For example, /x/. Upon a cursory examination, the expression /x/ seems like it should be equal to x. Since there is no sign in front of the x, the absolute value bars should be able to be removed without jeopardizing the "guarantee of positive." What this line of reasoning fails to account for, however, is that x itself could be negative! ***When dealing with absolute value expressions that contain variables, two scenarios must be considered: (1) the scenario whereby the expression inside the bars is positive and (2) the scenario whereby the expression inside the bars is negative.***

In this example, for scenario (1) if x > 0, the expression /x/ can simply be represented as x; for scenario (2) if x < 0, the expression /x/ must be represented as (-x). ***Notice that in the negative scenario, we don't simply remove the absolute value bars. We remove the absolute value bars and negate the entire expression within.***

Let's look at a more complicated example: the expression /x - 3/. As always, we must consider both the positive and negative scenarios. When is the expression inside the absolute value bars positive? Not simply when x > 0, but when x - 3 > 0 or when x > 3. Likewise the expression will be negative when x < 3.

To recap, the two scenarios are:
(1) /x - 3/ can be rewritten as x - 3 when x > 3
(2) /x - 3/ can be rewritten as -(x - 3) or 3 - x when x < 3

One more for the road: /3x + y/.
(1) /3x + y/ can be rewritten as 3x + y when 3x + y > 0
(2) /3x + y/ can be rewritten as -(3x + y) when 3x + y < 0

To finish our discussion of the definition of absolute value expressions, let's return to the concept of the "measure of distance from zero." In our first example with an algebraic expression, /x/, the absolute value expression rather straightforwardly represents the distance of x from zero. But what about in the expression /x - 3/? We might think of this as the distance of "x - 3" from zero; however, this will rarely be useful on the GMAT. A more sophisticated way of looking at this is "the distance of x from three."

Let's see how this conceptualization is put into practice. If x is to the right of the number 3 on the number line (let's say x is four units to the right of 3, which would make x equal to 7), the expression x - 3 will be positive and the expression /x - 3/ (in this case 7 - 3 = 4 units) represents the distance between x and 3. If x is to the left of 3 on the number line (keeping x four units away from 3, would make x equal to -1), the expression x - 3 will be negative and the expression /x - 3/ (in this case 3 - (-1) = 4 units) again represents the distance between x and 3.

On 700+ level GMAT questions, it is sometimes useful to be able to think of the general expression /x - y/ as the distance of x from y. In data sufficiency questions, this line of thinking is sometimes faster than running through the different algebraic scenarios

Absolutely Positive: A close-up look at absolute values expressions on the GMAT (Part 2 of 4)

This week we continue our discussion of absolute values on the GMAT with a look at solving absolute value equations. The first step to solving an equation containing an absolute value expression is to remove the absolute value bars. To do so, however, we must consider both the positive and negative scenarios (i.e. see last week's discussion).

Consider the following example: /x/ + 8 = 12.
SCENARIO 1: If x is positive, then the absolute value bars can simply be dropped and the equation can be rewritten as: x + 8 = 12, so x = 4. Therefore, if x > 0, x = 4.

SCENARIO 2: If, however, x is negative, then the absolute value bars are dropped and the expression is negated, so the equation can be rewritten as (-x) + 8 = 12, so x = -4. Therefore, if x < 0, x = -4.

Now consider a slightly more difficult example: /x + 1/ = 8
SCENARIO 1: If (x + 1) is positive, x + 1 = 8 so x = 7.
Notice that the condition here is that x + 1 > 0 or x > -1
Therefore, if x > -1, x = 7

SCENARIO 2: If (x + 1) is negative, -(x + 1) = 8, so x = -9 |
Therefore, if x < -1, x = -9

Notice that in both of these examples, there were two solutions to the absolute value equation. That's because a variable absolute value expression behaves differently depending on the sign of the variable expression it contains within. Each variable absolute value expression has a breaking point, known as the critical point, where the behavior changes. This critical point is the "zeroing point" of the expression in the absolute value bars. For the simple expression /x/, the critical point is 0. All x's to the right of 0 on the number line reduce the expression /x/ to x; all x's to the left of 0 on the number line reduce the expression /x/ to (-x). For the expression /x - 2/, the critical point is 2; for the expression /2x + 3/ the critical point is -3/2. The critical point is important because it defines the range whereby each equation is true.

Now we can look at a more complicated example: /x + 1/ + /x - 3/ = 6.

Notice that this equation has two absolute value expressions, both in terms of the variable x. With two scenarios per absolute value expression, solving this equation could potentially involve solving four different equations! Because the absolute value expressions contain the same variable x, however, it turns out there are only three possible equations.

The easiest way to see the various scenarios is to track the critical points of the two absolute value expressions on a number line. The critical point of the expression /x + 1/ is -1 and the critical point of the expression /x - 3/ is 3. How many distinct regions are there on the number line with the points -1 and/or 3 as boundaries? The answer is three: x < -1, -1 < x < 3, and x > 3. (The theoretical "fourth" scenario here is not a possibility, i.e. 3 < x < -1.)

So what does the equation look like for each one of these three scenarios?

SCENARIO 1: when x < -1
(x + 1) is negative in this region and (x - 3) is also negative, so the equation becomes:
-(x + 1) - (x - 3) = 6, which simplifies to x = -2
With complex absolute value questions, the solution (x = -2) must be checked against the range for which the equation holds true (x < -1). In this case there is no conflict (since -2 is less than -1) so this is an actual solution.

SCENARIO 2: when -1 < x < 3 (x + 1) is positive in this region and (x - 3) is negative, so the equation becomes:
(x + 1) - (x - 3) = 6, which simplifies to 4 = 6, i.e. mathematical jibberish!
This means that there is no solution for the equation in this range.

SCENARIO 3: when x > 3 (x + 1) is positive in this region and (x - 3) is also positive, so the equation becomes:
(x + 1) + (x - 3) = 6, which simplifies to x = 4
This solution (x = 4) check out since it is in the range for which the equation hold true (x > 3).

Therefore, there are two potential solutions to this absolute value equation, x = -2 and 4. Conceptually, it is important to understand that because this equation had two absolute value expressions for the same variable x, there were two critical points and therefore three regions of "different behavior." This created the need for the investigation of three different scenarios/equations.

Absolutely Positive: A close-up look at absolute values expressions on the GMAT (Part 3 of 4)

We continue our discussion of absolute values on the GMAT this week with a look at inequalities containing absolute values expressions.

In general, inequalities are not that different from equations. In fact, most of the algebraic manipulations that can be used to solve equations can be applied to inequalities. There is however, one notable exception: when you multiply or divide an inequality by a negative value, you must change the direction of the inequality symbol. This may seem trivial, but the ramifications are numerous:

(1) You cannot multiply/divide an inequality by a variable unless you know the sign of that variable because you need to know whether or not to switch the direction of the inequality symbol. The inequality p/q > 1 does not necessarily imply that p > q (unless q is positive).

(2) You cannot take the square root of both sides of an inequality expression. x2 < 4 is not simply equivalent to x < 2 (it's actually equivalent to x < -2 OR x > 2).

(3) You can't square both sides of an inequality. x > y does not necessarily imply that x2 > y2.

(4) Finally, absolute value expressions must be treated with special care, since as we learned last week, absolute value expressions have two possible scenarios. Special care must be taken when considering the negative scenario.

This last point is of primary interest to us in our discussion of absolute values. Let's look at an example of a simple inequality containing an absolute value: /x/ <

2. Just as we did when working with equations, we must consider two scenarios for the absolute value expression.

Scenario 1: when $x > 0$, the inequality simply becomes $x < 2$.
Scenario 2: when $x < 0$, the inequality becomes $-x < 2$, which can be simplified to $x > -2$. Notice that the direction of the inequality symbol switches when we multiply by a negative.
The solution is $-2 < x < 2$

Now let's take a slightly more difficult example. $/x - 4 / > 8$
Scenario 1: when $x - 4 > 0$ (i.e. when $x > 4$), $x - 4 > 8$, which simplifies to $x > 12$
Scenario 2: when $x - 4 < 0$ (i.e. when $x < 4$), $-(x - 4) > 8$ which simplifies to $x < -4$
The solution is $x > 12$ OR $x < -4$.

More often than not, absolute value and inequalities show up as data sufficiency questions on the GMAT. Now that we have covered the basics of inequalities and absolute values, let's take a look at a sample GMAT data sufficiency question.

Is $x > 0$?
(1) $/x - 3/ < 5$
(2) $/x + 2/ > 5$

The first statement can be solved using the method described above.
Scenario 1: When $x > 3$, $x - 3 < 5$, which can be simplified to $x < 8$
Scenario 2: When $x < 3$, $-(x - 3) < 5$ or $x > -2$
Statement (1) can be simplified as $-2 < x < 8$, which is NOT sufficient to answer the question "is $x > 0$?"

The second statement can be solved in a similar manner.
Scenario 1: When $x > -2$, $x + 2 > 5$ or $x > 3$.
Scenario 2: When $x < -2$, $-(x + 2) > 5$ or $x < -7$.
Statement (2) can be simplified as $x > 3$ or $x < -7$, which is NOT sufficient to answer the question "is $x > 0$?"

Together, the two statements ARE SUFFICIENT to answer the question "is $x > 0$?" Since both statements are true (remember this is always the case in Data Sufficiency), x must be greater than 3 and less than 8. This is the only overlapping region between the ranges from statements (1) and (2). All of the values between 3 and 8 are positive so TOGETHER the statements are SUFFICIENT and the answer is C.

Absolutely Positive: A close-up look at absolute values expressions on the GMAT (Part 4 of 4)

This week we will conclude our discussion of absolute values on the GMAT with a more conceptual way of solving these expressions. In more difficult questions involving absolute value expressions on the GMAT, considering the positive and negative scenarios for each absolute value expression can be either overly tedious or altogether not useful. In these cases, it is preferable to take a more conceptual approach.

Consider the following sample data sufficiency question.

If $y = /x + 7/ + /2 - x/$, is $y = 9$?

(1) $x < 2$
(2) $x > -7$

This question could be solved using a systematic algebraic approach. It would start like this: the two absolute value expressions each have two scenarios that must be considered. If we consider the intersection of these four scenarios, we will come up with three different equations (Theoretically there should be four different equations given the two scenarios for each expression, however, one of the equations would never hold since x cannot be both less than -7 and greater than 2). The process of considering these three different equations would be quite tedious and time consuming.

If we apply a conceptual understanding of absolute values, we can greatly simplify the solution to this problem. As we learned earlier in the series, each absolute value expression has two scenarios, a positive and a negative one. Recall that these scenarios are true for a given range of values. For example, the first expression $/x + 7/$ has two scenarios: one when x is greater than -7 and one when x is less than -7. -7 is called the critical point of this absolute value expression. What is the critical point of the second absolute value expression in the above equation? 2. Notice that the critical point is the value of x that would cause the absolute value expression to be zero.

Now that we have the critical points of the two absolute value expressions in this equation, we are poised for a strategic, conceptual solution to this question. Because there are two critical points, the equation $y = /x + 7/ + /2 - x/$ will have three different forms, one for each of three critical regions: $x < -7$, $-7 < x < 2$, and $x > 2$.

The question asks us if this equation can be simplified to $y = 9$. Let's look at the statements. Statement (1) says that $x < 2$. Is $y = 9$ for all x's less than 2? If we plugged a few values for x less than 2, i.e. $x = 1, 0, -1$, we might come to the hasty decision that in fact y is always equal to 9 when $x < 2$. The problem with this set of numbers, however, is that it doesn't do justice to the critical regions in the problem. For statement (1) we must check values not only less than 2 but also less than -7. Otherwise we are artificially restricting the statement to one of the three critical ranges for the problem. In fact when we check $x = -10$, we see that y is not equal to 9, so statement (1) is not sufficient.

The same process can be applied when looking at statement (2) on its own. We must try values of x that fall in the two critical regions which meet the criteria in the statement. Thus, we must test values in the region $-7 < x < 2$ and values in the region $x > 2$. When we do, we find that y is not always equal to 9.

When we look at the two statements together, the only critical region to be considered is $-7 < x < 2$. Now the two statements are sufficient (y is always 9 in this region) and the correct answer is C, statements (1) and (2) TAKEN TOGETHER are sufficient to answer the question, but NEITHER statement ALONE is sufficient.

Let's take a look at another example.

If $/x/ + /y/ = /x + y/$ then which of the following must be true?

(A) $x + y > 0$
(B) $x + y < 0$
(C) $x - y > 0$
(D) $xy > 0$
(E) $xy < 0$

At first glance, one might consider writing out the different scenarios for the absolute value expressions in this equation, however, that would be a mistake. A conceptual approach is much better. Let's use the "guarantee of positive" that we spoke about in the first week to think about the equation $/x/ + /y/ = /x + y/$. The left side of the equation takes the variables x and y and makes them positive before they are added. The right side of the equation adds the variables first and then makes the result positive.

What must be true about x and y for the two sides of the equation to be equal? If x and y are both positive ($x = 2$, $y = 3$), both sides of the equation equal 5. If x and y are both negative ($x = -2$, $y = -3$), both sides of the equation still equal 5. If, however, one value is positive and the other is negative ($x = -2$, $y = 3$), the left side of the equation is 5, but the right side of the equation is 1. We see that for the two sides of the equation to be equal, x and y must have the same sign. This means that the product of x and y must be greater than zero, so the correct answer is D. Notice that the solution to this question did not involve breaking the absolute value expressions up into two scenarios, but instead involved a conceptual understanding of absolute value expressions.

In this strategy series we have covered a variety of approaches to solving absolute value expressions on the GMAT. From the algebraic to the conceptual, these methods should aid you in tackling even the most difficult absolute value questions.