THE GRE *Math Book*

prepclubforgre

THE GRE MATH BOOK

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CHAPTER 1. ARITHMETIC

1.1- INTEGERS

Frequency of the concepts tested: Very High

Definition

Integers are defined as: all negative natural numbers $\{\ldots, -4, -3, -2, -1\}$, zero $\{0\}$, and positive natural numbers $\{1,2,3,4, \ldots\}$. Note that integers do not include decimals or fractions - just whole numbers.

GRE Number Types

GRE deals with only Real Numbers: Integers, Fractions and Irrational Numbers.

Even and Odd Numbers

An even number is an *integer* that is "evenly divisible" by 2, i.e., divisible by 2 without a remainder. An even number is an integer of the form $n = 2k$, where k is an integer.

An odd number is an *integer* that is not evenly divisible by 2. An odd number is an integer of the form $n = 2k + 1$, where k is an integer. **Zero is an even number.**

Addition / Subtraction:

even +/- even = even;

even +/- odd = odd;

odd +/- odd = even.

Multiplication:

even * even = even;

even * odd = even;

odd * odd = odd.

Division of two integers can result into an even/odd integer or a fraction.

ZERO:

1. 0 is an integer.

2. 0 is an even integer. An even number is an integer that is "evenly divisible" by 2, i.e., divisible by 2 without a remainder and as zero is evenly divisible by 2 then it must be even.

3. 0 is neither positive nor negative integer (the only one of this kind).

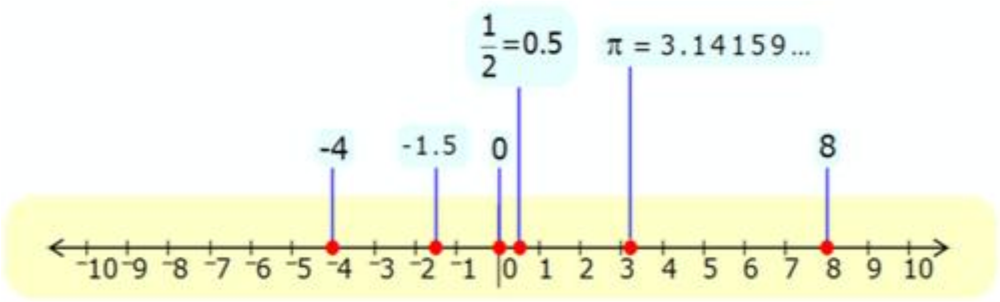
4. 0 is divisible by EVERY integer except 0 itself.

IRRATIONAL NUMBERS

Fractions (also known as rational numbers) can be written as *terminating* (ending) or *repeating* decimals (such as 0.5, 0.76, or 0.333333....). On the other hand, all those numbers that can be written as non-terminating, non-repeating decimals are non-rational, so they are called the "irrationals". Examples would be $\sqrt{2}$ ("the square root of two") or the number pi π =~3.14159..., from geometry). The rationals and the irrationals are two totally separate number types: there is no overlap.

Putting these two major classifications, the rationales and the irrationals, together in one set gives you the "real" numbers.

POSITIVE AND NEGATIVE NUMBERS



The numbers to the right of zero are called positive numbers and those to the left are called negative. The first ones must be written with the positive (+) sign in front and the second ones with the negative sign in front (-). These numbers are also identified as **signed numbers**.

Important facts are also being true and are worth remembering as cold.

1) for any number a, one and only one of the following statements are true

- a is negative
- a is = to zero
- a is positive

A positive number is a real number that is greater than zero.
A negative number is a real number that is smaller than zero.

Zero is not positive, nor negative.

Right away is important to denote that the absolute value of a number a, | a |, is the distance between a and zero on the number line, both on the positive side and negative one. The number a, for instance, 5, is 5 units apart from zero and -5 is 5 units apart from zero on the negative side. As such, their absolute value is |5|. In this scenario 5 and - 5 are opposite numbers. The only number that is equal to its opposite is **zero**

2) The product of any number * zero is zero. And if the product of two numbers is zero, then at least one of the two is zero.

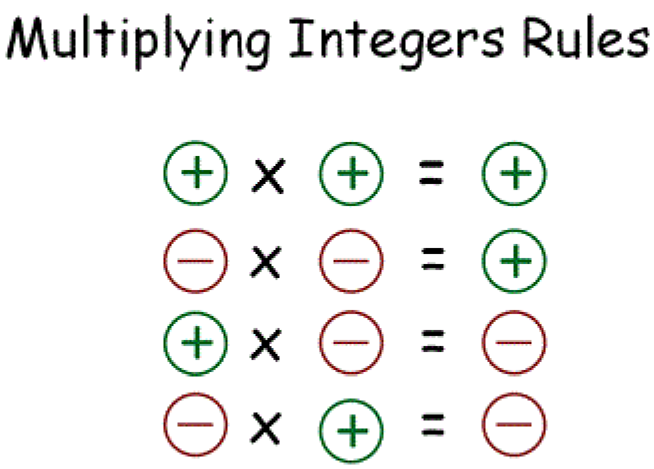
We need to consider that our numbers on the number line, and we are talking not only of integers but also NOT integers, can be the product or quotient of two positive numbers or two negative ones; or a mix: one positive or one negative.

3) The product and quotient of two positive numbers or two negative numbers are positive; the product and quotient of a positive number and a negative number are negative.

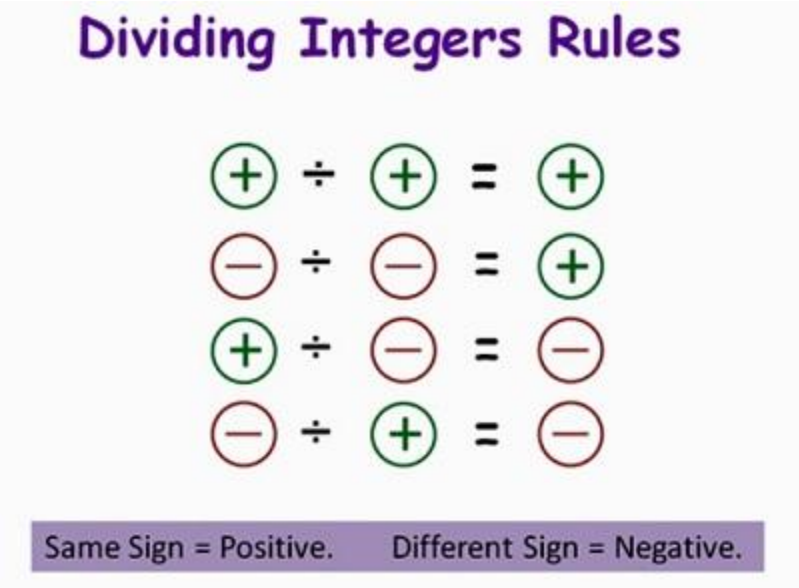
Multiplication:
positive * positive = positive

positive * negative = negative

negative * negative = positive



Division:
positive / positive = positive
positive / negative = negative
negative / negative = positive



Prime Numbers

A Prime number is a natural number with exactly two distinct natural number divisors: 1 and itself. Otherwise, a number is called a *composite* number. Therefore, **1 is not a prime**, since it only has one divisor, namely 1. A number $n > 1$ is prime if it cannot be written as a product of two factors a and b, both of which are greater than 1: n = ab.

• **The first twenty-six prime numbers are:**

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101

Note: only positive numbers can be primes.

• **There are infinitely many prime numbers.**

• The **only even prime** number is 2, since any larger even number is divisible by 2. Also **2 is the smallest prime**.

• **All prime numbers except 2 and 5 end in 1, 3, 7 or 9**, since numbers ending in 0, 2, 4, 6 or 8 are multiples of 2 and numbers ending in 0 or 5 are multiples of 5. Similarly, **all prime numbers above 3 are of the form** $6n - 1$ **or** $6n + 1$, because all other numbers are divisible by 2 or 3.

• Any nonzero natural number n can be factored into primes, written as a product of primes or powers of primes. Moreover, this **factorization is unique** except for a possible reordering of the factors.

• **Prime factorization:** every positive integer greater than 1 can be written as a product of one or more prime integers in a way which is unique. For instance, integer n with three unique prime factors a, b, and c can be expressed as $n = a^p * b^q * c^r$, where p, q, and r are powers of a, b, and c, respectively and are ≥ 1.

Example: $4200 = 2^3 * 3 * 5^2 * 7$.

• **Verifying the primality** (checking whether the number is a prime) of a given number n can be done by trial division, that is to say dividing n by all integer numbers smaller than $\sqrt{n}$, thereby checking whether n is a multiple of $m \leq \sqrt{n}$.

Example: Verifying the primality of 161: $\sqrt{161}$ is little less than 13, from integers from 2 to 13, 161 is divisible by 7, hence 161 is not prime.

Note that, it is only necessary to try dividing by *prime* numbers up to $\sqrt{n}$, since if *n* has any divisors at all (besides 1 and *n*), then it must have a prime divisor.

• If n is a positive integer greater than 1, then there is always a prime number p with $n < p < 2n$.

Factors

A divisor of an *integer* n, also called a factor of n, is an *integer* which evenly divides n without leaving a remainder. In general, it is said m is a factor of n, for non-zero integers m and n, if there exists an integer k such that $n = km$.

• 1 (and -1) are divisors of every integer.

• Every integer is a divisor of itself.

• Every integer is a divisor of 0, except, by convention, 0 itself.

• Numbers divisible by 2 are called even and numbers not divisible by 2 are called odd.

• A positive divisor of n which is different from n is called a *proper divisor*.

• An integer n > 1 whose only proper divisor is 1 is called a prime number. Equivalently, one would say that a prime number is one which has exactly two factors: 1 and itself.

• Any positive divisor of n is a product of prime divisors of n raised to some power.

• If a number equals the sum of its proper divisors, it is said to be a *perfect number*.

Example: The proper divisors of 6 are 1, 2, and 3: 1+2+3=6, hence 6 is a perfect number.

There are some elementary rules:

• If a is a factor of b and a is a factor of c, then a is a factor of $(b + c)$. In fact, a is a factor of$(mb + nc)$ for all integers m and n.

• If a is a factor of b and b is a factor of c, then a is a factor of c.

• If a is a factor of b and b is a factor of a, then $a = b$ or $a = -b$.

• If a is a factor of bc, and $gcd(a, b) = 1$, then a is a factor of c.

• If p is a prime number and p is a factor of ab then p is a factor of a *or* p is a factor of b.

Finding the Number of Factors of an Integer

First make prime factorization of an integer $n = a^p * b^q * c^r$, where a, b, and c are prime factors of n and p, q, and r are their powers.

The number of factors of n will be expressed by the formula $(p + 1)(q + 1)(r + 1)$.

NOTE: this will include 1 and *n* itself.

Example: Finding the number of all factors of 450: $450 = 2^1 * 3^2 * 5^2$

Total number of factors of 450 including 1 and 450 itself is $(1 + 1) * (2 + 1) * (2 + 1) = 2 * 3 * 3 = 18$ factors.

Greatest Common Factor (Divisor) - GCF (GCD)

The greatest common divisor (gcd), also known as the greatest common factor (gcf), or highest common factor (hcf), of two or more non-zero integers, is the largest positive integer that divides the numbers without a remainder.

To find the GCF, you will need to do prime-factorization. Then, multiply the common factors (pick the lowest power of the common factors).

• Every common divisor of a and b is a divisor of gcd(a, b).

• a*b=gcd(a, b)*lcm(a, b)

Lowest Common Multiple – LCM

The lowest common multiple or lowest common multiple (lcm) or smallest common multiple of two integers a and b is the smallest positive integer that is a multiple both of a and of b. Since it is a multiple, it can be divided by a and b without a remainder. If either a or b is 0, so that there is no such positive integer, then lcm(a, b) is defined to be zero.

To find the LCM, you will need to do prime-factorization. Then multiply all the factors (pick the highest power of the common factors).

Perfect Square

A perfect square, is an integer that can be written as the square of some other integer. For example, 16=4^2, is a perfect square. There are some tips about the perfect square:

• The number of distinct factors of a perfect square is ALWAYS ODD.
• The sum of distinct factors of a perfect square is ALWAYS ODD.
• A perfect square ALWAYS has an ODD number of Odd-factors, and EVEN number of Even-factors.
• Perfect square always has even number of powers of prime factors.

Divisibility Rules (Bonus content)

2 - If the last digit is even, the number is divisible by 2.

3 - If the sum of the digits is divisible by 3, the number is also.

4 - If the last two digits form a number divisible by 4, the number is also.

5 - If the last digit is a 5 or a 0, the number is divisible by 5.

6 - If the number is divisible by both 3 and 2, it is also divisible by 6.

7 - Take the last digit, double it, and subtract it from the rest of the number, if the answer is divisible by 7 (including 0), then the number is divisible by 7.

8 - If the last three digits of a number are divisible by 8, then so is the whole number.

9 - If the sum of the digits is divisible by 9, so is the number.

10 - If the number ends in 0, it is divisible by 10.

11 - If you sum every second digit and then subtract all other digits and the answer is: 0, or is divisible by 11, then the number is divisible by 11.

Example: to see whether 9,488,699 is divisible by 11, sum every second digit: 4+8+9=21, then subtract the sum of other digits: 21-(9+8+6+9) = -11, -11 is divisible by 11, hence 9,488,699 is divisible by 11.

12 - If the number is divisible by both 3 and 4, it is also divisible by 12.

25 - Numbers ending with 00, 25, 50, or 75 represent numbers divisible by 25.

Consecutive Integers

Consecutive integers are integers that follow one another, without skipping any integers. 7, 8, 9, and -2, -1, 0, 1, are consecutive integers.

• Sum of n consecutive integers equal the mean multiplied by the number of terms, n. Given consecutive integers $\{-3,-2,-1,0,1,2\}$, $mean = \frac{-3+2}{2} = -\frac{1}{2}$, (mean equals to the average of the first and last terms), so the sum equals to $-\frac{1}{2} * 6 = -3$.

• If n is odd, the sum of consecutive integers is always divisible by n. Given $\{9,10,11\}$, we have $n = 3$ consecutive integers. The sum of 9+10+11=30, therefore, is divisible by 3.

• If n is even, the sum of consecutive integers is never divisible by n. Given $\{9,10,11,12\}$, we have $n = 4$ consecutive integers. The sum of 9+10+11+12=42, therefore, is not divisible by 4.

• The product of nn consecutive integers is always divisible by n!.

Given $n = 4$ consecutive integers: {3,4,5,6}. The product of 3*4*5*6 is 360, which is divisible by 4!=24.

Evenly Spaced Set

Evenly spaced set or an arithmetic progression is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. The set of integers $\{9,13,17,21\}$ is an example of evenly spaced set. Set of consecutive integers is also an example of evenly spaced set.

• If the first term is a_1 and the common difference of successive members is d, then the n_{th} term of the sequence is given by: $a_n = a_1 + d(n-1)$.

• In any evenly spaced set the arithmetic **mean (average) is equal to the median** and can be calculated by the formula $mean = median = \frac{a_1+a_n}{2}$, where a_1 is the first term and a_n is the last term. Given the set $\{7,11,15,19\}$, $mean = median = \frac{7+19}{2} = 13$.

• The sum of the elements in any evenly spaced set is given by: $Sum = \frac{a_1+a_n}{2} * n$, the mean multiplied by the number of terms. OR, $Sum = \frac{2a_1+d(n-1)}{2} * n$

• Special cases:

Sum of n first positive integers: $1 + 2 + \ldots + n = \frac{1+n}{2} * n$

Sum of n first positive odd numbers: $a_1 + a_2 + \ldots + a_n = 1 + 3 + \ldots + a_n = n^2$, where a_n is the last, n_{th} term and given by: $a_n = 2n - 1$.

Given $n = 5$ first odd positive integers, then their sum equals to $1 + 3 + 5 + 7 + 9 = 5^2 = 25$.

Sum of n first positive even numbers: $a_1 + a_2 + \ldots + a_n = 2 + 4 + \ldots + a_n = n(n+1)$, where a_n is the last, n_{th} term and given by: $a_n = 2n$.

Given $n = 4$ first positive even integers, then their sum equals to $2 + 4 + 6 + 8 = 4(4+1) = 20$.

• If the evenly spaced set contains odd number of elements, the mean is the middle term, so the sum is middle term multiplied by number of terms. There are five terms in the set {1, 7, 13, 19, 25}, middle term is 13, so the sum is 13*5 =65.

ORDER OF OPERATIONS – PEMDAS

Perform the operations inside a **P**arenthesis first (absolute value signs also fall into this category), then **E**xponents, then **M**ultiplication and **D**ivision, from left to right, then **A**ddition and **S**ubtraction, from left to right - PEMDAS.

Special cases:

• An exclamation mark indicates that one should compute the factorial of the term immediately to its left, before computing any of the lower-precedence operations, unless grouping symbols dictate otherwise. But $3^2!$ means $(3^2)! = 9!$ While $2^{5!} = 2^{120}$; a factorial in an exponent applies to the exponent, while a factorial not in the exponent applies to the entire power.

• If exponentiation is indicated by stacked symbols, the rule is to work from the top down, thus: $a^{m^n} = a^{(m^n)}$ and not (a^m)^n

Divisibility Rules

Theory

Find Factors: Divisibility Rules help us in quickly identifying if a number is a factor of another number or not.

Save Time: If we use divisibility rules then we do not have to go for Long Division method to find out the factors of the number.

Divisibility Rule for divisibility by 2

There are multiple ways of checking if a number is divisible by 2 or not, 3 of them are listed below

- Number should be even
- Last digit of the number should be divisible by 2
- Units digit should be 0,2,4,6,8

Q1: Check if 360 is divisible by 2 or not.

Solution: There are multiple ways of checking it. 3 or them are given below:

360 is even so 360 divisible by 3
Last digit of 360 which is 0 is divisible by 2, so 360 divisible by 3
Unit digit of 360 is 0, so 360 divisible by 3

Divisibility Rule for divisibility by 3

Sum of all the digits of the number should be divisible by 3

Q2: Check if 360 is divisible by 3 or not.

Solution: Sum of all the digits of 360 = 3 + 6 + 0 =9
We know that 9 is divisible by 3 => 360 is divisible by 3

Divisibility Rule for divisibility by 4

Number formed by last two digits should be divisible by 4

Q3: Check if 360 is divisible by 4 or not.

Solution: Number formed by last two digits of 360 is 60.
We know that 60 is divisible by 4 => 360 is divisible by 4

Divisibility Rule for divisibility by 5

Number should end with 0 or 5

Q4: Check if 360 is divisible by 5 or not.

Solution: Since 360 ends with a 0 => 360 is divisible by 5

Divisibility Rule for divisibility by 6

Number should be divisible by both 2 and 3

Q5: Check if 360 is divisible by 6 or not.

Solution: 360 is divisible by both 2 and 3 (Check above problems)
=> 360 is divisible by 6

Divisibility Rule for divisibility by 7

Remove the last digit and double it and subtract it from the rest of the number.
If the result is divisible by 7 then number is divisible by 7, else it is not

Divisibility Rule for divisibility by 8

Number formed by last three digits should be divisible by 8

Q7: Check if 1360 is divisible by 8 or not.

Solution: Number formed by last three digits of 1360 is 360.
We know that 360 is divisible by 8 => 1360 is divisible by 8

Divisibility Rule for divisibility by 9

Sum of all the digits of the number should be divisible by 9

Q8: Check if 9360 is divisible by 9 or not.

Solution: Sum of all the digits of 9360 = 9 + 3 + 6 + 0 = 18
We know that 18 is divisible by 9 => 9360 is divisible by 9

Divisibility Rule for divisibility by 10

Number should end with 0

Q9: Check if 360 is divisible by 10 or not.

Solution: Since 360 ends with a 0 => 360 is divisible by 10

Divisibility Rule for divisibility by 11

If the difference of the sum of odd place digits and the sum of even place digits of the number is divisible by 11, then the number is divisible by 11, else it is not

Q10: Check if 1320 is divisible by 11 or not.

Solution: In 1320
Sum of odd places = 1 + 2 = 3
Sum of even places = 3 + 0 = 3

Sum of odd places - sum of even places = 3-3 =0
And 0 is divisible by all the numbers
=> 1320 is divisible by 11

Divisibility Rule for divisibility by 12

Number should be divisible by both 3 and 4

Q11: Check if 360 is divisible by 12 or not.

Solution: 360 is divisible by both 3 and 4 (Check above problems)
=> 360 is divisible by 12

Similarity in Divisibility rule for 2, 4 and 8

2 can be written as 2^1-> Rule -> Number formed by last 1 digit(s) should be divisible by 2
4 can be written as 2^2-> Rule -> Number formed by last 2 digit(s) should be divisible by 4
8 can be written as 2^3-> Rule -> Number formed by last 3 digit(s) should be divisible by 8

Similarity in Divisibility rule for 3 and 9

Divisibility Rule for 3: Sum of all the digits of the number should be divisible by 3
Divisibility Rule for 9: Sum of all the digits of the number should be divisible by 9

How to Solve: LCM, GCD and Properties of LCM and GCD

Properties of LCM and GCD

Prop 1:
LCM of 0 and any number does not exist
• LCM (0, a) = Does not exist

Prop 2:
GCD of 0 and any number is equal to the number itself. (Because all whole numbers are factors of 0. So, a will be also be a factor of 0.)
• GCD (0, a) = a

Prop 3:
LCM of 1 and any number is equal to the number itself
• LCM (1, a) = a

Prop 4:
GCD of 1 and any number is equal to 1
• GCD (1, a) = 1

Prop 5:
Product of two numbers = Product of their LCM and GCD
• a*b = LCM(a, b) * GCD(a, b)

Prop 6:
LCM of two numbers is always a multiple of their GCD
• LCM(a, b) = GCD(a, b) * k [where k is an integer]

Prop 7:
LCM of two numbers always lies between the larger of the two numbers and the product of those two numbers
• Larger (a, b) <= LCM (a, b) <= a*b

Prop 8:
GCD of two numbers is always smaller than or equal to the smaller of those two numbers
• GCD (a, b) <= Smaller (a, b)

Prop 9:
LCM of two numbers is equal to the larger of the two numbers when one number is multiple of other
• LCM(a, b) = Larger(a, b) => either a is a multiple of b or b is a multiple of a

Prop 10:
LCM of two numbers is equal to the product of those two numbers when the numbers are co-prime
• LCM(a, b) = a*b => a and b are co-prime numbers
Co-prime numbers are numbers which have only one factor in common (i.e. 1)

Prop 11:
GCD of two numbers is equal to the smaller of the two numbers when one number is multiple of other
• GCD(a, b) = Smaller(a, b) => either a is a multiple of b or b is a multiple of a

Prop 12:
GCD of two numbers is 1 when the numbers are co-prime
• GCD(a, b)=1 => a and b are co-prime numbers

Prop 13:
All numbers can be written as a multiple of their GCD
• a and b can be written as a multiple of their GCD
• a = GCD(a, b) * k [where k is an integer]
• b = GCD(a, b) * t [where t is an integer]

Prop 14:
All multiples of LCM are divisible by the numbers whose LCM we have calculated
• Multiples of LCM(a, b) will be divisible by both a and b

Prop 15:
a + b and a -b will always be divisible by GCD(a, b)

• $\frac{a+b}{GCD(a,b)} = k$[where k is an integer]

• $\frac{a-b}{GCD(a,b)} = t$[where t is an integer]

<u>Sample Word Problems:</u>

Prob 1: Joe has 12 A-grade projects and Chris has 18 B-grade projects. They have to divide up the projects into small teams with equal number of projects in each team; each team can have either A-grade project or B-grade project only. If there is no remainder, find the largest possible number of projects in each team.

Prob 2: Joe and Chris were each given a piece of wood of equal length. Joe cuts his wood piece into equal lengths of 12 cm, while Chris cuts his wood piece into equal lengths of 18 cm. If there was no remainder in both cases, find the shortest possible length of wood given to them.

Prob 3: Two steel pieces of length 340 cm and 408 cm are to be cut into pieces of all the same length without remainder. Find the greatest possible length of the pieces.

Solutions:

Prob 1: GCD (12,18) =6

Prob 2: LCM(12,18) = 36

Prob 3: GCD(340,408) = 68

1.2 - ABSOLUTE VALUE/MODULUS

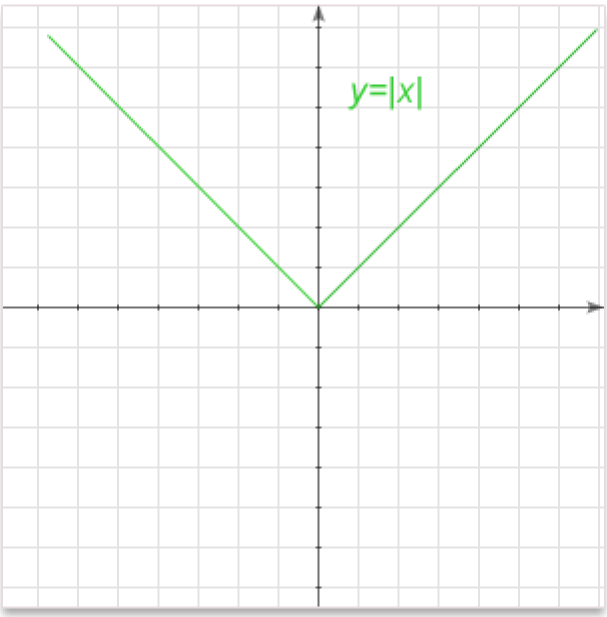
Frequency of the concepts tested: **Medium**

Definition

The absolute value (or modulus) $|x|$ of a real number x is x's numerical value without regard to its sign.

For example, $|3| = 3$; $|-12| = 12$; $|-1.3| = 1.3$

Graph:



Important properties:

- $|x| \geq 0$
- $|0| = 0$
- $|-x| = |x|$
- $|x| + |y| \geq |x + y|$
- $|x| \geq 0$

How to approach equations with moduli

It's not easy to manipulate with moduli in equations. There are two basic approaches that will help you out. Both of them are based on two ways of representing modulus as an algebraic expression.

1) $|x| = \sqrt{x^2}$. This approach might be helpful if an equation has × and /.

2) |x| equals x if x>=0 or -x if x<0. It looks a bit complicated but it's very powerful in dealing with moduli and the most popular approach too (see below).

3-steps approach:

General approach to solving equalities and inequalities with absolute value:

1. Open modulus and set conditions.

To solve/open a modulus, you need to consider 2 situations to find all roots:

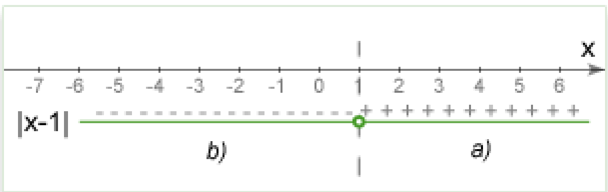
- Positive (or rather non-negative)
- Negative

For example, $|x - 1| = 4$

a) Positive: if $(x - 1) \geq 0$, we can rewrite the equation as: $x - 1 = 4$

b) Negative: if $(x - 1) < 0$, we can rewrite the equation as: $-(x - 1) = 4$

We can also think about conditions like graphics. $x = 1$ **This** is a key point where the expression under the modulus equals zero. All points right are the first condition $(x > 1)$ and all points left are second condition $(x < 1)$.



2. Solve new equations:

a) x−1=4$x - 1 = 4$ --> x=5

b) −x+1=4$-x + 1 = 4$ --> x=-3

3. Check conditions for each solution:

a) $x = 5$ It has to satisfy the initial condition $x - 1 >= 0.$ $5 - 1 = 4 > 0.$ It satisfies. Otherwise, we would have to reject x=5.

b) $x = -3$ It has to satisfy the initial condition $x - 1 < 0.$ $-3 - 1 = -4 < 0.$ It satisfies. Otherwise, we would have to reject x=-3.

3-steps approach for complex problems

Let's consider following examples,

Example #1

Q.: $|x + 3| - |4 - x| = |8 + x|$. How many solutions does the equation have?

Solution: There are 3 key points here: -8, -3, 4. So we have 4 conditions:

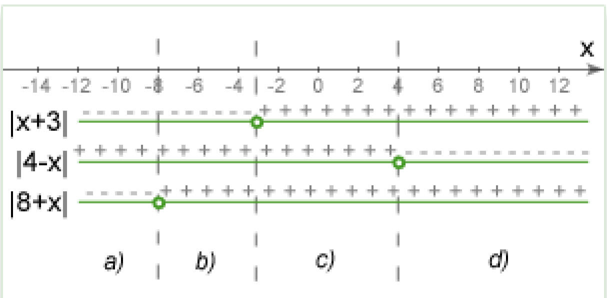
a) $x < -8.$ $-(x + 3) - (4 - x) = -(8 + x)$ --> $x = -1$. We reject the solution because our condition is not satisfied (-1 is not less than -8)

b) $-8 \leq x < -3.$ $-(x + 3) - (4 - x) = (8 + x)$ --> $x = -15$. We reject the solution because our condition is not satisfied (-15 is not within (-8,-3) interval.)

c) $-3 \leq x < 4.$ $(x + 3) - (4 - x) = (8 + x)$ --> $x = 9$. We reject the solution because our condition is not satisfied (9 is not within (-3,4) interval.)

d) $x \geq 4.$ $(x + 3) + (4 - x) = (8 + x)$ --> $x = -1$. We reject the solution because our condition is not satisfied (-1 is not more than 4)

(Optional) The following illustration may help you understand how to open modulus at different conditions



Answer: 0

Example #2

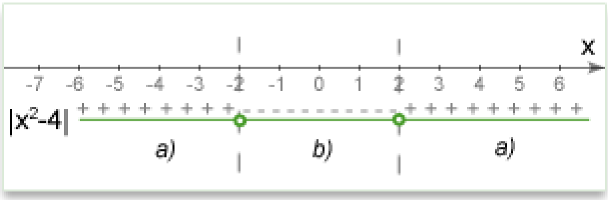
Q.: |x2−4|=1$|x^2 - 4| = 1$. What is x?

Solution: There are 2 conditions:

a) $(x^2-4) \geq 0$ --> $x \leq -2$ or $x \geq 2$. $x^2-4=1$ --> $x^2=5$. xe $\{-\sqrt{5}, \sqrt{5}\}$ and both solutions satisfy the condition.

b) $(x^2-4) < 0$ --> $-2 < x < 2$. $-(x^2-4)=1$ --> $x^2=3$. xe $\{-\sqrt{3}, \sqrt{3}\}$ and both solutions satisfy the condition.

(Optional) The following illustration may help you understand how to open modulus at different conditions.



Answer: $-\sqrt{5}, -\sqrt{3}, \sqrt{3}, \sqrt{5}$

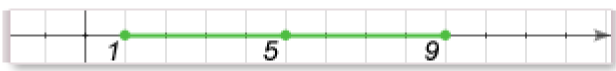
Tip & Tricks

The 3-steps method works in almost all cases. At the same time, often there are shortcuts and tricks that allow you to solve absolute value problems in 10-20 sec.

1. **Thinking of inequality with modulus as a segment at the number line.**

   For example,

   Problem: 1<x<9. What inequality represents this condition?



A. |x|<3

B. |x+5|<4

C. |x-1|<9

D. |-5+x|<4

E. |3+x|<5

Solution: 10sec. Traditional 3-steps method is too time-consume technique. First of all, we find length (9-1) =8 and center (1+8/2=5) of the segment represented by 1<x<9. Now, let's look at our options. Only B and D have 8/2=4 on the right side and D had left site 0 at x=5. Therefore, answer is D.

II. Converting inequalities with modulus into a range expression.

In many cases, especially in DS problems, it helps avoid silly mistakes.

For example, |x|<5 is equal to x e (-5,5). |x+3|>3 is equal to x e (-inf,-6) & (0,+inf)

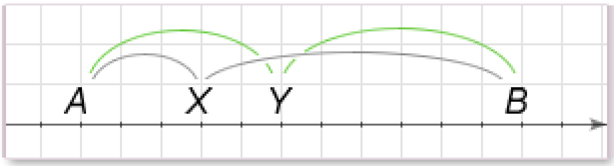
III. Thinking about absolute values as the distance between points at the number line.

For example,

Problem: A<X<Y<B. Is |A-X| <|X-B|?

1) |Y-A|<|B-Y|

Solution:



We can think of absolute values here as the distance between points. Statement 1 means than the distance between Y and A is less than that between Y and B. Because X is between A and Y, |X-A| < |Y-A| and at the same time the distance between X and B will be larger than that between Y and B (|B-Y|<|B-X|). Therefore, statement 1 is sufficient.

Pitfalls

The most typical pitfall is ignoring the third step in opening modulus - always check whether your solution satisfies conditions.

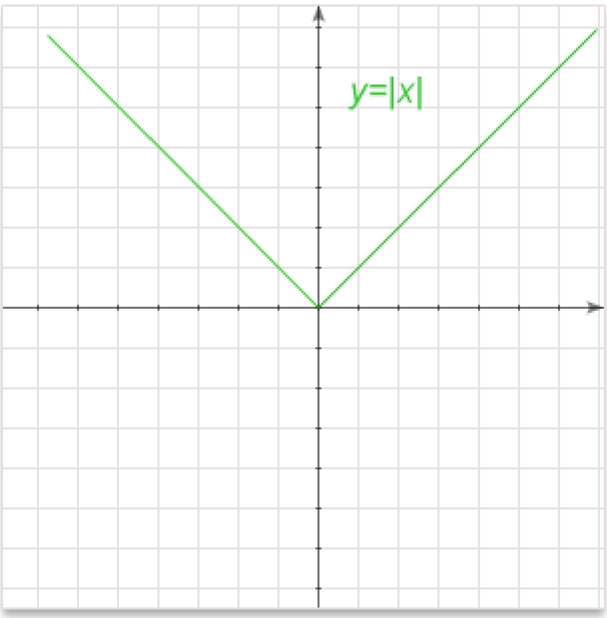
The absolute value of a number is the value of a number without regard to its sign.

For example, |3|=3 $|3| = 3$; |−12| = 12 $|-12| = 12$; |−1.3| = 1.3$|-1.3| = 1.3$...

Another way to understand absolute value is as the distance from zero. For example, $|x|$ is the distance between x and 0 on a number line.

From that comes the most important property of an absolute value: since the distance cannot be negative, **an absolute value expression is ALWAYS more than or equal to zero**.

Graph of y = |x|



As you can see for any value of x, the value of y, which is |x|, is ALWAYS more than or equal to zero.

IMPORTANT PROPERTY

When $x \leq 0$, then $|x| = -x$, or more generally when some expression ≤ 0 then |some expression = − (some expression). For example: $|-5| = 5 = -(-5)$.

Notice that in the negative scenario, we don't simply remove the absolute value bars. *We remove the absolute value bars and negate the entire expression contained within, thus making it positive again*;

When x ≥ 0 $x \geq 0$, then |x| = x $|x| = x$, or more generally when some expression ≥ 0 then |some expression| = some expression. For example: $|5| = 5$.

OTHER IMPORTANT PROPERTIES

1. $|x| \geq 0$
2. $\sqrt{x^2} = |x|$
3. $|0| = 0$

4. $|-x| = |x|$

5. $|x - y| = |y - x|$. |x - y| represents the distance between x and y, so naturally it equals to |y- x|, which is the distance between y and x.

6. $|x| + |y| \geq |x + y|$. Note that "=" sign holds for $xy \geq 0$ (or simply when x and y have the same sign). So, the strict inequality (>) holds when $xy < 0$;

7. $|x| - |y| \leq |x - y|$. Note that "=" sign holds for $xy > 0$ (so when x and y have the same sign) **and** $|x| \geq |y|$ (simultaneously).

1.3- FRACTIONS

Frequency of the concepts tested: **High**

Definition

Fractional numbers are ratios (divisions) of integers. In other words, a fraction is formed by dividing one integer by another integer. Set of Fraction is a subset of the set of Rational Numbers.

Fraction can be expressed in two forms *fractional representation* $(\frac{m}{n})$ and *decimal representation* $(a.bcd)$.

Fractional representation

Fractional representation is a way to express numbers that fall in between integers (note that integers can also be expressed in fractional form). A fraction expresses a part-to-whole relationship in terms of a numerator (the part) and a denominator (the whole).

• The number on top of the fraction is called *numerator* or *nominator*. The number on bottom of the fraction is called *denominator*. In the fraction, $\frac{9}{7}$, 9 is the numerator and 7 is denominator.

• Fractions that have a value between 0 and 1 are called *proper fraction*. The numerator is always smaller than the denominator. $\frac{1}{3}$ is a proper fraction.

• Fractions that are greater than 1 are called *improper fraction*. Improper fraction can also be written as a mixed number. $\frac{5}{2}$ is improper fraction.

• An integer combined with a proper fraction is called *mixed number*. $4\frac{3}{5}$ is a mixed number. This can also be written as an improper fraction: $\frac{23}{5}$

Converting Improper Fractions

Converting Improper Fractions to Mixed Fractions:
1. Divide the numerator by the denominator
2. Write down the whole number answer
3. Then write down any remainder above the denominator

Example #1: Convert $\frac{11}{4}$ to a mixed fraction.

Solution: Divide $\frac{11}{4} = 2$ with a remainder of 3. Write down the 2 and then write down the remainder 3 above the denominator 4, like this: $2\frac{3}{4}$

• Converting Mixed Fractions to Improper Fractions:
1. Multiply the whole number part by the fraction's denominator
2. Add that to the numerator
3. Then write the result on top of the denominator

Example #2: Convert $3\frac{2}{5}$ to an improper fraction.

Solution: Multiply the whole number by the denominator: $3 * 5 = 15$. Add the numerator to that: $15 + 2 = 17$. Then write that down above the denominator, like this: $\frac{17}{5}$

Reciprocal

Reciprocal for a number x, denoted by $\frac{1}{x}$ or x^{-1}, is a number which when multiplied by x yields 1. The reciprocal of a fraction $\frac{a}{b}$ is $\frac{b}{a}$. To get the reciprocal of a number, divide 1 by the number. For example reciprocal of 3 is $\frac{1}{3}$, reciprocal of $\frac{5}{6}$ is $\frac{6}{5}$.

Operation on Fractions

• **Adding/Subtracting fractions:**

To add/subtract fractions with the same denominator, add the numerators and place that sum over the common denominator.

To add/subtract fractions with the different denominator, find the Least Common Denominator (LCD) of the fractions, rename the fractions to have the LCD and add/subtract the numerators of the fractions.

• **Multiplying fractions:** To multiply fractions just place the product of the numerators over the product of the denominators.

• **Dividing fractions:** Change the divisor into its reciprocal and then multiply.

Example #1: $\frac{3}{7} + \frac{2}{3} = \frac{9}{21} + \frac{14}{21} = \frac{23}{21}$

Example #2: Given $\frac{\frac{3}{5}}{2}$, take the reciprocal of 2. The reciprocal is $\frac{1}{2}$. Now multiply: $\frac{3}{5} * \frac{1}{2} = \frac{3}{10}$.

Decimal Representation

The decimals have ten as its base. Decimals can be *terminating* (ending) (such as 0.78, 0.2) or *repeating* (recuring) decimals (such as 0.333333....).

Reduced fraction $\frac{a}{b}$ (meaning that fraction is already reduced to its lowest term) can be expressed as terminating decimal *if and only* b (denominator) is of the form $2^n 5^m$, where m and n are non-negative integers. For example: $\frac{7}{250}$ is a terminating decimal 0.028, as 250 (denominator) equals to $2 * 5^3$. Fraction $\frac{3}{30}$ is also a terminating decimal, as $\frac{3}{30} = \frac{1}{10}$ and denominator $10 = 2 * 5$.

Converting Decimals to Fractions

• To convert a terminating decimal to fraction:

1. Calculate the total numbers after decimal point
2. Remove the decimal point from the number
3. Put 1 under the denominator and annex it with "0" as many as the total in step 1
4. Reduce the fraction to its lowest terms

Example: Convert 0.56 to a fraction.

1: Total number after decimal point is 2.

2 and 3: $\frac{56}{100}$.

4: Reducing it to lowest terms: $\frac{56}{100} = \frac{14}{25}$

To convert a recurring decimal to fraction:

1. Separate the recurring number from the decimal fraction
2. Annex denominator with "9" as many times as the length of the recurring number
3. Reduce the fraction to its lowest terms

Example #1: Convert.$0.393939\ldots$ to a fraction.

1: The recurring number is 39.

2: $\frac{39}{99}$, the number 39 is of length 2 so we have added two nines.

3: Reducing it to lowest terms: $\frac{39}{99} = \frac{13}{33}$.

• To convert a mixed-recurring decimal to fraction:

1. Write down the number consisting with non-repeating digits and repeating digits.

2. Subtract non-repeating number from above.

3. Divide 1-2 by the number with 9's and 0's: for every repeating digit write down a 9, and for every non-repeating digit write down a zero after 9's.

Example #2: Convert $0.2512(12)$ to a fraction.

1. The number consisting with non-repeating digits and repeating digits is 2512;

2. Subtract 25 (non-repeating number) from above: 2512-25=2487;

3. Divide 2487 by 9900 (two 9's as there are two digits in 12 and 2 zeros as there are two digits in 25): 2487/9900=829/3300.

Rounding

Rounding is simplifying a number to a certain place value. To round the decimal, drop the extra decimal places, and if the first dropped digit is 5 or greater, round up the last digit that you keep. If the first dropped digit is 4 or smaller, round down (keep the same) the last digit that you keep.

Example:

5.3485 rounded to the nearest tenth = 5.3, since the dropped 4 is less than 5.

5.3485 rounded to the nearest hundredth = 5.35, since the dropped 8 is greater than 5.

5.3485 rounded to the nearest thousandth = 5.349, since the dropped 5 is equal to 5.

Ratios and Proportions

Given that $\frac{a}{b} = \frac{c}{d}$, where a, b, c and d are non-zero real numbers, we can deduce other proportions by simple Algebra. These results are often referred to by the names mentioned along each of the properties obtained.

$\frac{b}{a} = \frac{d}{c}$ – *invertendo*

$\frac{a}{c} = \frac{b}{d}$ – *alternendo*

$\frac{a+b}{b} = \frac{c+d}{d}$ – *componendo*

$\frac{a-b}{b} = \frac{c-d}{d}$ – *dividend*

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$ - *componendo & dividend*

1.4- EXPONENTS

Frequency of the concepts tested: **High**

Exponents are a "shortcut" method of showing a number that was multiplied by itself several times. For instance, number a multiplied n times can be written as a^n, where a represents the base, the number that is multiplied by itself n times and n represents the exponent. The exponent indicates how many times to multiple the base, a, by itself.

Exponents one and zero:
$a^0 = 1$ Any nonzero number to the power of 0 is 1.
For example: $5^0 = 1$ and $(-3)^0 = 1$

• Note: the case of 0^0 is not tested on the GRE.
$a^1 = a$ Any number to the power 1 is itself.

Powers of zero:
If the exponent is positive, the power of zero is zero: $0^n = 0$, where $n > 0$.

If the exponent is negative, the power of zero (0^n, where $n < 0$) is undefined, because division by zero is implied.

Powers of one:
$1^n = 1$ The integer powers of one are one.

Negative powers:
$$a^{-n} = \frac{1}{a^n}$$

Powers of minus one:
If n is an even integer, then $(-1)^n = 1$.
If n is an odd integer, then $(-1)^n = -1$.

Operations involving the same exponents:
Keep the exponent, multiply or divide the bases $a^n * b^n = (ab)^n$

$$\frac{a^n}{b^n} = (\frac{a}{b})^n$$

$$(a^m)^n = a^{mn}$$

$a^{m^n} = a^{(m^n)}$ and not $(a^m)^n$

Operations involving the same bases:

Keep the base, add or subtract the exponent (add for multiplication, subtract for division)

$a^n * a^m = a^{n+m}$

$\frac{a^n}{a^m} = a^{n-m}$

Fraction as power:

$a^{\frac{1}{n}} = \sqrt[n]{a}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Exponential Equations:

When solving equations with *even exponents*, we must consider both positive and negative possibilities for the solutions.
For instance, $a^2 = 25$, the two possible solutions are 5 and -5.

When solving equations with *odd exponents*, we'll have only one solution.
For instance, for $a^3 = 8$, solution is $a = 2$ and for $a^3 = -8$, solution is $a = -2$.

Exponents and divisibility:

$a^n - b^n$ is ALWAYS divisible by $a - b$.
$a^n - b^n$ is divisible by $a + b$ if n is even.
$a^n + b^n$ is divisible by $a + b$ if n is odd, and not divisible by a+b if n is even.

LAST DIGIT OF A PRODUCT

Last n digits of a product of integers are last n digits of the product of last n digits of these integers.

For instance last 2 digits of 845*9512*408*613 would be the last 2 digits of 45*12*8*13=5**40***1**04**=40*4=1**60**=60

Example: The last digit of 8594**5***8**9***5830**7**=5*9*7=4**5***7=3**5**=5?

LAST DIGIT OF A POWER

Determining the last digit of $(xyz)^n$:

1. Last digit of $(xyz)^n$ is the same as that of z^n;

2. Determine the cyclicity number c of z;

3. Find the remainder r when n divided by the cyclicity;

4. When $r > 0$, then last digit of $(xyz)^n$ is the same as that of z^r and when $r = 0$, then last digit of $(xyz)^n$ is the same as that of z^c, where c is the cyclisity number.

• Integer ending with 0, 1, 5 or 6, in the integer power k>0, has the same last digit as the base.

• Integers ending with 2, 3, 7 and 8 have a cyclicity of 4.

• Integers ending with 4 (eg. $(xy4)^n$) have a cyclisity of 2. When n is odd $(xy4)^n$ will end with 4 and when n is even $(xy4)^n$ will end with 6.

• Integers ending with 9 (eg. $(xy9)^n$) have a cyclisity of 2. When n is odd $(xy9)^n$ will end with 9 and when n is even $(xy9)^n$ will end with 1.

Example: What is the last digit of 127^{39}?

Solution: Last digit of 127^{39} is the same as that of 7^{39}.

Now we should determine the cyclicity of 7:

1. 7^1=7 (last digit is 7)
2. 7^2=9 (last digit is 9)
3. 7^3=3 (last digit is 3)
4. 7^4=1 (last digit is 1)
5. 7^5=7 (last digit is 7 again!)

...

So, the cyclicity of 7 is 4.

Now divide 39 (power) by 4 (cyclicity), remainder is 3. So the last digit of 127^{39} is the same as that of the last digit of 7^{39}, is the same as that of the last digit of 7^3, which is 3.

ROOTS

Roots (or radicals) are the "opposite" operation of applying exponents. For instance, $x^2 = 16$ and square root of 16=4.

General rules:

- $\sqrt{x}\sqrt{y} = \sqrt{xy}$ and $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$.

- $(\sqrt{x})^n = \sqrt{x^n}$

- $x^{\frac{1}{n}} = \sqrt[n]{x}$

- $x^{\frac{n}{m}} = \sqrt[m]{x^n}$

- $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

- $\sqrt{x^2} = |x|$, when $x \leq 0$, then $\sqrt{x^2} = -x$ and when $x \geq 0$, then $\sqrt{x^2} = x$

- When the GRE provides the square root sign for an even root, such as $\sqrt{x}$ or $\sqrt[4]{x}$, then the only accepted answer is the positive root.

That is, $\sqrt{25} = 5$, NOT +5 or -5. In contrast, the equation $x^2 = 25$ has TWO solutions, +5 and -5.

Even roots have only a positive value on the GRE.

- Odd roots will have the same sign as the base of the root. For example, $\sqrt[3]{125} = 5$ and $\sqrt[3]{-64} = -4$.

- For GRE it's good to memorize following values:

$\sqrt{2} \approx 1.41$
$\sqrt{3} \approx 1.73$
$\sqrt{5} \approx 2.24$
$\sqrt{6} \approx 2.45$
$\sqrt{7} \approx 2.65$
$\sqrt{8} \approx 2.83$
$\sqrt{10} \approx 3.16$

Some rules of engagement with exponents are given below:

Laws of Exponents:

- $x^A * x^B = x^{(A+B)}$
- $\frac{x^A}{x^B} = x^{(A-B)}$
- $x^a * y^a = (xy)^a$
- $x^{(-a)} = \frac{1}{x^a}$
- $x^0 = 1$
- $x^1 = x$
- $(x^A)^B = x^{(AB)}$
- $x^{\frac{a}{b}} == \sqrt[b]{x^a}$

The above rules are the exhaustive lists of concepts you can use to tackle any exponents problem. However, there are some tricky versions of the above rules which may fool you during your exams which are listed below:

Negative Bases

- \(x^A = **+ve**\) value; if x is -ve and A is **even**
- \(x^A = **-ve**\) value; if x is -ve and A is **odd**

For example: $(-3)^2 = 9$, $(-3)^3 = -27$

Keep in mind $(-3)^2 = 9$, whereas $-3^2 = -9$.

-3^2 simply means negative of square of 3.

Square Roots

Square root is basically the inverse of a square. In terms of functions if we write, $f(x) = x^2$, then the inverse of the function will be $f^{-1}(x) = \sqrt{x}$

Algebraic way of writing a square root is simply powering it to the power of $\frac{1}{2}$, i.e, $x^{1/2} = \sqrt{x}$

However, there are some pitfalls when it comes to square roots, like the following example:
$\sqrt{x} = r$; such that $r^2 = x$, where r is nonnegative.
Keep in mind even though $(-r)^2 = x$, symbol $\sqrt{x}$ is used denote the non-negative root of a number. For example: $\sqrt{9} = 3$; not -3.

Some basic properties of square roots are given below:

- $(\sqrt{x})^2 = x$
- $\sqrt{x^2} = x$
- $\sqrt{x}\sqrt{y} = \sqrt{xy}$
- $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$

1.5- PERCENT %

Frequency of the concepts tested: **High**

Definition

A percentage is a way of expressing a number as a fraction of 100 (*per cent* meaning "per hundred"). It is often denoted using the percent sign, "%", or the abbreviation "pct". Since a percent is an amount per 100, percents can be represented as fractions with a denominator of 100. For example, 25% means 25 per 100, 25/100 and 350% means 350 per 100, 350/100.

A percent can be represented as a decimal. The following relationship characterizes how percents and decimals interact. Percent Form / 100 = Decimal Form

For example: What is 2% represented as a decimal?
Percent Form / 100 = Decimal Form: 2%/100=0.02

Percent change

General formula for percent increase or decrease, (percent change):

$$Percent = \frac{Change}{Original} * 100$$

Example: A company received $2 million in royalties on the first $10 million in sales and then $8 million in royalties on the next $100 million in sales. By what percent did the ratio of royalties to sales decrease from the first $10 million in sales to the next $100 million in sales?

Solution: Percent decrease can be calculated by the formula above:

$$Percent = \frac{Change}{Original} * 100 =$$

$= \frac{\frac{2}{10} - \frac{8}{100}}{\frac{2}{10}} * 100 = 60$, so, the royalties decreased by 60%.

Simple Interest

Simple interest = principal * interest rate * time, where *"principal"* is the starting amount and *"rate"* is the interest rate at which the money grows per a given period of time (note: express the rate as a decimal in the formula). Time must be expressed in the same units used for time in the Rate.
Example: If $15,000 is invested at 10% simple annual interest, how much interest is earned after 9 months?

Solution: $15,000*0.1*9/12 = $1125

Compound Interest

$Balance(final) = principal * (1 + \frac{interest}{C})^{time*C}$, where C = the number of times compounded annually.

If C=1, meaning that interest is compounded once a year, then the formula will be: $Balance(final) = principal * (1 + interest)^{time}$, where time is number of years.

Example: If $20,000 is invested at 12% annual interest, compounded quarterly, what is the balance after 2 year?

Solution: $Balance = 20{,}000 * (1 + \frac{0.12}{4})^{2*4} =$
$= 20{,}000 * (1.03)^8 = 25{,}335.4$

Percentile

If someone's grade is in x_{th} percentile of the n grades, this means that x of people out of n has the grades less than this person.

Example: Lena's grade was in the 80th percentile out of 120 grades in her class. In another class of 200 students there were 24 grades higher than Lena's. If nobody had Lena's grade, then Lena was what percentile of the two classes combined?

Solution:

Being in 80th percentile out of 120 grades means Lena outscored 120∗0.8=96$120 * 0.8 = 96$ classmates.

In another class she would outscore 200−24=176$200 - 24 = 176$ students.

So, in combined classes she outscored $96 + 176 = 272$. As there are total of $120 + 200 = 320$ students, so Lena is in $\frac{272}{320} = 0.85 = 85$, or in 85th percentile.

Percentage

The term percentage means **per hundred**. Percentages are fractions with a denominator as 100. Percentages are always written with symbol %. Hence, 7% means $\frac{7}{100}$.

Percentages are ratios, that represent the **part of a whole**. Meaning, 12% is 12 parts of 100.

Basic Rules of Percents

Compute percentage, given part and the whole. This is done by dividing the part over whole and multiply by 100.

$\% = \frac{part}{whole}100$

For example,
To find what percent of 120 is 24, we just divide 24 over 120 and multiply the value by 100. Hence, it is 20%.

In the same way, to find 13 is 24% of what number, we divide 13 over 24%. Hence, the value is, 54.16.

Percentage change

When the value of a product changes from an old to the new value, its percentage change is calculated by, finding the difference between old and new value and dividing the value by **old value** and multiplying by 100.

percentage change = $\frac{old-new}{old}$ 100. Keep in mind, If the value of percentage change is negative, it means the value of the product has reduced.

Consecutive Percentage Changes

Till now we have seen how to calculate the percentage, now we will look into how to calculate the final value with two consecutive percentage changes.

In this case, we have a simple formula to calculate the net percentage change. If the successive percentage changes are x% and y% then, the net percentage change is $(x + y + \frac{xy}{100})$.

Keep in mind, if at all there is a decrease in change in value, the formula is changed accordingly. For example, the value of a product is increased by a % and then decreased by b%, then the updated formula is, $(a - b - \frac{ab}{100})$.

Example, if the value of product is increased by 12% and further increased by 3%, then the net percentage change is, $(12 + 3 + \frac{36}{100})$ that is 15.36%.

In the same way, if the value of a product is increased by 14% and then given a discount of 10%, then the net percentage change is $(14 - 10 - \frac{140}{100})$ that is 2.6%.

Undoing Percentage Changes

The next step in percentage is, how to find the original value. Meaning, if a product is discounted and we know the final value and the percentage of discount, how to find its initial value.

If a is the percentage by which the value is decreased or increased and y is the final amount, then the initial value x is calculated using the following formula, $x = y(\frac{100}{100+a})$

Keep in mind, **If the value is discounted, then change the sign in the formula to negative**. For example, If a product is being sold for 270 after giving a discount of 15%, then what is the original price of the product?

In order to answer this question, we use the above formula with **negative sign** as the value is being discounted. Hence, on applying the above formula we get, $x = 270(\frac{100}{100-15})$ On solving the above equation we obtain 317.64 as the value.

Population Formula

Example:
The population of a city was 2 years ago 24000, population decreases every year at the rate of 5 %. Find the present population of the city.

Solution:
The population decreased at a particular rate so Present population of that city is

$= 24000 * (1 - 5)$

$= 24000 * (1 - 5/100)^2$

$= 24000 * (19/20) * (19/20)$

=21660.

1.6 - REMAINDER BASIC AND ADVANCED

Frequency of the concepts tested: **Low**

Definition: What is "Remainder"?

Remainder is the integer which is left over in a division, when the divisor cannot evenly divide the dividend. Dividend = Divisor * Quotient + Remainder

Ex: 13 when divided by 3 gives 1 remainder. (13 = 3*4 + 1)
13 -> Dividend
3 -> Divisor
4 -> Quotient
1 -> Remainder

Remainder Notation

1. A number when divided by 5 gives 3 as remainder.
Let the number be n using Dividend = Divisor * Quotient + Remainder n = 5k + 3
[where k is quotient and is an integer]

2. A number when divided by 13 gives 5 as remainder.
Let the number be n using Dividend = Divisor * Quotient + Remainder n = 13k + 5
[where k is quotient and is an integer]

3. A number, when divided by 16, gives 3 as remainder.
Let the number be n using Dividend = Divisor * Quotient + Remainder n = 16k + 3
[where k is quotient and is an integer]

Remainder Range
"A number, when divided by a number k, can give remainder from 0 to k-1"
Ex: If we are trying to divide a number by 6 then possible values of remainder are from 0-5

Q1. x when divided by "a" gives 3 as remainder. y when divided by "b" gives 4 as remainder. Find min(a+b).

Sol: As a and b are positive numbers so min(a+b) = min(a) + min(b) x when divided by "a" gives 3 as remainder -> Since a is giving 3 remainder that means that a >= 4. So, min a = 4y when divided by "b" gives 4 as remainder -> Since b is giving 4 remainder that means that b >= 5. So, min b = 5 => min(a+b) = min(a) + min(b) = 4+ 5 =9

Practice Questions
Q1. Find the remainder when 200 is divided by 3.
Q2. Find the remainder when 80 is divided by 7.
Q3. Find the remainder when 100 is divided by 9.

Ans:
Q1. 2, Q2 3, Q3. 1

• PT2: Find divisor when remainder is given

Q1. 20 when divided by which number will give 4 as remainder?

Sol: 20 = nk + 4 => n
k= 20-4 = 16 => n = 16/k
Now, n is giving 4 remainder that means that n>=5. Let's start putting values of k and get values of n.

k=1 => n=16
k=2 => n=16/2 = 8
k=3 => n not integer
k=4 => n= 16/4 not possible as n>=5
So, possible values of the number (n) are 8 and 16

Q2. 50 when divided by which number will give 7 as remainder?

Sol: 50 = nk + 7 => nk = 50-7 = 43 [note 43 is prime so it has only two factors 1 and 43 itself]
=> n = 43/k
k= 1 => n = 43
k = 43 not possible as n becomes 1 [but we know that n>=8 as n is giving 7 remainder]
So, Answer is 43

• PT3: Find dividend when one divisor and remainder are given

Q1. n when divided by 7 gives 4 as remainder. Find the possible values of the number.

Sol: n when divided by 7 gives 4 as remainder.
n = 7k + 4

We will start taking values of k starting from k=0 and find values of n correspondingly
k=0 n=7*0 + 4 = 4 k=5 n=7*5 + 4 = 39
k=1 n=7*1 + 4 = 11 k=6 n=7*6 + 4 = 46
k=2 n=7*2 + 4 = 18 k=7 n=7*7 + 4 = 53
k=3 n=7*3 + 4 = 25 k=8 n=7*8 + 4 = 60
k=4 n=7*4 + 4 = 32 k=9 n=7*9 + 4 = 67

Q2. n when divided by 5 gives 2 as remainder. Find the possible values of n.

n when divided by 5 gives 2 as remainder.
n = 5k + 2

We will start taking values of k starting from k=0 and find values of n correspondingly

k=0 n=5*0 + 2 = 2 k=5 n=5*5 + 2 = 27
k=1 n=5*1 + 2 = 7 k=6 n=5*6 + 2 = 32
k=2 n=5*2 + 2 = 12 k=7 n=5*7 + 2 = 37
k=3 n=5*3 + 2 = 17 k=8 n=5*8 + 2 = 42
k=4 n=5*4 + 2 = 22 k=9 n=5*9 + 2 = 47

• PT4: Find dividend when two divisors and remainders are given

Q1. n when divided by 7 gives 3 remainder and when divided by 5 gives 3 remainder. Find first 2 non-negative values of n

Sol: Method-1. n when divided by 7 gives 3 remainder
n = 7k + 3

We will start taking values of k starting from k=0 and find values of n correspondingly
k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
n = 3, 10, 17, 24, 31, 38, 45, 52, 59, 66

n when divided by 5 gives 3 remainder
n = 5t + 3

We will start taking values of t starting from t=0 and find values of n correspondingly
t = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
n = 3, 8, 13, 18, 23, 28, 33, 38, 43, 48
First two common values are 3 and 38

Sol: Method-2
n when divided by 7 gives 3 remainder
n = 7k + 3

n when divided by 5 gives 3 remainder
n = 5t + 3

7k+3 = 5t+3 => t = 7k/5
So, only those values of k will give us common values of n for which t is integer too.
k = 0 => t=7*0/5 = 0
k = 5 => t=7*5/5 = 7
So, k = 0 => n = 7*0 + 3 = 3
k = 5 => n = 7*5 + 3 = 38

Q2. n when divided by 6 gives 4 remainder and when divided by 4 gives 2 remainder. Find first 2 non-negative values of n

Sol: Method-1

n when divided by 6 gives 4 remainder
n = 6k + 4

We will start taking values of k starting from k=0 and find values of n correspondingly
k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
n = 4, 10, 16, 22, 28, 34, 40, 46, 52, 58

n when divided by 4 gives 2 remainder
n = 4t + 2

We will start taking values of t starting from t=0 and find values of n correspondingly
t = 0 , 1, 2, 3, 4, 5, 6, 7, 8, 9
n = 2, 6, 10, 14, 18, 22, 26, 30, 34, 38

First two common values are 10 and 22

Sol: Method-2
n when divided by 6 gives 4 remainder
n = 6k + 4
n when divided by 4 gives 2 remainder
n = 4t + 2
6k+4 = 4t+2 => t = (6k+2)/4 = (3k+1)/2

So, only those values of k will give us common values of n for which t is integer too.

k = 1 => t=(6*1 + 2)/4 = 2
k = 3 => t=(6*3 + 2)/4 = 5
So, k = 1 => n = 6*1 + 4 = 10
k = 3 => n = 6*3 + 4 = 22

Note: When the numerator is smaller than the denominator then the remainder is the numerator itself

Ex: If 2 is divided by 3 then remainder is 2

Remainder of sum of two numbers by a number
13 when divided by 3 gives us 1 remainder

When we split 13 as 8 and 5 and divide 8 and 5 individually by 3 then we still get the same remainder

13/3 = (8+5)/3 = 8/3 + 5/3
8/3 will give 2 remainder
5/3 will give 2 remainder

Total remainder is 2+2 = 4, but remainder can't be greater than 3 so remainder will be 4-3 = 1 which is same as the remainder for 13/3

Q1. "A" when divided by 12 gives 3 as remainder. What is the remainder when A is divided by 4?

Sol: A = 12k + 3
When A is divided by 4 then 12k will give 0 remainder and 3 will give 3 remainder. Total remainder is 3

Q2. "B" when divided by 15 gives 6 as remainder. What is the remainder when B is divided by 5?

Sol: B = 15k + 6
When B is divided by 5 then 15k will give 0 remainder and 6 will give 1 remainder. Total remainder is 1
Remainder of difference of two numbers by a number

13 when divided by 3 gives us 1 remainder
When we split 13 as 15 - 2 and divide 15 and 2 individually by 3 then we still get the same remainder
13/3 = (15-2)/3 = 15/3 - 2/3
15/3 will give 0 remainder
2/3 will give 2 remainder
Total remainder = 0-2 = -2

But remainder cannot be negative so we are going to keep on adding 3 to -2 till the time the sum comes in the range of 0 and 2 [3-1]
=> -2 + 3 = 1
So, remainder is 1

Remainder of product of two numbers by a number

21 when divided by 5 gives us 1 remainder. Now if we break 21 into product of two numbers and find the remainder of these individual numbers by 5 and multiply the remainders then we are going to get the same remainder as we got when we divided 21 by 5

Let's write 21 = 3*7 and divided 3 and 7 by 5
3/5 * 7/5 will give us 3 * 2 remainder respectively = 6
But we divided set of numbers by 5 so remainder cannot be more than 5, so we divide 6 again by 5 to get final remainder as 1

Q1. Find the remainder of 136 * 148 * 298 by 5.

Sol: 136/5 remainder is 1
148/5 remainder is 3
298/5 remainder is 3
Total remainder = 1*3*3 = 9 [>=5]
=> Final remainder = remainder of 9/5 = 4

Q2. Find the remainder of 1205 * 1208 * 2404 by 12.

Sol: 1205/12 remainder is 5
1208/12 remainder is 8
2404/12 remainder is 4
Total remainder = 5*8*4 = 160 [>=12]
=> Final remainder = remainder of 160/12 = 4

Remainder of numbers when divided by 2
When we try to find remainder of a number by 2 then we just need to know if the number is odd or even. If the number is odd then remainder is 1. If the number is even then the remainder is 0

Remainder of numbers when divided by 3
Remainder of a number by 3 is same as the remainder of sum of digits of the number by 3

Remainder of numbers when divided by 5
Remainder of a number by 5 is same as the remainder of the unit's digit of the number by 5

Remainder of numbers when divided by 9
Remainder of a number by 9 is same as the remainder of sum of digits of the number by 9

Remainder of numbers when divided by 10
Remainder of a number by 10 is same as the unit's digit of the number.

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CHAPTER 2. ALGEBRA

2.1 - OPERATIONS WITH ALGEBRAIC EXPRESSIONS

Frequency of the concepts tested: **Very High**

Scope

Manipulation of various algebraic expressions
Equations in 1 & more variables
Dealing with non-linear equations
Algebraic identities

Notation & Assumptions

Here, lower case roman alphabets will be used to denote variables such as *a, b, c, x, y, z, w*

In general it is assumed that the GRE will only deal with real numbers ($\mathbb{R}$) or subsets of $\mathbb{R}$ such as Integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$) etc

Concept of variables

A variable is a place holder, which can be used in mathematical expressions. They are most often used for two purposes:

(a) **In Algebraic Equations**: To represent unknown quantities in known relationships. For e.g.: "Mary's age is 10 more than twice that of Jim's", we can represent the unknown "Mary's age" by x and "Jim's age" by y and then the known relationship is $x = 2y + 10$

(b) **In Algebraic Identities**: These are generalized relationships such as $\sqrt{x^2} = |x|$, which says for any number, if you square it and take the root, you get the absolute value back. So, the variable acts like a true placeholder, which may be replaced by any number.

Basic rules of manipulation

- When switching terms from one side to the other in an algebraic expression + becomes - and vice versa. E.g. $x + y = 2z \Rightarrow x = 2z - y$
- When switching terms from one side to the other in an algebraic expression * becomes/ and vice versa. E.g. $4 * x = (y + 1)^2 \Rightarrow x = \frac{(y+1)^2}{4}$

- You can add/subtract/multiply/divide both sides by the same amount. E.g. $x + y = 10z \Rightarrow \frac{x+y}{43} = \frac{10z}{43}$

- You can take to the exponent or bring from the exponent as long as the base is the same.
  E.g. 1.$x^2 + 2 = z \Rightarrow 4^{x^2+2} = 4^z$
  E.g. 2.$2^{4x} = 8^y \Rightarrow 2^{4x} = 2^{3y} \Rightarrow 4x = 3y$

It is important to note that all the operations above are possible not just with constants but also with variables themselves. So, you can "add x" or "multiply with y" on both sides while maintaining the expression. But what you need to be very careful about is when dividing both sides by a variable. **When you divide both sides by a variable (or do operations like "canceling x on both sides") you implicitly assume that the variable cannot be equal to 0, as division by 0 is undefined.**

Degree of an expression
The degree of an algebraic expression is defined as the highest power of the variables present in the expression.

- Degree 1 : Linear
- Degree 2 : Quadratic
- Degree 3 : Cubic
- Degree 4 : Bi-quadratic

Eg: $x + y$ the degree is 1
$x^3 + x + 2$ the degree is 3
$x^3 + z^5$ the degree of x is 3, degree of z is 5, degree of the expression is 5

2.2 - SOLVING LINEAR EQUATIONS

Frequency of the concepts tested: **Low**

Solving equations of degree 1: LINEAR

Degree 1 equations or linear equations are equations in one or more variable such that degree of each variable is one. Let us consider some special cases of linear equations:

One variable
Such equations will always have a solution. General form is $ax = b$ and solution is $x = (b/a)$

One equation in Two variables

This is not enough to determine x and y uniquely. There can be infinitely many solutions.

Two equations in Two variables

If you have a linear equation in 2 variables, you need at least 2 equations to solve for both variables. The general form is:

$$ax + by = c$$
$$dx + ey = f$$

If $(a/d) = (b/e) = (c/f)$ then there are infinite solutions. Any point satisfying one equation will always satisfy the second.

If $(a/d) = (b/e) \neq (c/f)$ then there is no such x and y which will satisfy both equations. No solution

In all other cases, solving the equations is straight forward, multiply eq (2) by a/d and subtract from (1).

More than two equations in Two variables

Pick any 2 equations and try to solve them:

Case 1: No solution --> Then there is no solution for bigger set
Case 2: Unique solution --> Substitute in other equations to see if the solution works for all others
Case 3: Infinite solutions --> Out of the 2 equations you picked, replace any one with an un-picked equation and repeat.

More than 2 variables

This is not a case that will be encountered often on the GRE. But in general, for n variables, you will need at least n equations to get a unique solution. Sometimes you can assign unique values to a subset of variables using less than n equations using a small trick. For example, consider the equations:

$$x + 2y + 5z = 20$$
$$x + 4y + 10z = 40$$

In this case you can treat $2y + z$ as a single variable to get:

$$x + (2y + 5z) = 20$$
$$x + 2 * (2y + 5z) = 40$$

These can be solved to get x=0 and 2y+5z=20

There is a common misconception that you need n equations to solve n variables. This is not true.

2.3- SOLVING QUADRATIC EQUATIONS

Frequency of the concepts tested: **Medium**

The general form of a quadratic equation is $ax^2 + bx + c = 0$

The equation has no solution if

$$b^2 < 4ac$$

The equation has exactly one solution

$$b^2 = 4ac$$

This equation has 2 solutions given by

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} \text{if } b^2 > 4ac$$

$The\ sum\ of\ roots\ is\ \frac{-b}{a}$

$The\ product\ of\ roots\ is\ \frac{c}{a}$

If the roots are $r_1$and r_2 the equation can be written as:
$r_2(x - r_1)(x - r_2) = 0$

A quick way to solve a quadratic, without the above formula is to factorize it:

Step 1: Divide throughout by coeff of x^2 to put it in the form $x^2 + dx + e = 0$

Step 2: Sum of roots = -d and Product = e. Search for 2 numbers which satisfy this criteria, let them be f,g

Step 3: The equation may be re-written as (x-f)(x-g)=0. And the solutions are f,g

Eg. $x^2 + 11x + 30 = 0$

The sum is -11 and the product is 30. So numbers are -5,-6

$x^2 + 11x + 30 = x^2 + 5x + 6x + 30 = x(x + 5) + 6(x + 5) = (x + 5)(x + 6)$

You will never be asked to solved higher degree equations, except in some cases where using simple tricks these equations can either be factorized or be reduced to a lower degree or both.

What you need to note is that an equation of degree n has at most n unique solutions.

Factorization

This is the easiest approach to solving higher degree equations. Though there is no general rule to do this, generally a knowledge of algebraic identities helps. The basic idea is that if you can write an equation in the form:

$A * B * C = 0$

Eg. $x^3 + 11x^2 + 30x = 0$

$x * (x^2 + 11x + 30) = 0$

$x * (x + 5) * (x + 6) = 0$

So, the solution is x=0, -5, -6

Reducing to lower degree

This is useful sometimes when it is easy to see that a simple variable substitution can reduce the degree.

Eg. $x^6 - 3x^3 + 2 = 0$

here let, $y = x^3$

$y^2 - 3y + 2 = 0$

$(y - 2)(y - 1) = 0$

$So\ the\ solution\ is\ y = 1\ or\ 2\ or\ x^3 = 1\ or\ 2\ or\ x = 1\ or\ \sqrt[3]{2}$

Other tricks

Sometimes we are given conditions such as the variables being integers which make the solutions much easier to find. When we know that the solutions are integral, often times solutions are easy to find using just brute force.

Eg. $a^2 + b^2 = 116$ and we know a and b are integers such that a<b

We can solve this by testing values of a and checking if we can find b

a=1 b=root(115) Not integer

a=2 b=root(112) Not integer

a=3 b=root(107) Not integer

a=4 b=root(100)=10,

a=5 b=root(91) Not integer,

a=6 b=root(80) Not integer

a=7 b=root(67) Not integer,

a=8 b=root(52)<a

So the answer is (4,10)

Algebraic Identities

These can be very useful in simplifying & solving a lot of questions:

- $(x+y)^2 = x^2 + y^2 + 2xy$
- $(x-y)^2 = x^2 + y^2 - 2xy$
- $x^2 - y^2 = (x+y)(x-y)$
- $(x+y)^2 - (x-y)^2 = 4xy$
- $(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$
- $x^3 + y^3 = (x+y)(x^2 + y^2 - xy)$

2.4 - SOLVING LINEAR INEQUALITIES

Frequency of the concepts tested: **Medium**

Four types of Inequality Problems solved using two methods

- Method 1: Algebra
- Method 2: Sine Wave Method / Wave Method / Wavy method

What is Inequality and Types of Inequalities

Usually we are given discrete values of variables like x=2, y=3 etc. In case of inequalities, we are given a range of values. Let's take some examples to understand this:

$x > 3$ => x can take all real values which are greater than 3, i.e. 3.001, 4, 5, 6, 8, 100, etc...
So, instead of giving a single value in case of inequalities we are given a set of values for the variables.

Let's understand various types of inequalities now:

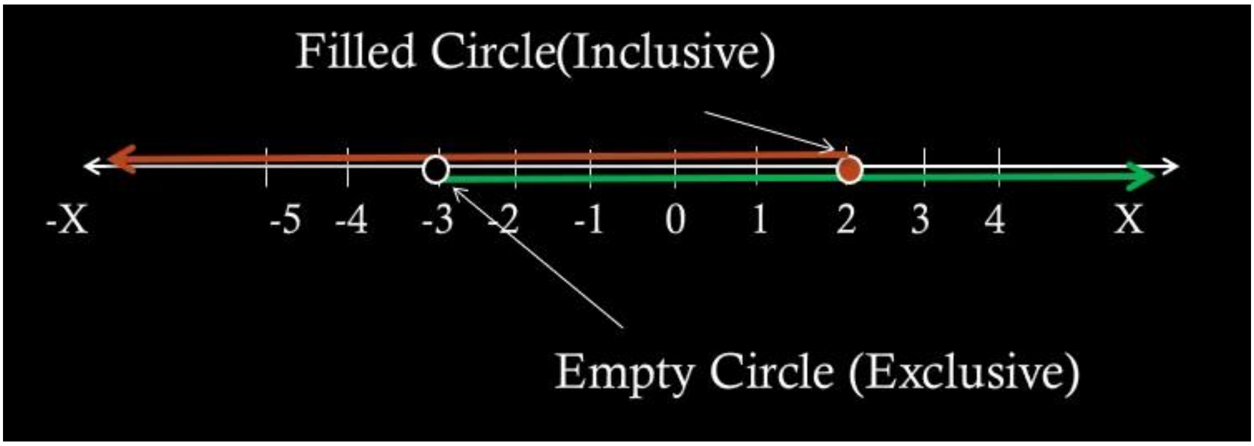
-Greater Than Inequality (>): Ex: $x > 3$ (We have seen above)
-Less Than Inequality (<): Ex: $y < 2$ => Y can take all read values which are less than 2
-Greater Than or Equal to Inequality (≥): Ex: $x \geq 5$ (x can take all real values greater than or equal to 5)
-Less Than or Equal to Inequality (≤): Ex: $y \leq 3$ (y can take all real values less than or equal to 3)
-In-Between Inequality ($-2 \leq x < 5$): Ex: x can take all values which are greater than or equal to -2 and less than 5

Graphing Inequalities

Now let's talk about how to plot and inequality on the number line

Ex 1: Graph $a \leq 2$ and $b > -3$ on a number line
Sol:
To plot $a \leq 2$ we need to draw a line starting at 2 and extending till $-\infty$ on the left-hand side (Refer Orange line in below image). Note that we need to darken point 2 because it is included (as $a \leq 2$, so 2 is included)

To plot $b > -3$ we need to draw a line starting at -3 and extending till $+\infty$ on the right-hand hand. (Refer Green line in below image). Note that we DO NOT darken point -3 as it is excluded (As $b > -3$ and not ≥ -3)



PROPERTIES OF INEQUALITIES

PROP 1: Adding or Subtracting the same number from both the sides of the inequality DOES NOT change the sign of the inequality.

Ex 1: 7 > 3
Add 4 on both the sides we get 7 + 4 > 3 + 4 => 11 > 7
[Which is True and Note that sign of inequality which was > is still >]

Ex 2: 8 > 4
Subtract 9 from both the sides we get 8 -9 > 4 -9 => -1 > -5
[Which is True and Note that sign of inequality which was < is still <]

Ex 3: a > b
Add k on both the sides we get a + k > b + k
[which is true and sign of inequality did not change]

PROP 2: Multiplying / Dividing an inequality equation with a positive number DOES NOT change the sign of the inequality

Ex 1: 7 > 3
Multiply both the sides by +2 we get 7 * 2 > 3 * 2 => 14 > 6
[which is true and sign of inequality did not change]

Ex 2: 8 > 4
Divide both the sides by +2 we get

$\frac{8}{2} > \frac{4}{2}$ => 4 > 2 [which is true and sign of inequality did not change]

Ex 3: a > b
Multiply both the sides with a positive variable k we get ak > bk

PROP 3: Multiplying / Dividing an inequality equation with a negative number REVERSES the sign of the inequality

Ex 1: 7 > 3
Multiply both the sides by -2 we get 7 * -2 < 3 * -2 => -14 < -6
[note the sign of inequality has changed from > to <]

Ex 2: 8 > 4
Divide both the sides by -2 we get

$\frac{8}{-2} < \frac{4}{-2}$ => -4 < -2 [note the sign of inequality has changed from > to <]

Ex 3: a > b
Multiply both the sides with a negative variable t we get at < bt
[note the sign of inequality has changed from > to <]

PROP 4: We can add two inequalities which have the same inequality sign

Ex 1: 7 > 3 and 8 > 2, Since the two inequalities have same sign of (>) so we can add both of them to get 7 + 8 > 3 + 2 => 15 > 5

Ex 2: a > b and c > d
Since the two inequalities have same sign of (>) so we can add both of them to get
a + c > b + d [Note that this is true irrespective of the signs of a, b, c and d]

Ex 3: If two inequalities have different signs then we can multiply one of them to make the signs same and then add them
a > b
c < d
we can multiple c < d with -1 to get -c > -d and now we can add a > b and -c > -d to get a - c > b - d

PROP 5: Taking Square Root on both sides of an inequality DOES NOT Change the sign of the inequality (provided it is possible to take square root on both the sides and get real values).

Ex 1: $a^2 > b^2$ [given that a and b are positive numbers]
Taking square root on both the sides we will get a > b

PROP 6: Square of a number is always non-negative

Ex 1: $a^2 \geq 0$ [this is true for all real values of a]
this will be equal to 0 only when a itself is zero

Types of Inequality Problems

Type 1: x * y > 0
If product of two variables > 0 that means that the two variables have SAME SIGN
Either Both are Positive => x > 0 and y > 0 Or Both are Negative => x < 0 and y < 0

Type 2: x / y > 0
If division of two variables > 0 that means that the two variables have SAME SIGN
Either Both are Positive => x > 0 and y > 0 Or Both are Negative => x < 0 and y < 0

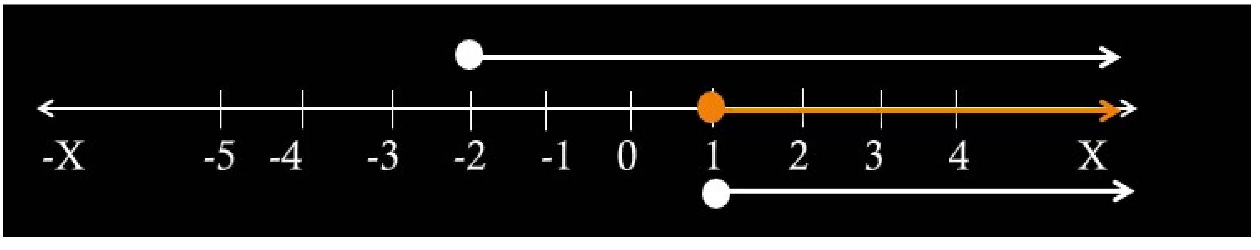
Type 3: x * y < 0
If product of two variables < 0 that means that the two variables have DIFFERENT SIGN
Either x > 0 and y < 0 Or x < 0 and y > 0

Type 4: x / y < 0
If division of two variables < 0 that means that the two variables have DIFFERENT SIGN
Either x > 0 and y < 0 Or x < 0 and y > 0

Basic Problems on Inequalities

Given that y = 4 + $(1-x)2$ Find yMin(Minimum Value of y) and find the value of x for which y = yMin

Sol:
y = 4 + $(1-x)^2$
$(1-x)^2$= 0 when 1-x = 0 or when x = 1
So, yMin = 4 when x = 1
Combining Inequalities
We are discussing this because we will use this in solving problems using the algebra method



If after solving an inequality equation we are getting x ≥ -2 and x ≥ 1 as two solutions then our final solution will be x ≥ 1. As it is the intersection/common part of both the inequalities

(As shown in orange in above figure)

4 types of Inequality Problems solved using Algebra and Sine Wave Method

There are mainly four types of inequality problems which you would need to solve:--

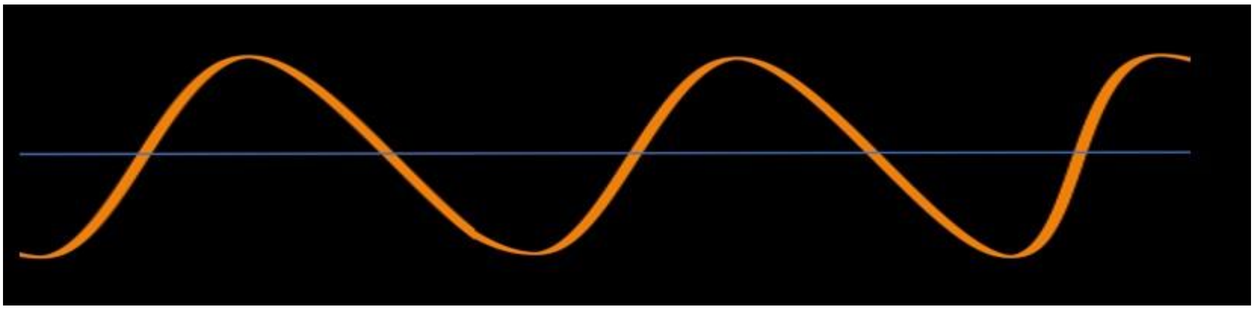
<u>TYPE 1</u>: x*y > 0
When xy > 0 then we know that both x and y can be either positive or both can be negative i.e. both x and y have the same sign
so, we have
x>0, y>0 or x<0,y<0

Example Problem
(x-1)*(x-2) > 0
Method 1: Algebra
So, we have two cases
Case 1
both (x-1) and (x-2) are positive
so, x-1 > 0 => x > 1
and x-2 > 0 => x > 2
Intersection of the two cases is x >2

Case 2
both (x-1) and (x-2) are negative
so, x-1 < 0 => x < 1
and x-2 < 0 => x < 2
Intersection of the two cases is x < 1
So, Solution to the question is x < 1 or x > 2

Method 2: Sine Wave Method / Wave Method / Wavy method

In this method we are going to use a sine wave method to solve the problem. Just a quick preview, sine wave is a continuous curve which oscillates between a minimum and a maximum value below and above the base line respectively. Sample image below:

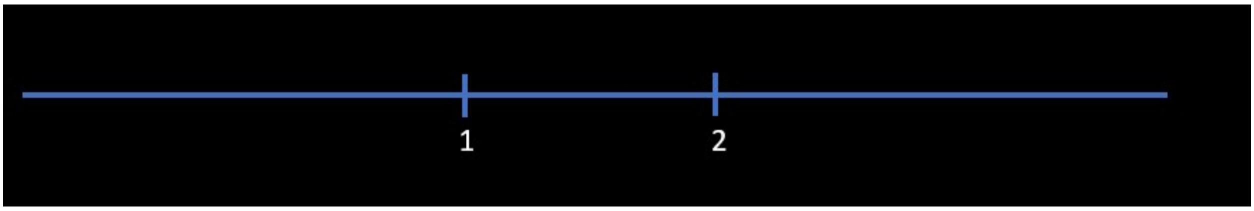


Let's attempt to solve (x-1)*(x-2) > 0 using Sine Wave Method

* Remember that in order to solve the problems using the sine wave method we need to have the coefficient of x positive.

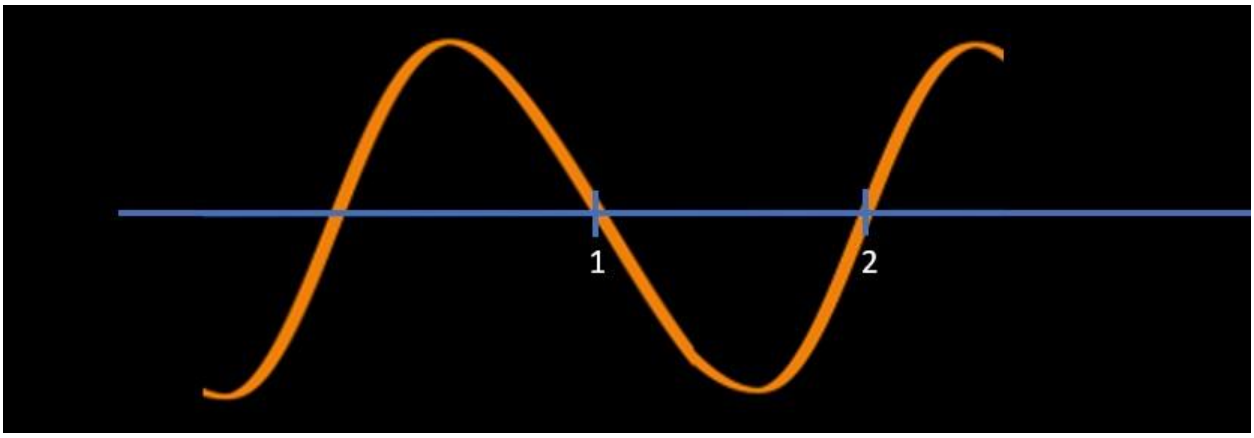
To solve an inequality using this method we find out the intersection points by equating the inequality to 0
=> (x-1)*(x-2) = 0 => x = 1 or 2

Now, we plot these two points on the number line as shown below



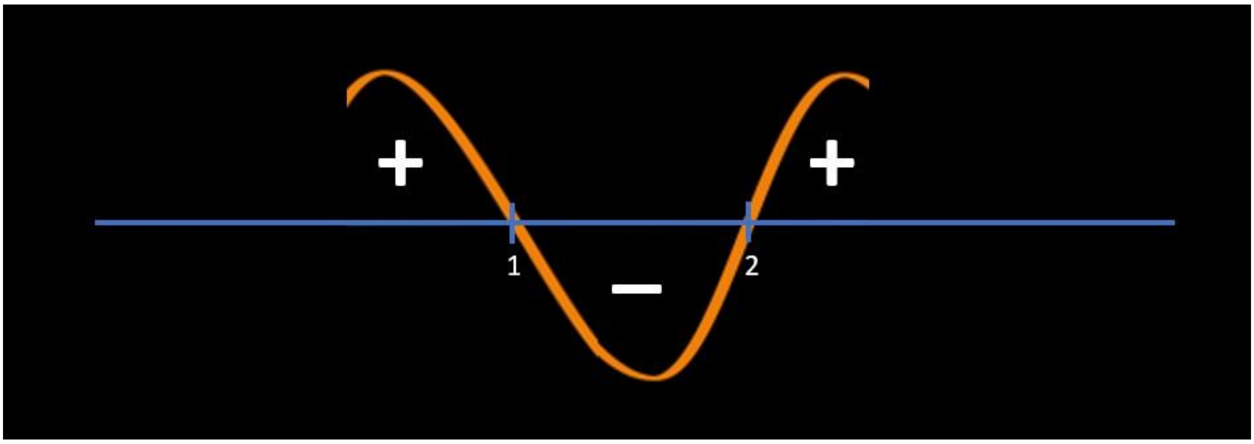
Then we are going to draw a sine curve

Starting from right top
Going down at the first solution which is 2 in this case and then
Coming up in the second solution which is 1 in this case and
Going down in the third solution if it is there (in this it is not there



Now we will start marking + and - as mentioned below:

Any Area (in-between) above the number line and below the sine curve is marked as "+" and
Any Area (in-between) below the number line and above the sine curve is marked as "-" as shown below



Now, get your answer as below:

If the inequality in the question is > 0 then pick all the ranges which are "+"
If the inequality in the question is < 0 then pick all the ranges which are "-"

In our case the question was (x-1)*(x-2) > 0 so we will pick all "+" areas which are
x > 2 and x < 1

If the question was (x-1)*(x-2) < 0 then we will pick all "-" areas which are
1 < x < 2

Note that if the question has ≥ or ≥ then we need to check for the border conditions too
Ex: if question was (x-1)*(x-2) ≥ 0 then we need to check the border condition of x = 1 and x = 2 manually and see if we want to include it in the answer or not.

TYPE 2: x/y > 0
When x/y > 0 then we know that both x and y can be either positive or both can be negative i.e. both x and y have the same sign
so, we have
x>0, y>0 or x<0,y<0

Example Problem
$\frac{(x-3)}{(x-4)} > 0$
Method 1: Algebra
So, we have two cases
Case 1
Both (x-3) and (x-4) are positive
=> x-3 > 0 => x>3
And x-4 > 0 => x>4

Intersection of the two cases is x > 4

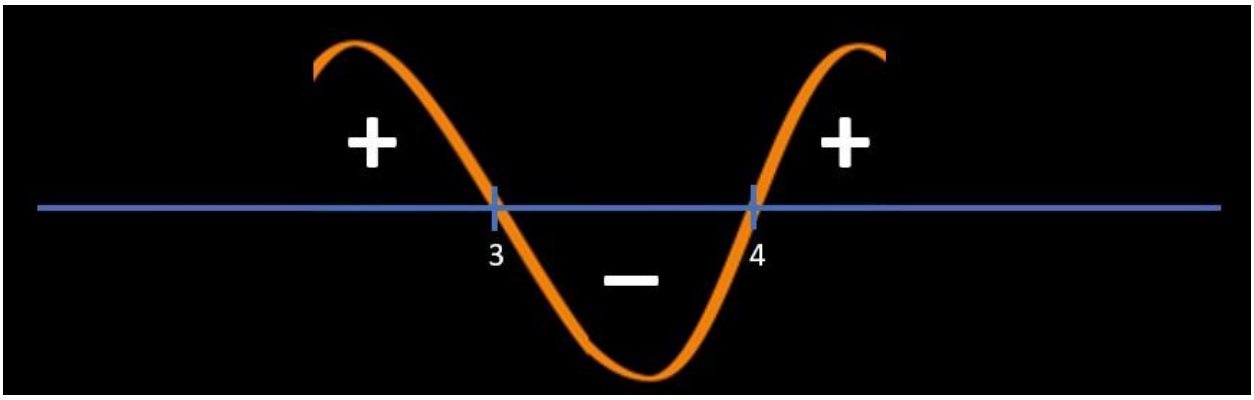
Case 2
Both (x-3) and (x-4) are negative
=> x-3 < 0 => x < 3
and x-4 < 0 => x < 4
Intersection of the two cases is x < 3

So, solution to the question is x < 3 or x > 4

Method 2: Sine Wave Method / Wave Method / Wavy method

Point of intersections:
x - 3 = 0 and x-4 = 0
=> x = 3, 4

Refer below image



Since question is $\frac{(x-3)}{(x-4)} > 0$
So, we will pick "+" area regions
So, answer is x < 3 and x > 4

<u>TYPE 3</u>: x*y < 0
When x*y < 0 then we know that that
(x can be positive and y will be negative) or (x can be negative and y will be positive)
i.e. x and y have opposite signs
so, we have
x>0, y<0 or x<0,y>0

Example Problem
(x+1)(x-1) < 0
Method 1: Algebra
So, we will have two cases

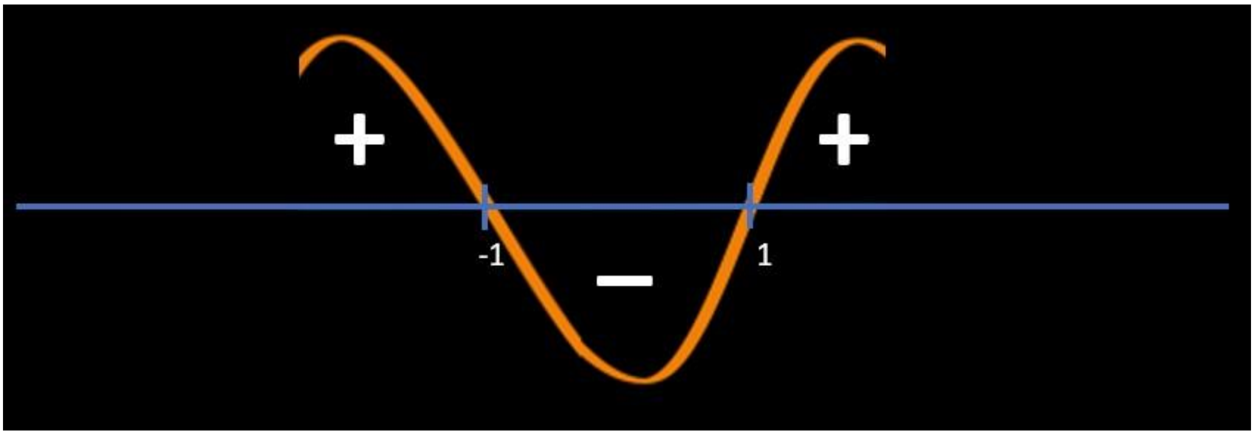
Case 1
(x+1) is positive and (x-1) is negative
=> x + 1 > 0 => x > -1
And x - 1 < 0 => x < 1
Intersection of the two cases is
-1 < x < 1

Case 2
(x+1) is negative and (x-1) is positive
=> x+1 < 0 => x < -1
And x-1 > 0 => x > 1
The two cases have no intersection. So, no solution from this case

So, solution of the problem is -1 < x < 1
Method 2: Sine Wave Method / Wave Method / Wavy method

Point of intersections:
x + 1 = 0 and x - 1 = 0
=> x = -1, 1

Refer below image



Since question is (x+1)(x-1) < 0
So, we will pick "-" area regions
So, answer is -1 < x < 1

<u>TYPE 4</u>: x/y < 0
When x/y < 0 then we know that that
(x can be positive and y will be negative) or (x can be negative and y will be positive)
i.e. x and y have opposite signs
so, we have
x>0, y<0 or x<0,y>0

Example Problem

$\frac{(x-2)}{(x+3)} < 0$

Method 1: Algebra

So, we will have two cases

Case 1

(x-2) is positive and (x+3) is negative

=> x-2 > 0 => x > 2

And x+3 < 0 or x < -3

There is no intersection of the two cases. So, no solution from this case

Case 2

(x-2) is negative and (x+3) is positive

=> x-2 < 0 => x < 2

And x+3 > 0 => x > -3

Intersection of the two cases is -3 < x < 2

So, Solution of the question is -3 < x < 2

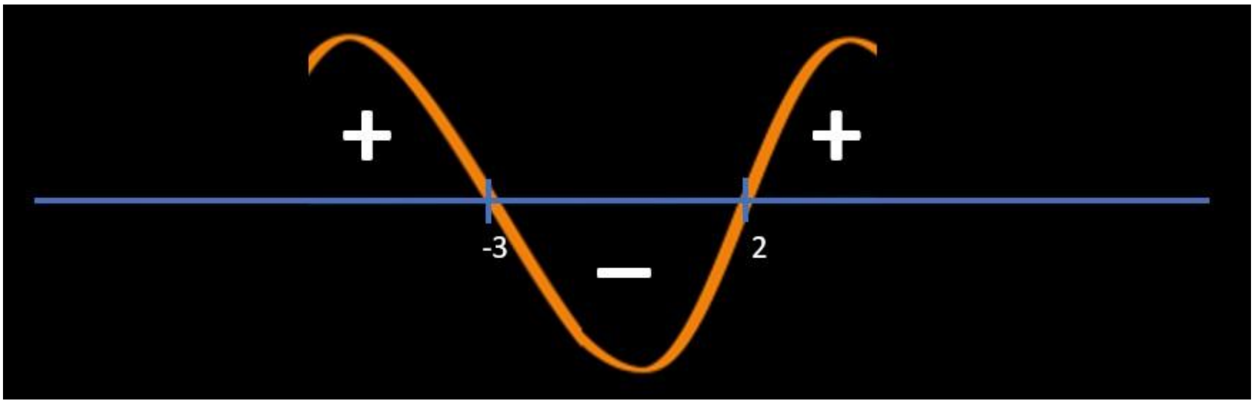
Method 2: Sine Wave Method / Wave Method / Wavy method

Point of intersections:

x - 2 = 0 and x + 3 = 0

=> x = 2, -3

Refer below image



Since question is $\frac{(x-2)}{(x+3)} < 0$

So, we will pick "-" area regions

So, answer is -3 < x < 2

Problems:

1. If $x^5 + x^2 < 0$

(A) $x < -1$
$x < 0$
$x > 0$
$x > 1$
$x^4 < x^2$
$x^5 + x^2 < 0$
x^2common on the left-hand side we have
$x^2(x^3 + 1) < 0$

$x^2(x^3 + 1) < 0$that means that product of two terms $x^2$and $(x^3 + 1)$is < 0 so one of them is positive and other is negative. We also know that $x^2$is a square of a number so it can never be < 0 => $(x^3 + 1)$< 0 or x^3< -1

That means x is a negative number less than 1 as $x^3 x^3$

So, Answer will be A

Discussed here: Problem Link

2. Which of the following describes all the values of y for which y < y^2?

A. 1 < y
B. −1 < y < 0
C. y < −1
D. 1/y < 1
E. 0 < y < 1

Solution:
The question can be written as
y^2- y > 0
=> y*(y-1) > 0
It is of the form xy > 0
So, we will have two cases
Case 1
Both y and y-1 are positive
=> y > 0
And y-1 > 0 => y > 1
Intersection of the two cases is y > 1

Case 2
Both y and y-1 are negative
=>y < 0
And y -1 < 0 => y < 1
Intersection of the two cases is y <0

So, solution to the problem is y < 0 or y > 1

So, Answer will be D
(As option D can be written as
$\frac{1}{y}$- 1 < 0
or, $\frac{(1-y)}{y}$< 0
or $\frac{(y-1)}{y}$> 0
And solution to this will be same as that of $y * (y - 1)$> 0

3. Which of the following describes all values of x for which 1–x^2 > = 0?

(A) x >= 1
(B) x <= –1
(C) 0 <= x <= 1
(D) x <= –1 or x >= 1
(E) –1 <= x <= 1

Solution:
Question can be written as
x^2- 1 <=0
=> (x+1)*(x-1) <=0
Case 1
x+1 is positive or 0 and x-1 is negative or 0
=> x+1 >= 0 => x >= -1
And x-1 <= 0 => x <= 1
Intersection is -1 <= x <= 1

Case 2
x+1 is negative or 0 and x-1 is positive or 0
x+1 <=0 => x <= -1
And x-1 >= 0 => x >= 1
No intersection in this case

So, solution to the problem is -1 <= x <= 1
So, Answer will be E

4. If y>0>x, and $\frac{(3+5y)}{(x-1)} < -7$, then which of the following must be true?

A. 5y−7x+4 < 0
B. 5y+7x−4 > 0
C. 7x−5y−4 < 0
D. 4+5y+7x > 0
E. 7x−5y+4 > 0

Solution:

$\frac{3+5y}{x-1} < -7$

=> $\frac{3+5y}{x-1} + 7 < 0$

=> $\frac{(3+5y)+7*(x-1)}{x-1} < 0$

=> $\frac{7x+5y-4}{x-1} < 0$

Now, we know that x < 0 so, x- 1 < 0

in $\frac{7x+5y-4}{x-1}$

we know that x - 1 < 0

=> (7x + 5y -4) > 0, So, Answer will be B

Solving Linear Inequalities

Rules of Operations in an Inequality

1. Multiplication/Division Properties for Inequalities: when multiplying/dividing by a positive value

If a < b AND c is positive, then $ac < bc$
If a < b AND c is positive, then $a/c < b/c$

2. Multiplication/Division Properties for Inequalities: when multiplying/dividing by a negative value
If a < b AND c is negative, then ac > bc
If a < b AND c is negative, then a/c > b/c

When solving inequalities, if you multiply or divide through by a negative, you must also flip the inequality sign.

3. Compound Inequality

Solve 10 < 3x + 4 < 19.

This is what is called a "compound inequality". It works just like regular inequalities, except that it has three "sides". So, for instance, when I go to subtract the 4, I will have to subtract it from all three "sides".

Three Step Solving Strategy:

Step 1: Simplify each side, if needed.
This would involve things like removing ( ), removing fractions, adding like terms, etc.

Step 2: Use Add./Sub.
Properties to move the variable term on one side and all other terms to the other side.

Step 3: Use Mult./Div.
Properties to remove any values that are in front of the variable.

Example:

$2(y+1) <= y-4$
$2y+2 <= y-4$ Step 1
or $2y+2-y <= y-4-y$ Step 2
or $y+2 <= -4$
$y+2-2 \leq -4-2$ Step 2
or $y <= -6$

Absolute value equations and inequalities

An absolute value of a number a is represented as, $|a|$ $an\ absolute\ equation, |x| = a$ consists of two solutions, $x = a$ and $x = -a$. Hence, care should be taken while solving an equation consisting of an absolute value.

For example, the equation, $|x-17| = 45$ is solved in two different parts.

Part I: $x-17 = 45$ making $x = 62$
Part II: $x-17 = -45$ making $x = -28$

An inequality equation with an absolute value, $|x| > a$ means $x > a$and $x < -a$
For example,
$5|3x+18| < 45$

In order to solve this equation we first divide it by 5, $|3x + 18| < 9$
Now comes the most important step of removing absolute sign and rewriting the equation as, $-9 < 3x + 18 < 9$
This equation is finally solved as, $-27 < 3x < -9. Hence, -9 < x < 3$ is the solution.

Inequalities Tips

ADDING/SUBTRACTING INEQUALITIES

1. You can only add inequalities when their signs are in the same direction:

If $a > b$ and $c > d$ (signs in same direction: $>$ and $>$ $a + c > b + d$
$3 < 4$and $2 < 5$--> $3 + 2 < 4 + 5$

2. You can only apply subtraction when their signs are in the opposite directions:

If $a > b$ and $c < d$ (signs in opposite direction: $>$ and $<$) $\rightarrow a - c > b - d$ (take the sign of the inequality you subtract from).
Example: $3 < 4$ and $5 > 1$--> $3 - 5 < 4 - 1$

RAISING INEQUALITIES TO EVEN/ODD POWER

1. We can raise both parts of an inequality to an even power if we know that both parts of an inequality are non-negative (the same for taking an even root of both sides of an inequality).
For example:

$2 < 4$--> we can square both sides and write: $2^2 < 4^2$
$0 \leq x < y$--> we can square both sides and write: $x^2 < y^2$

But if either of side is negative then raising to even power doesn't always work.
For example:
$1 > -2$ if we square, we'll get $1 > 4$ which is not right. So, if given that $x > y$ then we cannot square both sides and write $x^2 > y^2$ if we are not certain that both x and y are non-negative.

2. We can always raise both parts of an inequality to an odd power (the same for taking an odd root of both sides of an inequality).

For example:
$-2 < -1$--> we can raise both sides to third power and write: $-2^3 = -8 < -1 = -1^3$
or $-5 < 1$--> $-5^3 = -125 < 1 = 1^3$

$x < y$--> we can raise both sides to third power and write: $x^3 < y^3$

MULTIPLYING/DIVIDING TWO INEQUALITIES

1. If both sides of both inequalities are positive and the inequalities have the same sign, you can multiply them.
For example, for positive $x, y, a, b, if\ x < a$ and $y < b\ xy < ab$.

2. If both sides of both inequalities are positive and the signs of the inequality are opposite, then you can divide them.

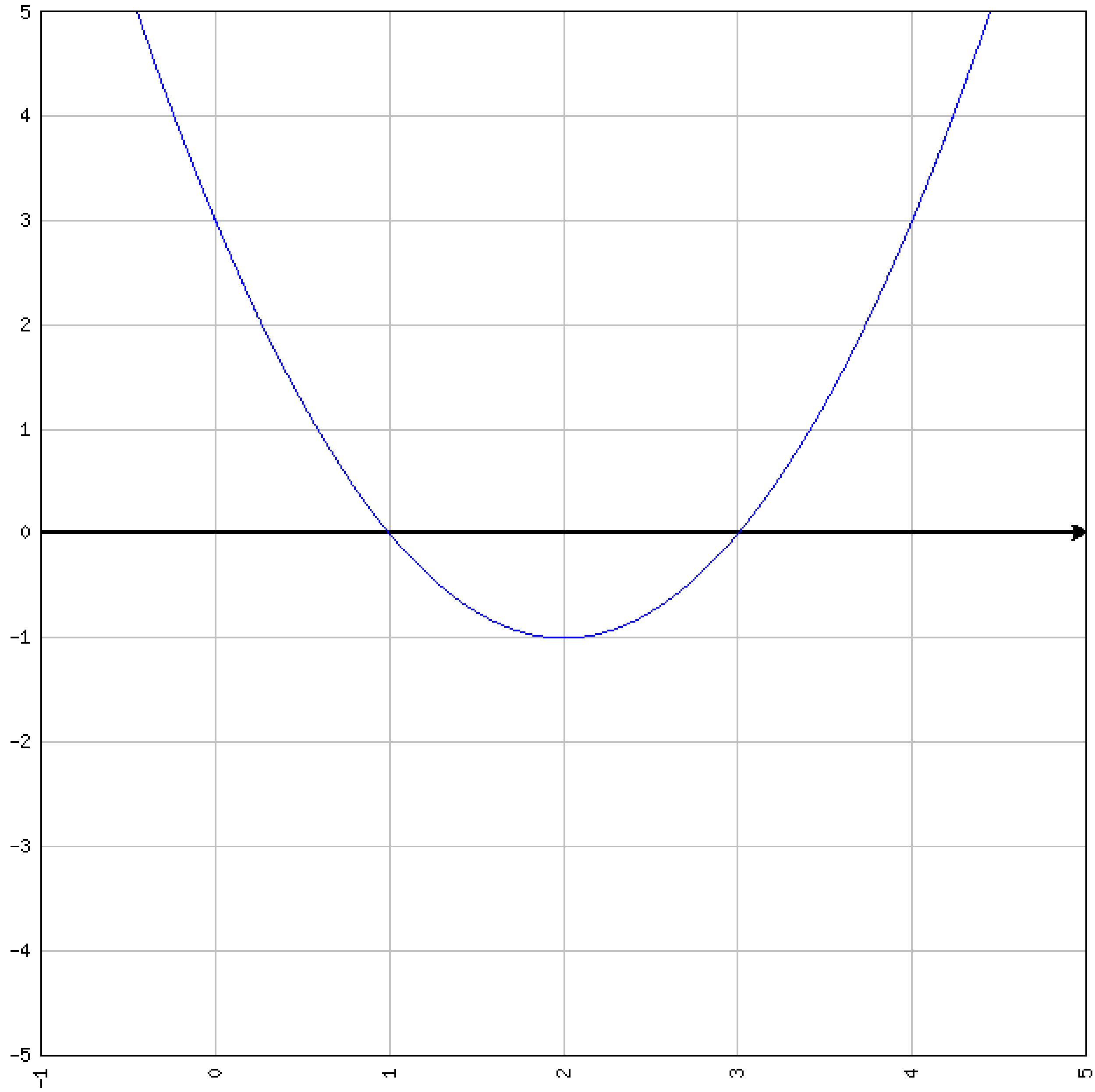
For example, for positive $x, y, a, b\ if\ x < a$ and $y > b\ then\ \frac{x}{y} < \frac{a}{b}$ (The final inequality takes the sign of the numerator).

MULTIPLYING/DIVIDING AN INEQUALITY BY A NUMBER

1. Whenever you multiply or divide an inequality by a *positive* number, you must *keep* the inequality sign.
2. Whenever you multiply or divide an inequality by a *negative* number, you must *flip* the inequality sign.
3. Never multiply (or reduce) an inequality by a variable (or the expression with a variable) if you don't know the sign of it or are not certain that variable (or the expression with a variable) doesn't equal to zero.

SOLVING QUADRATIC INEQUALITIES: GRAPHIC APPROACH

Say we need to find the ranges of $x$for $x^2 - 4x + 3 < 0$ $x^2 - 4x + 3 = 0$ is the graph of a parabola and it look likes this:



Intersection points are the roots of the equation $x^2 - 4x + 3 = 0$ $x_1 = 1$and $x_2 = 3$ "<" sign means in which range of $x$the graph is *below* x-axis. Answer is $1 < x < 3$ (between the roots).

If the sign were ">": $x^2 - 4x + 3 > 0$ $x_1 = 1$ and $x_2 = 3$ ">" sign means in which range of x the graph is *above* x-axis. Answer is $x < 1$ and $x > 3$ (to the left of the smaller root and to the right of the bigger root).

This approach works for any quadratic inequality. For example: $-x^2 - x + 12 > 0.$ $first\ rewrite\ this\ as\ x^2 + x - 12 < 0$ (so that the coefficient of x^2 to be positive. It's possible to solve without rewriting, but easier to master one specific pattern).
$x^2 + x - 12 < 0.\ Roots\ are\ x_1 = -4$ and $x_1 = 3$--> below ("<") the x-axis is the range for $-4 < x < 3$(between the roots).

Again, if it were $x^2 + x - 12 > 0. Then\ the\ answer\ would\ be\ x < -4$ and $x > 3$ (to the left of the smaller root and to the right of the bigger root).

2.5- FUNCTIONS

Frequency of the concepts tested: High

Definition: What is a Function?

A function is a relation which takes an input and gives a unique output for that input.

Ex: f(x) = 2x + 1
For each value of x we will get a unique value of f(x)
If x=1, then in f(x) we will replace x with 1 on both left and right hand side of f(x) = 2x+1 to get
f(1) = 2*1 + 1 = 3

• Domain and Range of a Function

Domain: The set of values which the independent variable [Ex: (x) in f(x)] can take for which the dependent variable [Ex: f(x)] has a valid value [Ex: value should not be undefined]

Ex: f(x) = $\frac{1}{x-2}$
Now, we know that a fraction becomes undefined when the denominator is zero. So, f(x) = $\frac{1}{x-2}$ will become undefined when denominator will become 0.

So, for x-2 = 0 or, x=2 f(x) is not defined

So, Domain of f(x) is x ∈ R (x belongs to all real numbers) and x ≠ 2

Range: The set of resulting values of the function f(x) for all possible values of x in the domain of f(x)

Ex: f(x) = $\frac{1}{x-2}$
Range of f(x) is the set of all values which f(x) takes when x takes the values in the domain (i.e x ∈ R and x ≠ 2) of f(x)

Ex. x=3 is in domain so, f(3) = $\frac{1}{3-2}$= 1 is in Range of the function.

Problem Types

• PT1: f(constant) Problems

In these type of problems F(x) will be given and we will be asked to find the value of f(constant), ex f(1) and so no. Let's take some examples:

Q1. f(x) = 3x + 1. Find f(2)
Sol: Compare the things inside the bracket in f(x) and f(2). So, we need to replace x with 2 in f(x) = 3x + 1 to get the value of f(2)
=> f(2) = 3*2 + 1 = 7

Q2. If f(x) = $\frac{(x+3)}{(2x-6)}$ where x ≠ 3, then find f(5).
Sol: Replace x with 5 in f(x) = $\frac{(x+3)}{(2x-6)}$ we get
f(5) = $\frac{(5+3)}{(2*5-6)}$= $\frac{8}{4}$= 2

Q3. If f(x) =$2x^2$+ 5 and f(a) = 13. Then find the value of a.

Sol: Replace x with a in f(x) =$2x^2$+ 5 we have
f(a) =$2a^2$+ 5 = 13 (given)
=> $2a^2$= 13 - 5 = 8
=> a^2= 4
=> a = ± 2

Q4. If f(x) = $\frac{(5x-1)}{(3x-7)}$ where x ≠ $\frac{7}{3}$ and f(a) = 7 then find the value of a.

Sol: Replace x with a in f(x) = $\frac{(5x-1)}{(3x-7)}$ we have
f(a) = $\frac{(5a-1)}{(3a-7)}$= 7 (given)
=> 5a - 1 = 7* (3a-7)
=> 5a - 1 = 21a - 49
=> 48 = 16a => a = 3

• PT2: Reciprocal Problems

In this type of problems we will be given f(x) and will be required to find the value of f($\frac{1}{x}$)
Let, take some examples.

Q1. If f(x) = 2x+1, then find the value of f($\frac{1}{x}$)

Sol: We will compare what is inside the bracket of f(x) and f($\frac{1}{x}$)

So, we need to replace x with $\frac{1}{x}$ in $\frac{1}{x}$ in f(x) = 2x+1 to get the value of f($\frac{1}{x}$)

=> f($\frac{1}{x}$) $= 2 * \frac{1}{X} + 1 = \frac{(2+x)}{x}$)

Q2. If f(x) = $\frac{(x-2)}{(3x+4)}$ $then\ find\ the\ value\ of\ f$($\frac{1}{x}$)

Sol: Replace x with $\frac{1}{x}$in f(x) = $\frac{(x-2)}{(3x+4)}$ we get

f($\frac{1}{x}$) $= \frac{1}{x}$ − 2) / (3$\frac{1}{x}$ + 4)

=> f($\frac{1}{x}$) $= \frac{(1-2x)}{(3+4x)}$

• PT3 : Nested Functions Problems

In this type of problems we will be given two functions, let's say f(x) and g(x) and will be required to find the value of f(g(x)) or g(f(x)). Let's take some examples:

Q1. If f(a) =$\boldsymbol{a^2}$, $\boldsymbol{g(a) = a^3}$, Find f(g(a))

Sol: f(g(a)) is called as nested function, as one function is inside the other.

Let's start by finding the value of g(a) first. g(a) = a^3[given]

So, f(g(a)) = f(a^3)

To find f(a^3) $we\ will\ we\ will\ replace\ a\ with\ a^3$in f(a) =$a^2$ to get

f(a^3) = ((a^3)^2)= a^6

=> f(g(a)) = a^6

Q2. If f(x) = 2x-3 and g(x) = x^3 Then find f(g(x))

Sol: Let's start by finding the value of g(x) first. g(x) = x^3[given]
So, f(g(x)) = f(x^3) $= 2 * x^3$- 3)

Q3. If f(x) = $\frac{(2x+3)}{(3x+5)}$ and g(x) = 3x – 2, then find the value of f(g(2)) and g(f(3))

Sol: f(g(2)) = f(3*2 -2) = f(4) = $\frac{(2*4+3)}{(3*4+5)} = \frac{11}{17}$

g(f(3)) = g($\frac{(2*3+3)}{(3*3+5)}$) $= g\left(\frac{9}{14}\right) = 3 * \frac{9}{14}$ - 2 = $\frac{-1}{14}$

Q4. If f(x) = 3x-2 and g(x) = x^2. Then for what value of x is f(g(x)) = g(f(x))

Sol: f(g(x)) = f(x^2) $= 3 * x^2$- 2

g(f(x)) = g(3x-2) = $(3x-2)^2$

x^2- 2 = $(3x-2)^2$

$f(g(x)) = g(f(x))$ => $3 * x^2$- 2 = (3x-2)^2

=> 3* x^2- 2 = 9x^2- 2*3x**2^2

=> 3*x^2- 2 = 9x^2-12x +2^2

=> 6x^2- 12x + 6 = 0

=> x^2- 2x + 1 = 0

=> $(x-1)^2$= 0

=> x = 1

• PT4 : Simultaneous Equations Problems

In this type of problems we will be given a function and some of the values for that function and we would be asked to find value of f(something). Let's take some examples:

Q1. If f(x) = a + bx and f(1) = 6 and f(2) = 10, then find the value of f(10)

Sol: f(1) = a + b*1 = a + b = 6 (given) ....(1)

f(2) = a + b*2 = a + 2b = 10 (given) .....(2)

(2) - (1) we get

(a+2b) - (a+b) = 10 - 6 =4

=> b= 4

a + b = 6 => a = 2

So, f(x) = 2 + 4x

f(10) = 2 + 4*10 = 42

Q2. If f(x) = ax^2+ bx and f(1)=10 and f(2)=30, then find the value of f(3)

Sol: f(1) = a1^2+ b*1 = a + b = 10 (given) ....(1)

f(2) = a2^2+ b*2 = 4a + 2b = 30 (given) ....(2)

(1) * 4 - (2) we get

4(a+b) - (4a + 2b) = 4*10 - 30 = 10

2b = 10 => b = 5

a + b = 10 => a = 5

f(x) = 5x^2+ 5x

f(3) = 5$3^2$+ 5*3 = 45 + 15 = 60

• PT5 : Symmetric Functions Problems

In these types of problems we will be given couple of f(x)'s and we will be asked for which of this function f(X) = f(1-x). Let's take some examples:

Q1. For which of the following functions is f(x) = f(1-x) for all values of x?

A. f(x) = x^2

B. f(x) = x + 1

C. f(x) = 2x + 1

D. f(x) = x*(1-x)

Sol: We need to find the value of f(1-x) using f(x) in all the four options and check for which option is the value of f(1-x) exactly equal to f(x)

A. f(x) = x^2

$f(1-x)^2$= 1 - 2x + $x^2 \neq x^2$for all values of x. So, f(x) ≠ f(1-x) for all x

B. f(x) = x + 1

f(1-x) = (1-x) + 1 = 2 - x ≠ x + 1 for all values of x. So, f(x) ≠ f(1-x) for all x

C. f(x) = 2x + 1

f(1-x) = 2(1-x) + 1 = 3 - 2x ≠ 2x + 1 for all values of x. So, f(x) ≠ f(1-x) for all x

D. f(x) = x*(1-x)

f(1-x) = (1-x)*(1-(1-x)) =(1-x)*x = x*(1-x) for all values of x. So, f(x) = f(1-x) for all x

So, Answer is D

Now, this is lengthy to do. But, there is a theory behind symmetry which can make these problems easier to solve:

<u>How to check if a function is symmetric in terms of x and 1-x</u>

• f(x) = f(1-x) when f(x) is symmetric in terms of x and (1-x), which means

• If there is a term of x in numerator then there should be a term of (1-x) in numerator with the same power and the same sign

• If there is a term of x in denominator then there should be a term of (1-x) in denominator with the same power and the same sign

Examples of Symmetric Functions

f(x) = f(1-x) in the below examples as the function is symmetric in terms of x and (1-x)

1. f(x) = $x^2 * (1-x)2$

The function has a term **x** and a term **(1 – x)** in the numerator with the same power (2) and the same sign (+)

2. f(x) = $x^2 + (1-x)^2$

The function has a term **x** and a term **(1 − x)** in the numerator with the same power (2) and the same sign (+)

3. f(x) = $\frac{1}{(-x^3 * -(1-x)^3)}$

The function has a term **x** and a term **(1 − x)** in the denominator with the same power (3) and the same sign (-)

Examples of Non-Symmetric Functions

f(x) ≠ f(1-x) in the below examples as function is non-symmetric in terms of x and (1-x)

1. f(x) = $x^2 * (1-x)^3$

Different Powers: Function is having a term of x and a term of (1-x) in the numerator and the same sign (+) but the power of x and (1-x) is different, 2 and 3 respectively

2. f(x) = x^2- $(1-x)^2$

Different Signs: Function is having a term of x and a term of (1-x) in the numerator with the same power (2) but different sign (+) and (-) respectively

3. f(x) = $\frac{1}{(-x^3+(1-x)^3)}$

Function is having a term of x and a term of (1-x) in the denominator with the same power (3) but different signs

4. f(x) = $\frac{x^2}{(1-x)^2}$

Function is having a term of x in numerator but no term of (1-x) in numerator and a term for (1-x) in denominator but no term of x in denominator, so not symmetric

• PT6: Custom Characters Problems

In these types of problems we will be given some custom characters like @their definition will be given too. We will to consider them as functions and solve the problems. Let's take some examples:

Q1. If $a@b$= a^2- 2ab, then find the value of 2 @ (3@1)

Sol: Here we will consider @a function of a and b

Let's find the value of 3@1 first.

we will compare 3@1with $a@b$=> a= 3 and b= 1

=> $3@1 = 3^2$- 2*3*1 = 9 - 6 = 3

=> 2 @ $(3@1) = 2@3 = 2^2$- 2*2*3 = 4 - 12 = -8

Q2. If a □ b = b □ a for all values of a and b, then what can be the possible operation □?

1. Addition
2. Multiplication

3. Division
4. Subtraction

Sol:
1. Addition => a + b = b + a which is true for all values of a and b. So, it is possible
2. Multiplication => a * b = b * a which is true for all values of a and b. So, it is possible
3. Division=> $\frac{a}{b}\frac{b}{a}$which does not have to be true for all values of a and b. So, it is NOT possible
4. Subtraction => a - b = b - a which does not have to be true for all values of a and b. So, it is NOT possible

So, answer is a 1 and 2

Q3. If a # b = a + b - 2ab, then find the value of (4 # 2) # (1 # 3)

1. 2 # 3
2. 3 # 4
3. 4 # 8
4. 1 # 2

Sol: 4 # 2 = 4 + 2 - 2*4*2 = -10
1 # 3 = 1 + 3 - 2*1*3 = -2
=> (4 # 2) # (1 # 3) = -10 # -2 = -10 -2 -2*(-10)*(-2) = -12 - 40 = -52
Now, the answer choices are also given in terms of the custom characters. So, we need to solve each answer to check which one will give us -52 answer.
1. 2 # 3 = 2+3 - 2*2*3 = 5 - 12 = -7, so not the answer
2. 3 # 4 = 3+4 - 2*3*4 = 7 - 24 = -17, so not the answer
3. 4 # 8 = 4+8 - 2*4*8 = 12 - 64 = -52, so THIS is the answer. We don't need to solve further but solving just to explain the process
4. 1 # 2 = 1+2 - 2*1*2 = 3 - 4 = -1

Q4. Solve for ◊ and □ in the below subtraction equation. (given that ◊ and □ are single digit numbers)

◊ ◊ 4
- □ ◊
----
5 7 □

Answer is ◊ = 6 and □ = 8

Q5. Solve for ◊ and □ in the below multiplication equation. (given that ◊ and □ are single digit numbers)

3 ◊
* □ 2
----
15 5 □

Answer is ◊ = 7 and □ = 4

2.6- MIXTURE, RATE, AND WORK PROBLEMS

Frequency of the concepts tested: **Medium**

The Following Points Outline a General Approach to Word Problems:

1) Read the entire question **carefully** and get a **feel** for what is happening. Identify what kind of word problem you're up against.

2) Make a note of **exactly** what is being asked.

3) **Simplify the problem** - this is what is usually meant by *'translating the English to Math'*. Draw a figure or table. Sometimes a simple illustration makes the problem much easier to approach.

4) It is not always necessary to start from the first line. Invariably, you will find it easier to **define what you have been asked for and then work backwards** to get the information that is needed to obtain the answer.

5) Use *variables* (a, b, x, y, etc.) or *numbers* (100 in case of percentages, any common multiple in case of fractions, etc.) **depending on the situation.**

6) Use **SMART** values. Think for a moment and choose the best possible value that would help you reach the solution in the quickest possible time. DO NOT choose values that would serve only to confuse you. Also, remember to make note of what the value you selected stands for.

7) Once you have the equations written down it's time to **do the math!** This is usually quite simple. Be very careful so as not to make any silly mistakes in calculations.

8) Lastly, after solving, **cross check** to see that the answer you have obtained corresponds to **what was asked.** The makers of these GRE questions love to trick students who don't pay careful attention to what is being asked. For example, if the question asks you to find 'what fraction of the remaining...' you can be pretty sure one of the answer choices will have a value corresponding to 'what fraction of the total…'

Translating Word Problems

These are a few common English to Math translations that will help you break down word problems. My recommendation is to refer to them only in the initial phases of study. With practice, decoding a word problem should come naturally. If, on test day, you still have to try and remember what the math translations to some English term is, you haven't practiced enough!
ADDITION: increased by; more than; combined; together; total of; sum; added to; and; plus

SUBTRACTION: decreased by; minus; less; difference between/of; less than; fewer than; minus; subtracted from

MULTIPLICATION: of ; times; multiplied by; product of; increased/decreased by a factor of (this type can involve both addition or subtraction and multiplication!)

DIVISION: per; out of; ratio of; quotient of; percent (divide by 100); divided by; each

EQUALS: is; are; was; were; will be; gives; yields; sold for; has; costs; adds up to; the same as; as much as

VARIABLE or VALUE: a number ; how much ; how many ; what

Some Tricky Forms:

'*per*' means '*divided by*'
Jack drove at a speed of 40 miles per hour OR 40 miles/hour.

'*a*' sometimes means '*divided by*'
Jack bought twenty-four eggs for $3 a dozen.

'*less than*'
In English, the 'less than' construction is **reverse** of what it is in math.
For example, '3 less than x' means 'x – 3' NOT '3 – x'
Similarly, if the question says 'Jack's age is 3 less than that of Jill', it means that Jacks age is 'Jill's age – 3'.

The '*how much is left*' construction
Sometimes, the question will give you a total amount that is made up of a number of smaller amounts of unspecified sizes. In this case, just assign a variable to the unknown amounts and the remaining amount will be what is left after deducting this named amount from the total. *Consider the following:*

A hundred-pound order of animal feed was filled by mixing products from Bins A, B and C, and that twice as much was added from Bin C as from Bin A.

Let "a" stand for the amount from Bin A. Then the amount from Bin C was "2a", and the amount taken from Bin B was the remaining portion of the hundred pounds: 100 – a – 2a.
In the following cases, order is important:

'*quotient/ratio of*' construction
If a problems says 'the ratio of x and y', it means 'x divided by y' NOT 'y divided by x'

‘*difference between/of*’ construction
If the problem says ‘the difference of x and y’ it means $|x - y|$

Now that we have seen how it is possible, in theory, to break down word problems, lets go through a few simple examples to see how we can apply this knowledge.

Example 1.
The length of a rectangular garden is 2 meters more than its width. Express its length in terms of its width.

Solution:
Key words: more than (implies addition); is (implies equal to)
Thus, the phrase ‘length is 2 more than width’ becomes:
Length = 2 + width

Example 2.
The length of a rectangular garden is 2 meters less than its width. Express its length in terms of its width.

Solution:
Key words: less than (implies subtraction but in reverse order); is (implies equal to)
Thus, the phrase ‘length is 2 less than width’ becomes:
Length = width - 2

Example 3.
The length of a rectangular garden is 2 times its width. Express its length in terms of its width.

Solution:
Key words: times (implies multiplication); is (implies equal to)
Thus, the phrase ‘length is 2 times width’ becomes:
Length = 2*width

Example 4.
The ratio of the length of a rectangular garden to its width is 2. Express its length in terms of its width.
Solution:
Key words: ratio of (implies division); is (implies equal to)
Thus, the phrase ‘ratio of length to width is 2’ becomes:
Length/width = 2 → Length = 2*width

Example 5.
The length of a rectangular garden surrounded by a walkway is twice its width. If difference between the length and width of just the rectangular garden is 10 meters, what will be the width of the walkway if just the garden has width 6 meters?

Solution:
Ok this one has more words than the previous examples, but don't worry, lets break it down and see how simple it becomes.
Key words: and (implies addition); twice (implies multiplication); difference between (implies subtraction where order is important); what (implies variable); is, will be (imply equal to)

Since this is a slightly more complicated problem, let us first define what we want.

'What will be the width of the walkway' implies that **we should assign a variable for width of the walkway** and find its value.

Thus, let width of the walkway be 'x'.

Now, in order to find the width of walkway, we need to have some relation between the **total length/width of the rectangular garden + walkway and the length/width of just the garden**.

Notice here that if we assign a variables to the width and length of either garden+walkway or just garden, we can express every thing in terms of just these variables.

So, let length of the garden+walkway = L

And width of garden+walkway = W

Thus length of just garden = L – 2x

Width of just garden = W - 2x

Note: Remember that the walkway completely surrounds the garden. Thus its width will have to be accounted for twice in both the total length and total width.

Now let's see what the question gives us.

'Garden with width 6 meters' translates to:
Width of garden = 6
W – 2x = 6
Thus, if we know W we can find x.

'Length of a rectangular garden surrounded by walkway is twice its width' translates to:
Length of garden + length of walkway = 2*(width of garden + width of walkway)
$L = 2*W$

'Difference between the length and width of just the rectangular garden is 10 meters' translates to:
Length of garden – width of garden = 10
$(L - 2x) - (W - 2x) = 10$
$L - W = 10$

Now, since we have two equations and two variables (L and W), we can find their values. Solving them we get: L = 20 and W = 10.

Thus, since we know the value of W, we can calculate 'x'

$10 - 2x = 6$
$2x = 4$
$x = 2$

Thus, the width of the walkway is 2 meters.

Easy wasn't it?

With practice, writing out word problems in the form of equations will become second nature. How much you need to practice depends on your own individual ability. It could be 10 questions or it could be 100. But once you're able to effortlessly translate word problems into equations, more than half your battle will already be won.

What is a 'Work' Word Problem?

- It involves a number of people or machines working together to complete a task.
- We are usually given individual rates of completion.
- We are asked to find out how long it would take if they work together.

Sounds simple enough, doesn't it? Well, it is!

There is just one simple concept you need to understand in order to solve any 'work' related word problem.

The 'Work' Problem Concept

STEP 1: Calculate how much work each person/machine does in one unit of time (could be days, hours, minutes, etc).

How do we do this? Simple. If we are given that A completes a certain amount of work in X hours, simply reciprocate the number of hours to get the per hour work. Thus, in one hour, A would complete $\frac{1}{X}$ of the work. But what is the logic behind this? Let me explain with the help of an example.

Assume we are given that Jack paints a wall in 5 hours. This means that in every hour, he completes a fraction of the work so that at the end of 5 hours, the fraction of work he has completed will become 1 (that means he has completed the task).

Thus, if in **5** hours the fraction of work completed is **1**, then in **1** hour, the fraction of work completed will be **(1*1)/5**

STEP 2: Add up the amount of work done by each person/machine in that one unit of time.

This would give us the total amount of work completed by both of them in one hour. For example, if **A completes** $\frac{1}{X}$ of the work in one hour and **B completes** $\frac{1}{Y}$ of the work in one hour, then **TOGETHER**, they can complete $\frac{1}{X}+\frac{1}{Y}$ of the work **in one hour**.

STEP 3: Calculate total amount of time taken for work to be completed when all persons/machines are working together.

The logic is similar to one we used in **STEP 1**, the only difference being that we use it in reverse order. Suppose $\frac{1}{X}+\frac{1}{Y}=\frac{1}{Z}$ **in one hour**, A and B **working together** will complete $\frac{1}{Z}$ of the work. **Therefore, working together, they will complete the work in Z hours.**

Advice here would be: DON'T go about these problems trying to remember some formula. Once you understand the logic underlying the above steps, you will have all the information you need to solve any 'work' related word problem. (You will see that the formula you might have come across can be very easily and logically deduced from this concept).

Now, let's go through a few problems so that the above-mentioned concept becomes crystal clear. Let's start off with a simple one:

Example 1.
Jack can paint a wall in 3 hours. John can do the same job in 5 hours. How long will it take if they work together?

Solution:
This is a simple straightforward question wherein we must just follow steps 1 to 3 in order to obtain the answer.

STEP 1: Calculate how much work each person does in one hour.
Jack → (1/3) of the work
John → (1/5) of the work

STEP 2: Add up the amount of work done by each person in one hour.
Work done in one hour when both are working together = $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$

If they complete $\frac{8}{15}$ of the work in **1** hour, then they would complete 1 job in $\frac{15}{8}$hours.

Example 2.
Working, independently X takes 12 hours to finish a certain work. He finishes 2/3 of the work. The rest of the work is finished by Y whose rate is 1/10 of X. In how much time does Y finish his work?

Solution:
Now the only reason this is trickier than the first problem is because the sequence of events are slightly more complicated. The concept however is the same. So if our understanding of the concept is clear, we should have no trouble at all dealing with this.

'Working, independently X takes 12 hours to finish a certain work' This statement tells us that in one hour, X will finish $\frac{1}{12}$ of the work.
'He finishes 2/3 of the work' This tells us that $\frac{1}{3}$ of the work still remains.
'The rest of the work is finished by Y whose rate is (1/10) of X' Y has to complete $\frac{1}{3}$ of the work.
'Y's rate is (1/10) that of X'. We have already calculated rate at which X works to be $\frac{1}{12}\frac{1}{10} * \frac{1}{12} = \frac{1}{120}$
'In how much time does Y finish his work?' If Y completes $\frac{1}{120}$ of the work in **1** hour, then he will complete $\frac{1}{3}$ of the work in 40 hours.

So as you can see, even though the question might have been a little difficult to follow at first reading, the solution was in fact quite simple. We didn't use any new concepts. All we did was apply our knowledge of the concept we learnt earlier to the information in the question in order to answer what was being asked.

Example 3.
Working together, printer A and printer B would finish a task in 24 minutes. Printer A alone would finish the task in 60 minutes. How many pages does the task contain if printer B prints 5 pages a minute more than printer A?

Solution:
This problem is interesting because it tests not only our knowledge of the concept of word problems, but also our ability to 'translate English to Math'

'Working together, printer A and printer B would finish a task in 24 minutes' This tells us that A and B combined would work at the rate of $\frac{1}{24}$ per minute.
'Printer A alone would finish the task in 60 minutes' This tells us that A works at a rate of $\frac{1}{60}$ per minute.

At this point, it should strike you that with just this much information, it is possible to calculate the rate at which B works: Rate at which B works = $\frac{1}{24} - \frac{1}{60} = \frac{1}{40}$

'B prints 5 pages a minute more than printer A' This means that the difference between the amount of work B and A complete in one minute corresponds to 5 pages. So, let us calculate that difference. It will be $\frac{1}{40} - \frac{1}{60} = \frac{1}{120}$

If $\frac{1}{120}$ of the job consists of **5** pages, then the **1** job will consist of $\frac{(5*1)}{\frac{1}{120}} = 600$ pages.

Example 4.
Machine A and Machine B are used to manufacture 660 sprockets. It takes machine A ten hours longer to produce 660 sprockets than machine B. Machine B produces 10% more sprockets per hour than machine A. How many sprockets per hour does machine A produce?

Solution:
The rate of A is $\frac{660}{t+10}$ sprockets per hour;

The rate of B is $\frac{660}{t}$ sprockets per hour.

We are told that B produces 10% more sprockets per hour than A, thus $\frac{660}{t+10} * 1.1 = \frac{660}{t}$--> $t = 100$--> the rate of A is $\frac{660}{t+10} = 6$ sprockets per hour.

As you can see, the main reason the 'tough' problems are 'tough' is because they test a number of other concepts apart from just the ‘work’ concept. However, once you manage to form the equations, they are really not all that tough.

And as far as the concept of ‘work’ word problems is concerned – it is always the same!

What is a ‘D/S/T’ Word Problem?

- Usually involve something/someone moving at a constant or average speed.
- Out of the three quantities (speed/distance/time), we are required to find one.
- Information regarding the other two will be provided in the question stem.

The ‘D/S/T’ Formula: Distance = Speed x Time

I’m sure most of you are already familiar with the above formula (or some variant of it). But how many of you truly understand what it signifies?

When you see a ‘D/S/T’ question, do you blindly start plugging values into the formula without really understanding the logic behind it? If then answer to that question is yes, then you would probably have noticed that your accuracy isn’t quite where you’d want it to be.

My advice here, as usual, is to make sure you understand the concept behind the formula rather
than just using it blindly.

So, what’s the concept? Let’s find out!
The *Distance = Speed x Time* formula is just a way of saying that the distance you travel depends on the speed you go for any length of time.

If you travel at 50 mph for one hour, then you would have traveled 50 miles. If you travel for 2 hours at that speed, you would have traveled 100 miles. 3 hours would be 150 miles, etc.

If you were to double the speed, then you would have traveled 100 miles in the first hour and 200 miles at the end of the second hour.

We can figure out any one of the components by knowing the other two.

For example, if you have to travel a distance of 100 miles, but can only go at a speed of 50 mph, then you know that it will take you 2 hours to get there. Similarly, if a friend visits you from 100 miles away and tells you that it took him 4 hours to reach, you will know that he AVERAGED 25 mph. Right?

All calculations depend on AVERAGE SPEED.

Supposing your friend told you that he was stuck in traffic along the way and that he traveled at 50 mph *whenever he could move.* Therefore, although practically he never really traveled at 25 mph, you can see how the standstills due to traffic caused his average to reduce. Now, if you think about it, from the information given, you can actually tell how long he was driving and how long he was stuck due to traffic (assuming; what is false but what they never worry about in these problems; that he was either traveling at 50 mph or 0 mph). If he was traveling constantly at 50 mph, he should have reached in 2 hours. However, since he took 4 hours, he must have spent the other 2 hours stuck in traffic!

Now let's see how we can represent this using the formula.

We know that the **total distance is 100 miles** and that the total time is 4 hours. BUT his *rates were different* AND they were *different at different times.* However, can you see that no matter how many different rates he drove for various different time periods, his **TOTAL distance depended simply on the SUM of each of the different distances he drove during each time period?**

E.g., if you drive a half hour at 60 mph, you will cover 30 miles. Then if you speed up to 80 mph for another half hour, you will cover 40 miles, and then if you slow down to 30 mph, you will only cover 15 miles in the next half hour. But if you drove like this, you would have covered a total of 85 miles (30 + 40 + 15). It is fairly easy to see this looking at it this way, but it is more difficult to see it if we scramble it up and leave out one of the amounts and you have to figure it out going "backwards". That is what word problems do.

Further, what makes them difficult is that the components they give you, or ask you to find can involve variable distances, variable times, variable speeds, or any two or three of these. How you "reassemble" all this in order to use the d = s*t formula takes some reflection that is "outside" of the formula itself. **You have to think about how to use the formula.**

So, the trick is to be able to understand EXACTLY what they are giving you and EXACTLY what it is that is missing, but you do that from thinking, not from the formula, because the **formula only works for the COMPONENTS of any trip where you are going an average speed for a certain amount of time.** ONCE the conditions deal with different speeds or different times, you have to look at each of those components and how they go together. And that can be very difficult if you are not methodical in how you think about the components and how they go together. The formula doesn't tell you which components you need to look at and how they go together. For that, you need to think, and the thinking is not always as easy or straightforward as it seems like it ought to be.

In the case of your friend above, if we call the time he spent driving 50 mph, T1; then the time he spent standing still is (4 - T1) hours, since the whole trip took 4 hours. So, we have **100 miles = (50 mph x T1) + (0 mph x [4 - T1])** which is equivalent then to: **100 miles = 50 mph x T1**

So, T1 will equal 2 hours. And, since the time he spent going zero is (4 - 2), it also turns out to be 2 hours.

Sometimes the right answers will seem *counter-intuitive*, so it is really important to think about the components methodically and systematically.

There is a famous trick problem: To qualify for a race, you need to average 60 mph driving two laps around a 1-mile-long track. You have some sort of engine difficulty the first lap so that you only average 30 mph during that lap; how fast do you have to drive the second lap to average 60 for both of them?

I will go through THIS problem with you because, since it is SO tricky, it will illustrate a way of looking at almost all the kinds of things you have to think about when working any of these kinds of problems FOR THE FIRST TIME (i.e., before you can do them mechanically because you recognize the TYPE of problem it is). Intuitively it would seem you need to drive 90, but this turns out to be wrong for reasons I will give in a minute.

The answer is that NO MATTER HOW FAST you do the second lap, you can't make it. And this SEEMS really odd and that it can't possibly be right, but it is. The reason is that in order to average at least 60 mph over two one-mile laps, since 60 mph is one mile per minute, you will need to do the whole two miles in two minutes or less. But if you drove the first mile at only 30, you used up the whole two minutes just doing IT. So, you have run out of time to qualify.

To see this with the **d = s*t** formula, you need to look at the overall trip and break it into **components**, and that is the hardest part of doing this (these) problem(s), because (often) the components are difficult to figure out, and because it is hard to see which ones you need to put together in which way.

In the next section we will learn how to do just that.

Resolving the Components

When you first start out with these problems, the best way to approach them is by organizing the data in a tabular form.

Use a separate column each for distance, speed and time and a separate row for the different components involved (2 parts of a journey, different moving objects, etc.). The last row should represent total distance, total time and average speed for these values (although there might be no need to calculate these values if the question does not require them).

	DISTANCE (units)	AVERAGE SPEED (units)	TIME (units)
COMPONENT 1			
COMPONENT 2			
TOTAL			

Assign a variable for any unknown quantity.

If there is more than one unknown quantity, do not blindly assign another variable to it. Look for ways in which you can express that quantity in terms of the quantities already present. Assign another variable to it only if this is not possible.

In each row, the quantities of distance, speed and time will always satisfy d = s*t.
The distance and time column can be added to give you the values of total distance and total time but you CANNOT add the speeds.

Think about it: If you drive 20 mph on one street, and 40 mph on another street, does that mean you averaged 60 mph?

Once the table is ready, form the equations and solve for what has been asked!

Warning: Make sure that the units for time and distance agree with the units for the rate. For instance, if they give you a rate of feet per second, then your time must be in seconds and your distance must be in feet. Sometimes they try to trick you by using the wrong units, and you have to catch this and convert to the correct units.

A Few More Points to Note

Motion in Same Direction (Overtaking): The first thing that should strike you here is that at the time of overtaking, the distances traveled by both will be the same.

Motion in Opposite Direction (Meeting): The first thing that should strike you here is that if they start at the same time (which they usually do), then at the point at which they meet, the time will be the same. In addition, the total distance traveled by the two objects under consideration will be equal to the sum of their individual distances traveled.

Round Trip: The key thing here is that the distance going and coming back is the same.

Now that we know the concept in theory, let us see how it works practically, with the help of a few examples.

Note for tables: All values in black have been given in the question stem. All values in blue have been calculated.

Example 1.
To qualify for a race, you need to average 60 mph driving two laps around a 1-mile-long track. You have some sort of engine difficulty the first lap so that you only average 30 mph during that lap; how fast do you have to drive the second lap to average 60 for both of them?

Solution:
Let us first start with a problem that has already been introduced. You will see that by clearly listing out the given data in tabular form, we eliminate any scope for confusion.

	DISTANCE (miles)	AVERAGE SPEED (mph)	TIME (hours)
LAP 1	1	30	1/30
LAP 2	1	x	1/x
TOTAL	2	60	1/30

In the first row, we are given the distance and the speed. Thus, it is possible to calculate the time.

Time(1) = Distance(1)/Speed(1) = 1/30

In the second row, we are given just the distance. Since we have to calculate speed, let us give it a variable 'x'. Now, by using the 'D/S/T' relationship, time can also be expressed in terms of 'x'.

Time(2) = Distance(2)/Speed(2) = 1/x

In the third row, we know that the total distance is 2 miles (by taking the sum of the distances in row 1 and 2) and that the average speed should be 60 mph. Thus, we can calculate the total time that the two laps should take.

Time(3) = Distance(3)/Speed(3) = 2/60 = 1/30

Now, we know that the total time should be the sum of the times in row 1 and 2. Thus we can form the following equation:

Time(3) = Time(1) + Time(2) ---> 1/30 = 1/30 + 1/x

From this, it becomes clear that '1/x' must be 0.

Since 'x' is the reciprocal of 0, which does not exist, there can be no speed for which the average can be made up in the second lap.

Example 2.
An executive drove from home at an average speed of 30 mph to an airport where a helicopter was waiting. The executive boarded the helicopter and flew to the corporate offices at an average speed of 60 mph. The entire distance was 150 miles; the entire trip took three hours. Find the distance from the airport to the corporate offices.
Solution:
Since we have been asked to find the distance from the airport to the corporate office (that is the distance he spent flying), **let us assign that specific value as 'x'.**

Thus, the distance he spent driving will be '150 - x'
Now, *in the first row*, we have the distance in terms of 'x' and we have been given the speed. Thus, we can calculate the time he spent driving in terms of 'x'.

Time(1) = Distance(1)/Speed(1) = (150 - x)/30
Similarly, *in the second row*, we again have the distance in terms of 'x' and we have been given the speed. Thus we can calculate the time he spent flying in terms of 'x'.

Time(2) = Distance(2)/Speed(2) = x/60
Now, notice that we have both the times in terms of 'x'. Also, we know the total time for the trip. Thus, summing the individual times spent driving and flying and equating it to the total time, we can solve for 'x'.

Time(1) + Time(2) = Time(3) --> (150 - x)/30 + x/60 = 3 --> x = 120 miles
Answer: 120 miles

Note: In this problem, we did not calculate average speed for row 3 since we did not need it. Remember not to waste time in useless calculations!

Example 3.
A passenger train leaves the train depot 2 hours after a freight train left the same depot. The freight train is traveling 20 mph slower than the passenger train. Find the speed of the passenger train, if it overtakes the freight train in three hours.

Solution:
Since this is an 'overtaking' problem, the first thing that should strike us is that the distance traveled by both trains is the same at the time of overtaking.

Next, we see that we have been asked to find the speed of the passenger train at the time of overtaking. **So let us represent it by 'x'.**

Also, we are given that the freight train is 20 mph slower than the passenger train. **Hence its speed in terms of 'x' can be written as 'x - 20'.**

Moving on to the time, we are told that it has taken the passenger train 3 hours to reach the freight train. This means that the passenger train has been traveling for 3 hours.

We are also given that the passenger train left 2 hours after the freight train. This means that the freight train has been traveling for 3 + 2 = 5 hours.

Now that we have all the data in place, we need to form an equation that will help us solve for 'x'. Since we know that the distances are equal, let us see how we can use this to our advantage.

From the first row, we can form the following equation:

Distance(1) = Speed(1) * Time(1) = x*3
From the second row, we can form the following equation:

Distance(2) = Speed(2) * Time(2) = (x - 20)*5
Now, equating the distances because they are equal, we get the following equation:

3*x = 5*(x - 20) --> x = 50 mph.
Answer: 50 mph.

Example 4.
Two cyclists start at the same time from opposite ends of a course that is 45 miles long. One cyclist is riding at 14 mph and the second cyclist is riding at 16 mph. How long after they begin will they meet?

Solution:
Since this is a 'meeting' problem, there are two things that should strike you. First, since they are starting at the same time, when they meet, the time for which both will have been cycling will be the same. Second, the total distance traveled by the will be equal to the sum of their individual distances.

Since we are asked to find the time, let us assign it as a variable 't'. (which is same for both cyclists)

In the first row, we know the speed and we have the time in terms of 't'. Thus, we can get the following equation:

Distance(1) = Speed(1) * Time(1) = 14*t
In the second row, we know the speed and again we have the time in terms of 't'. Thus, we can get the following equation:

Distance(2) = Speed(2) * Time(2) = 16*t
Now we know that the total distance traveled is 45 miles and it is equal to the sum of the two distances. Thus, we get the following equation to solve for 't':

Distance(3) = Distance(1) + Distance(2) --> 45 = 14*t + 16*t --> t = 1.5 hours
Answer: 1.5 hours.

Example 5.
A boat travels for three hours with a current of 3 mph and then returns the same distance against the current in four hours. What is the boat's speed in calm water?

Solution:
Since this is a question on round trip, the first thing that should strike us is that the distance going and coming back will be the same.

Now, we are required to find out the boats speed in calm water. So let us assume it to be 'b'. Now if speed of the current is 3 mph, then the speed of the boat while going downstream and upstream will be 'b + 3' and 'b - 3' respectively.

In the first row, we have the speed of the boat in terms of 'b' and we are given the time. Thus, we can get the following equation:

Distance(1) = Speed(1) * Time(1) = (b + 3)*3
In the second row, we again have the speed in terms of 'b' and we are given the time. Thus, we can get the following equation:

Distance(2) = Speed(2) * Time(2) = (b - 3)*4
Since the two distances are equal, we can equate them and solve for 'b'.

Distance(1) = Distance(2) --> (b + 3)*3 = (b - 3)*4 --> b = 21 mph.
Answer : 21 mph.

Word Statement	Algebraic Translation
n is less than 15	$n < 15$
x less than 3	$3 - x$
3 less than a number x	$x - 3$
-8 is 5 less than x	$-8 = x - 5$ or $x = -3$
y is 5 less than twice the value of x	$y = 2x - 5$
Is x less than y?	Is $x < y$?
x is how much less than y?	What is the value of $y - x$?
If $x = \frac{z}{2}$ and $y = 3z$, then y is how many times x?	$y = 3z = 3(2x) = 6x$, y is six times x.
-8 is 5 more than x	$-8 = x + 5$ or $x = -13$
x is 10 more than y	$x = y + 10$
5 times the quantity $(x^2 + 3x)$	$5(x^2 + 2x)$
7 divided by x	$\frac{7}{x}$
7 divided into a number x	$\frac{x}{7}$
Is x at least 5?	Is $x \geq 5$?
x is positive and is at most 5	$0 < x \leq 5$
n is the square of an integer	$n = k^2$, where k is some integer.
The sum of a and b is 7	$a + b = 7$
The sum of a and b is at least 7	$a + b \geq 7$
n is an integer greater than 5 but less than 9	n can be 6, 7, or 8
x is 200% of y	$x = \left(\frac{200}{100}\right) y = 2y$
x is 200% greater than y	$x = y + \left(\frac{200}{100}\right) y = 3y$

Mixture problems show up frequently on the quantitative section of the GRE and fall into two basic categories. As each type of mixture question will be approached in fairly different ways, it is important that you know the difference between them.

First, there are mixture problems that ask you to alter the proportions of a single mixture. These questions could, for example, tell you that you have a 200-liter mixture that is 90% water and

10% bleach and ask how much water you would need to add to make it 5% bleach. The key in this type of question is the part of the mixture that is constant – in this case the bleach. While we are adding water, the amount of bleach stays the same. First, determine how much bleach we have. 10% of 200 is 20 liters. Next, we know we want those 20 liters to equal 5% of our total. Since 20 is 5% of 400, our new total should be 400 liters. To go from 200 liters to 400 liters, you would need to add 200 liters of water, which would be the answer (For yet another way to solve this type of GRE quantitative problem, check out this post).

The other type of mixture problem will ask you to combine two mixtures. For example, you could be told that mixture A is 20% bleach and 80% water, while mixture B is 5% bleach and 95% water. You could then be asked in what ratio these mixtures should be combined to achieve a mixture that is 10% bleach. You should solve problems such as this algebraically. Both sides of your equation will represent the amount of bleach in the combined mixture. On one side you will represent the amount of bleach in terms of the individual mixtures.

This will give you .2A + .05B. On the other side of the equation, you will represent the amount of bleach overall, which is .1(A + B). Note that in these expressions A represents the total amount of mixture A and B represents the total amount of mixture B. Because these expressions both represent the total amount of bleach, we can set them equal to each other.

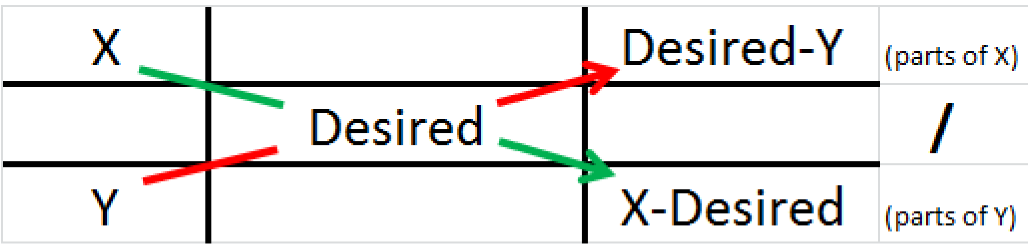
This gives us .2A + .05B = .1(A + B).

The ratio of A to B can be solved as follows:

.2A + .05B = .1(A + B)
.2A + .05B = .1A + .1B
.1A = .05B
A/B = .05/.1
A/B = ½

ANOTHER APPROACH to solve easily any mixture problem: it's called **ALLIGATION**.

It uses a simple table to solve any mixture problem, every answer to such problems can be obtained by looking at this table.



Please note: the X concentration is the highest, the Y is the lowest

The results that you get by subtracting, as I show you in the table, are the ratios of the substances in the desired mixture.

$$RATIO \frac{X}{Y} = \frac{Desired - Y}{X - Desired}$$

1) Seed mixture X is 40 percent ryegrass and 60 percent bluegrass by weight; seed mixture Y is 25 percent ryegrass and 75 percent fescue.

If a mixture of X and Y contains 30 percent ryegrass, what percent of the weight of this mixture is X?

(A) 10%
(B) 33.33 %
(C) 40%
(D) 50%
(E) 66.66 %

The question asks for the ryegrass so your table should look like this:

X mixture 40%		30-25=5
	30%	
Y mixture 25%		40-30=10

Solution: The final ratio is $\frac{X}{Y} = \frac{5}{10}$ or $(\frac{1}{2})$ so for every 1 part of X 2 parts of Y will be in the final mixture. So for a 3 kg mixture (for example)=> 1X and 2Y => $X = 33$ of the total **B**

This table can be used in other ways also, and this question is an example:

2)How many liters of pure alcohol must be added to a 100-liter solution that is 20 percent alcohol in order to produce a solution that is 25 percent alcohol?

(A) 7/2
(B) 5
(C) 20/3
(D) 8
(E) 39/4
Your table:

X pure alcohol 100%		25-20=5
	25%	
Y alcohol 20%		100-25=75

Final ratio: $\frac{X}{Y} = \frac{5}{75}$

$\frac{X}{100} = \frac{5}{75} X = \frac{20}{3}$ **C**

Easy!

As you see mixture problems start to look very easy if you consider this method, and for sure all this will save you valuable time.

IMPORTANT LINK!

GRE PREMIUM Quant Question Banks - Topic-Wise

CHAPTER 3. GEOMETRY

3.1 - LINES AND ANGLES

Frequency of the concepts tested: High

Definition

A. ***Lines, Line Segments, Rays, and Angles***

1. **Line**-is a set of points extending indefinitely in two directions.
2. **Line segment**-is a piece of a line with two endpoints.
3. **Ray**-is a part of a line with one endpoint.
4. **Angle**-is made up of two rays that share the same endpoint calls vertex.

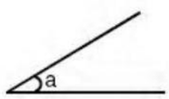
B. ***Angels as Acute, Right, Obtuse, or Straight***

1. **Acute Angle**-is an angle whose measure is between 0° and 90°.
2. **Obtuse Angle**- is an angle whose measure is between 90° and 180°.
3. **Right Angle**-is an angle that measures 90°.
4. **Straight Angle**-is an angle that measures 180°.

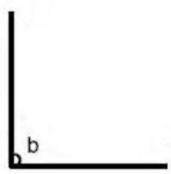
C. ***Complementary and Supplementary angles***

1. **Complementary Angle**—two angles that have a sum of 90°.
2. **Supplementary Angle**—two angles that have a sum of 180°.

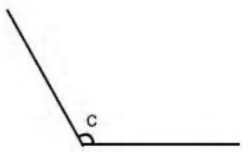
1) Two straight lines that meet at a point form an angle between them.



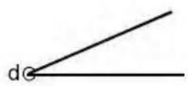
Acute angle : $0^\circ < a < 90^\circ$



Right angle : $b = 90^\circ$



Obtuse angle : $90^\circ < c < 180^\circ$

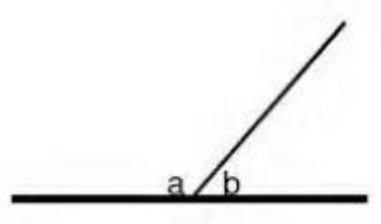


Reflex angle : $180^\circ < d < 360^\circ$

2) Theorems:

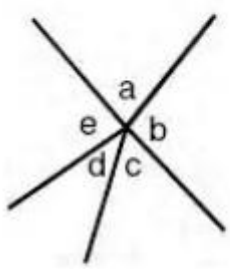
1. If AOB is a straight line,

$a + b = 180$



(Adjacent angles on a straight line)

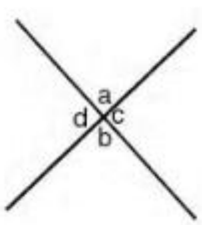
2. The sum of all the angles at a point, each being adjacent to the next, is 4 right angles.



(Angles at a point)

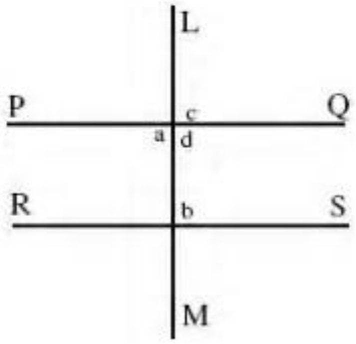
$a + b + c + d + e = 360$

3. If two straight lines intersect, the vertically opposite angles are equal.



(Vertically Opposite angles)

$a = b, c = d$

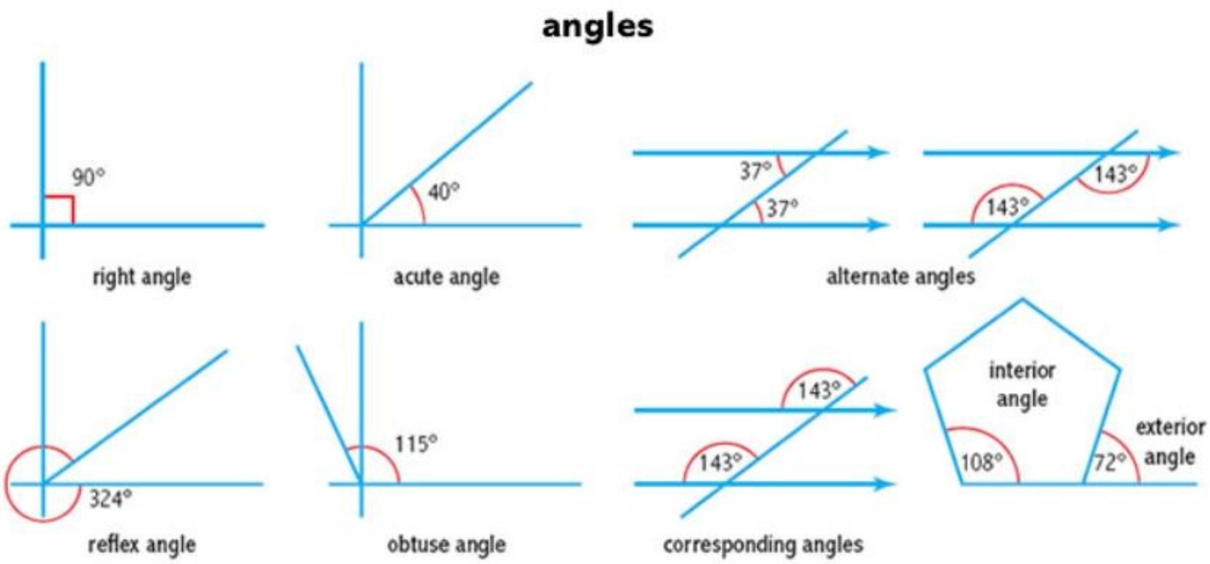


The corresponding angles are equal c = b **(Corresponding angles, PQ || RS)**

The alternate angles are equal a = b **(Alternate angles, PQ || RS)**

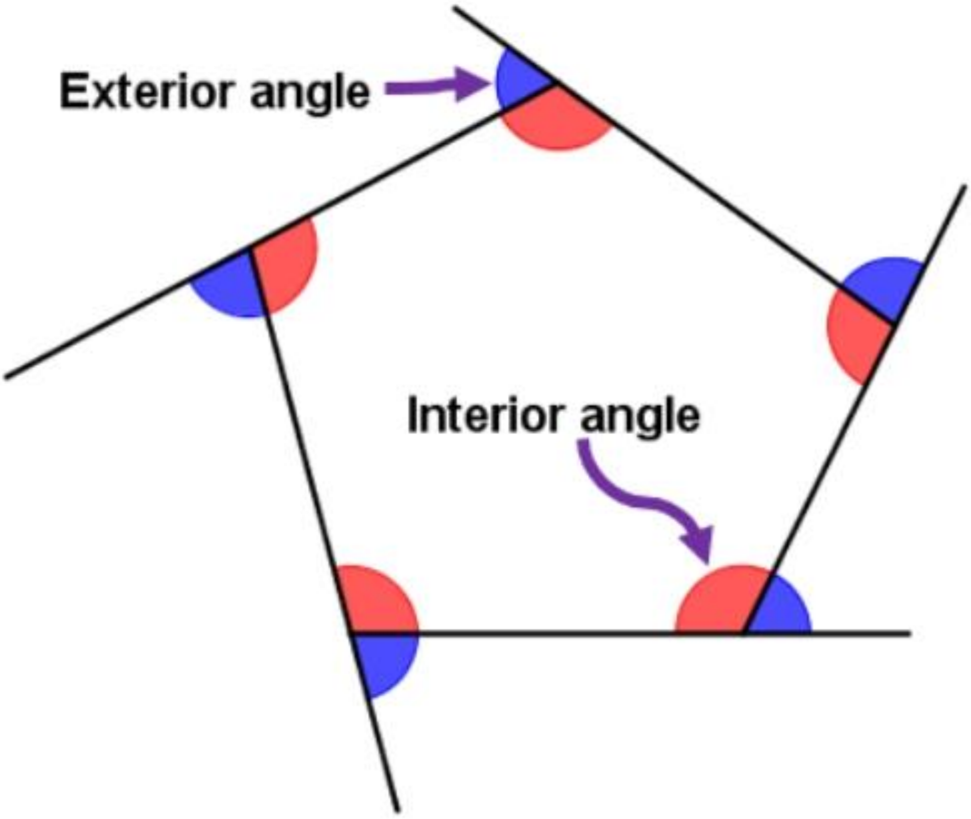
The interior angles are supplementary b + d = 2 right angles **(Interior angles, PQ || RS)**

To recap



Lines and angles may seem like basic concepts. However, they are at the foundation of all geometry questions.

Here is an example of how they are applied to a pentagon



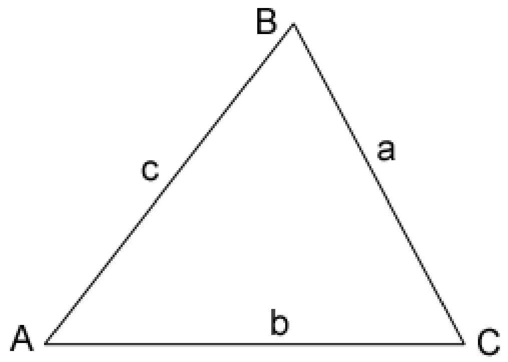
For an n sided polygon :

sum of interior angles = 180(n-2)
number of diagonals = n(n-3)/2 Applicable for any type of n sided polygon.
for a regular n sided polygon:
each angle = 180(n-2)/n
sum of exterior angles = 360

3.2- BASIC TRIANGLES

Frequency of the concepts tested: **Medium**

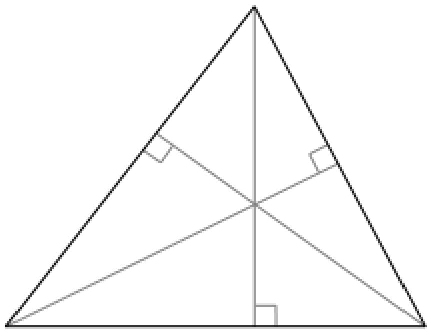
Triangle *A closed figure consisting of three-line segments linked end-to-end. A 3-sided polygon.*



Vertex *The vertex (plural: vertices) is a corner of the triangle. Every triangle has three vertices.*
Base *The base of a triangle can be any one of the three sides, usually the one drawn at the bottom.*

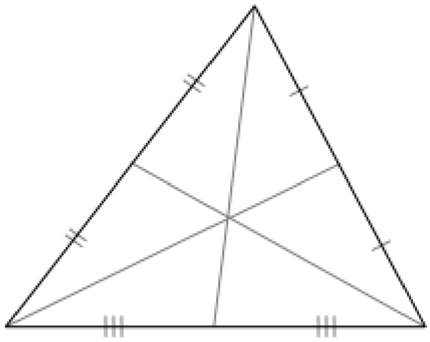
- You can pick any side you like to be the base.
- Commonly used as a reference side for calculating the area of the triangle.
- In an isosceles triangle, the base is usually taken to be the unequal side.

Altitude *The altitude of a triangle is the perpendicular from the base to the opposite vertex. (The base may need to be extended).*



- Since there are three possible bases, there are also three possible altitudes.
- The three altitudes intersect at a single point, called the orthocenter of the triangle.

Median *The median of a triangle is a line from a vertex to the midpoint of the opposite side.*



• The three medians intersect at a single point, called the centroid of the triangle.
• Each median divides the triangle into two smaller triangles which have the same area.
• Because there are three vertices, there are of course three possible medians.
• No matter what shape the triangle, all three always intersect at a single point. This point is called the **centroid** of the triangle.
• The three medians divide the triangle into six smaller triangles of equal area.
• The centroid (point where they meet) is the center of gravity of the triangle
• **Two-thirds of the length of each median is between the vertex and the centroid, while one-third is between the centroid and the midpoint of the opposite side.**
• $m = \sqrt{\frac{2b^2+2c^2-a^2}{4}}$ $ab$and $c$are the sides of the triangle and $a$is the side of the triangle whose midpoint is the extreme point of median m

Area *The number of square units it takes to exactly fill the interior of a triangle.*

Usually called "half of base times height", the area of a triangle is given by the formula below.

• $A = \frac{hb}{2}$

• $A = \frac{P*r}{2}$

$$A = \frac{abc}{4R}$$

b is the length of the base, $a$and $c$the other sides; $h$is the length of the corresponding altitude; R is the Radius of circumscribed circle; r is the radius of inscribed circle; P is the perimeter

• Heron's or Hero's Formula for calculating the area $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $a, b, c$are the three sides of the triangle and $s = \frac{a+b+c}{2}$ which is the semi perimeter of the triangle.

Perimeter *The distance around the triangle. The sum of its sides.*

• For a given perimeter equilateral triangle has the largest area.
• For a given area equilateral triangle has the smallest perimeter.

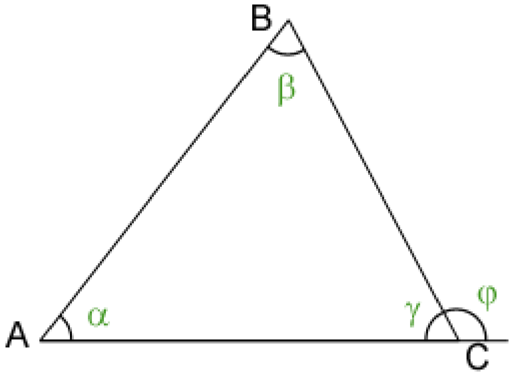
Relationship of the Sides of a Triangle

• The length of any side of a triangle must be larger than the positive difference of the other two sides, but smaller than the sum of the other two sides.

Interior angles *The three angles on the inside of the triangle at each vertex.*

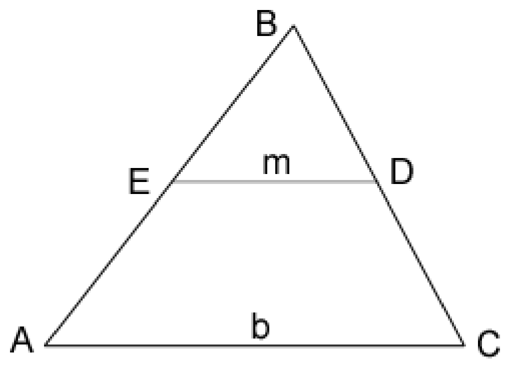
• The interior angles of a triangle always add up to 180°
• Because the interior angles always add to 180°, every angle must be less than 180°
• The bisectors of the three interior angles meet at a point, called the incenter, which is the center of the incircle of the triangle.

Exterior angles: T*he angle between a side of a triangle and the extension of an adjacent side*



• An exterior angle of a triangle is equal to the sum of the opposite interior angles.
• If the equivalent angle is taken at each vertex, the exterior angles always add to 360° In fact, this is true for any convex polygon, not just triangles.

Midsegment of a Triangle *A line segment joining the midpoints of two sides of a triangle*

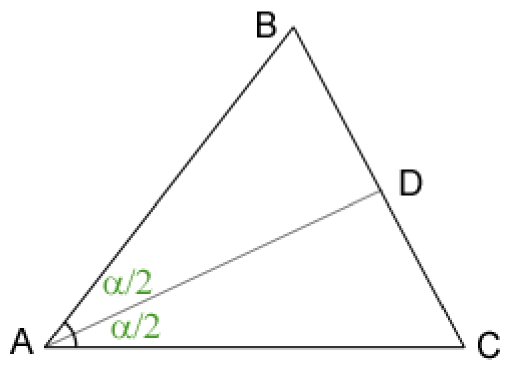


• A triangle has 3 possible midsegments.
• The midsegment is always parallel to the third side of the triangle.
• The midsegment is always half the length of the third side.
• A triangle has three possible midsegments, depending on which pair of sides is initially joined.

Relationship of sides to interior angles in a triangle

• The shortest side is always opposite the smallest interior angle
• The longest side is always opposite the largest interior angle

Angle bisector *An angle bisector divides the angle into two angles with equal measures.*



-An angle only has one bisector.
-Each point of an angle bisector is equidistant from the sides of the angle.
-The angle bisector theorem states that the ratio of the length of the line segment BD to the length of segment DC is equal to the ratio of the length of side AB to the length of side AC: $\frac{BD}{DC} = \frac{AB}{AC}$
The incenter is the point where the angle bisectors intersect. The incenter is also the center of the triangle's incircle - the largest circle that will fit inside the triangle.

Similar Triangles *Triangles in which the three angles are identical.*

• It is only necessary to determine that two sets of angles are identical in order to conclude that two triangles are similar; the third set will be identical because all of the angles of a triangle always sum to 180º.
• In similar triangles, the sides of the triangles are in some proportion to one another. For example, a triangle with lengths 3, 4, and 5 has the same angle measures as a triangle with lengths 6, 8, and 10. The two triangles are similar, and all of the sides of the larger triangle are twice the size of the corresponding legs on the smaller triangle.

• If two similar triangles have sides in the ratio $\frac{x}{y}\frac{x^2}{y^2}$

Congruence of triangles *Two triangles are congruent if their corresponding sides are equal in length and their corresponding angles are equal in size.*

1. **SAS (Side-Angle-Side):** If two pairs of sides of two triangles are equal in length, and **the included angles** are equal in measurement, then the triangles are congruent.

2. **SSS (Side-Side-Side):** If three pairs of sides of two triangles are equal in length, then the triangles are congruent.

3. **ASA (Angle-Side-Angle):** If two pairs of angles of two triangles are equal in measurement, and **the included sides** are equal in length, then the triangles are congruent.

So, knowing SAS or ASA <u>is sufficient to</u> determine unknown angles or sides.

NOTE IMPORTANT EXCEPTION:
The SSA condition (Side-Side-Angle) which specifies two sides and a non-included angle (also known as ASS, or Angle-Side-Side) does not always prove congruence, even when the equal angles are opposite equal sides.

Specifically, SSA does not prove congruence when the angle is acute and the opposite side is shorter than the known adjacent side but longer than the sine of the angle times the adjacent side. This is the ambiguous case. In all other cases with corresponding equalities, SSA proves congruence.

The SSA condition proves congruence if the angle is obtuse or right. In the case of the right angle (also known as the HL (Hypotenuse-Leg) condition or the RHS (Right-angle-Hypotenuse-Side) condition), we can calculate the third side and fall back on SSS.

To establish congruence, it is also necessary to check that the equal angles are opposite equal sides.

So, knowing two sides and non-included angle <u>is NOT sufficient</u> to calculate unknown side and angles.

Angle-Angle-Angle

AAA (Angle-Angle-Angle) says nothing about the size of the two triangles and hence proves only similarity and not congruence.

So, knowing three angles <u>is NOT sufficient</u> to determine lengths of the sides.

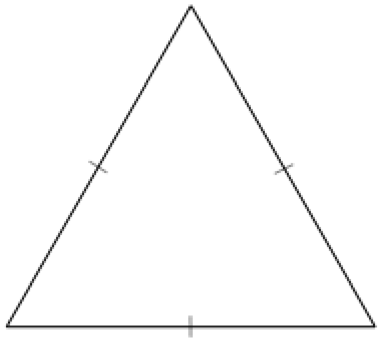
3.3 - SPECIAL TRIANGLES

Frequency of the concepts tested: Medium

Scalene triangle *all sides and angles are different from one another*

• All properties mentioned above can be applied to the scalene triangle, if not mentioned the special cases (equilateral, etc)

Equilateral triangle *all sides have the same length.*



• **An equilateral triangle is also a regular polygon with all angles measuring 60°.**

• **The area is** $A = a^2 * \frac{\sqrt{3}}{4}$

• **The perimeter is** $P = 3a$

• **The radius of the circumscribed circle is** $R = a * \frac{\sqrt{3}}{3}$

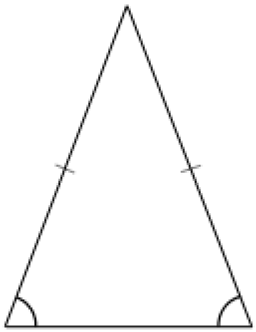
• **The radius of the inscribed circle is** $r = a * \frac{\sqrt{3}}{6}$

• And the altitude is $h = a * \frac{\sqrt{3}}{2}$(Where $a$is the length of a side.)

• For any point P within an equilateral triangle, the sum of the perpendiculars to the three sides is equal to the altitude of the triangle.

• For a given perimeter equilateral triangle has the largest area.
• For a given area equilateral triangle has the smallest perimeter.
• With an equilateral triangle, the radius of the incircle is exactly half the radius of the circumcircle.

Isosceles triangle *two sides are equal in length.*

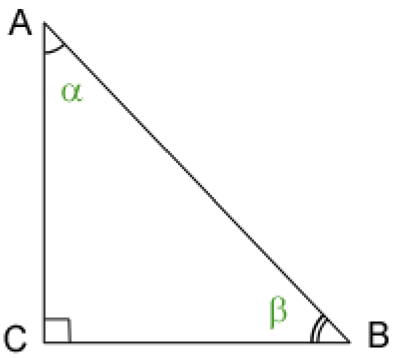


• An isosceles triangle also has two angles of the same measure; namely, the angles opposite to the two sides of the same length.
• For an isosceles triangle with given length of equal sides right triangle (included angle) has the largest area.
• **To find the base given the leg and altitude, use the formula:** $B = 2\sqrt{L^2 - A^2}$

• To find the leg length given the base and altitude, use the formula: $L = \sqrt{A^2 + (\frac{B}{2})^2}$

• To find the altitude given the base and leg, use the formula: $A = \sqrt{L^2 - (\frac{B}{2})^2}$ (Where: L is the length of a leg; A is the altitude; B is the length of the base)

Right triangle *A triangle where one of its interior angles is a right angle (90 degrees)*



• Hypotenuse: the side opposite the right angle. This will always be the longest side of a right triangle.
• The two sides that are not the hypotenuse. They are the two sides making up the right angle itself.
• **Theorem by Pythagoras** defines the relationship between the three sides of a right triangle: $a^2 + b^2 = c^2$ $c$is the length of the hypotenuse and $ab$are the lengths of the other two

sides.

• In a right triangle, the midpoint of the hypotenuse is equidistant from the three polygon vertices

• A right triangle can also be isosceles if the two sides that include the right angle are equal in length (AC and BC in the figure above)

• Right triangle with a given hypotenuse has the largest area when it's an isosceles triangle.

• A right triangle can never be equilateral, since the hypotenuse (the side opposite the right angle) is always longer than the other two sides.

• Any triangle whose sides are in the ratio 3:4:5 is a right triangle. Such triangles that have their sides in the ratio of whole numbers are called Pythagorean Triples. There are an infinite number of them, and this is just the smallest. If you multiply the sides by any number, the result will still be a right triangle whose sides are in the ratio 3:4:5. For example 6, 8, and 10.

• A Pythagorean triple consists of three positive integers abc $a^2 + b^2 = c^2(a, b, c)(3,4,5)$

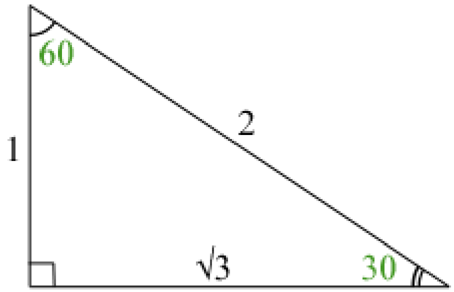
If (a, b, c) is a Pythagorean triple, then so is (ka, kb, kc) for any positive integer k

There are 16 primitive Pythagorean triples with $c \leq 100$

(3, 4, 5) (5, 12, 13) (7, 24, 25) (8, 15, 17) (9, 40, 41) (11, 60, 61) (12, 35, 37) (13, 84, 85) (16, 63, 65) (20, 21, 29) (28, 45, 53) (33, 56, 65) (36, 77, 85) (39, 80, 89) (48, 55, 73) (65, 72, 97). However, the GRE will suffice to memorize the following Pythagorean triples

TRIPLE	***Multiples***
3:4:5	6:8:10
5:12:13	10:24:26
7:24:25	14:48:50
8:15:17	16:30:34
9:40:41	
12:35:37	
20:21:29	

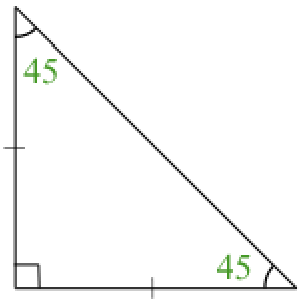
• A right triangle where the angles are 30°, 60°, and 90°.



This is one of the 'standard' triangles you should be able recognize on sight. A fact you should commit to memory is: The sides are always in the ratio $1:\sqrt{3}:2$

Notice that the smallest side (1) is opposite the smallest angle (30°), and the longest side (2) is opposite the largest angle (90°).

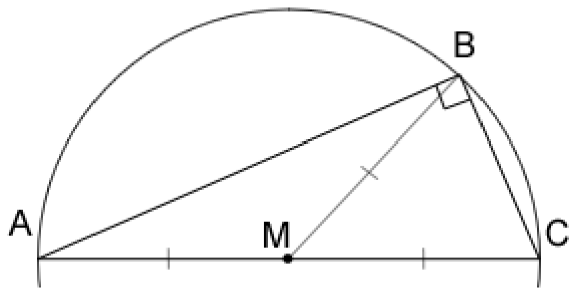
• A right triangle where the angles are 45°, 45°, and 90°.



This is one of the 'standard' triangles you should be able recognize on sight. A fact you should also commit to memory is: The sides are always in the ratio $1:1:\sqrt{2}\sqrt{2}$ being the hypotenuse (longest side). This can be derived from Pythagoras' Theorem. Because the base angles are the same (both 45°) the two legs are equal and so the triangle is also isosceles.

• Area of a 45-45-90 triangle. As you see from the figure above, two 45-45-90 triangles together make a square, so the area of one of them is half the area of the square. As a formula $A = \frac{S^2}{2}$

• **Right triangle inscribed in circle:**

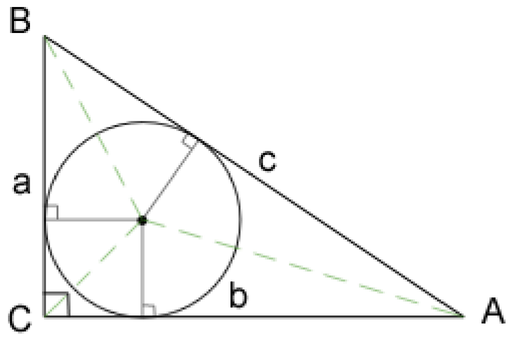


$R = \frac{AC}{2}$

• If M is the midpoint of the hypotenuse, then $BM = \frac{1}{2}\ AC$ One can also say that point B is located on the circle with diameter AC. Conversely, if B is any point of the circle with diameter AC (except A or C themselves) then angle B in triangle ABC is a right angle.

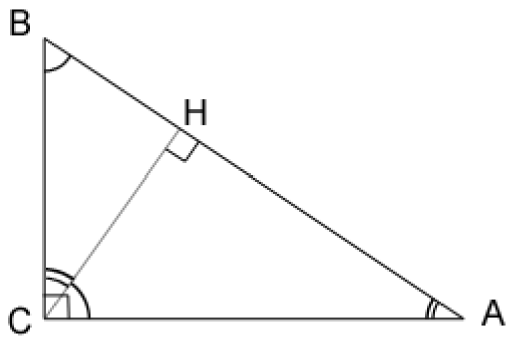
• A right triangle inscribed in a circle must have its hypotenuse as the diameter of the circle. The reverse is also true: if the diameter of the circle is also the triangle's hypotenuse, then that triangle is a right triangle.

• **Circle inscribed in right triangle:** $r = \frac{ab}{a+b+c} = \frac{a+b-c}{2}$



Note that in picture above the right angle is C.

• Given a right triangle, draw the altitude from the right angle.



Then the triangles ABC, CHB and CHA are similar. Perpendicular to the hypotenuse will always divide the triangle into two triangles with the same properties as the original triangle.

3.4 - QUADRILATERALS AND POLYGONS

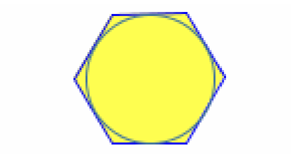
Frequency of the concepts tested: Medium

Types of Polygons
Regular A polygon with all sides and interior angles the same. Regular polygons are always convex.
Convex All interior angles less than 180°, and all vertices 'point outwards' away from the interior. The opposite of concave. Regular polygons are always convex.

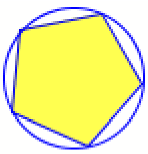
Definitions, Properties and Tips
• **Sum of Interior Angles** $180(n-2)$where n is the number of sides

• For a regular polygon, the total described above is spread evenly among all the interior angles, since they all have the same values. So, for example the interior angles of a pentagon always add up to 540°, so in a regular pentagon (5 sides), each one is one fifth of that, or 108°. Or, as a formula, each interior angle of a regular polygon is given by: $\frac{180(n-2)}{n}$ n is the number of sides.

• The apothem of a polygon is a line from the center to the midpoint of a side. This is also the inradius - the radius of the encircle.

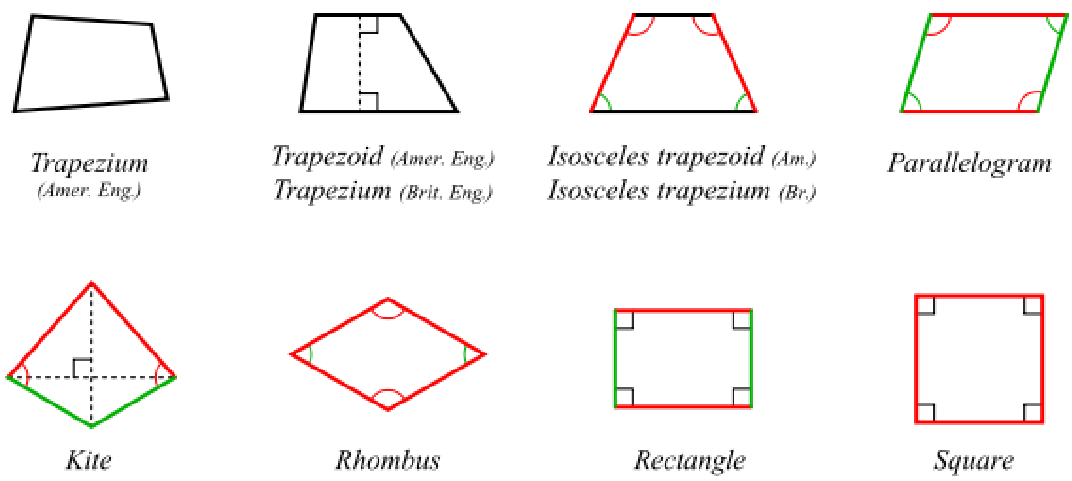


• The radius of a regular polygon is a line from the center to any vertex. It is also the radius of the circumcircle of the polygon.



GRE is dealing mainly with the following polygons:

Quadrilateral *A polygon with four 'sides' or edges and four vertices or corners.*



Types of quadrilaterals:

Square All sides equal, all angles 90°. See Definition of a square.

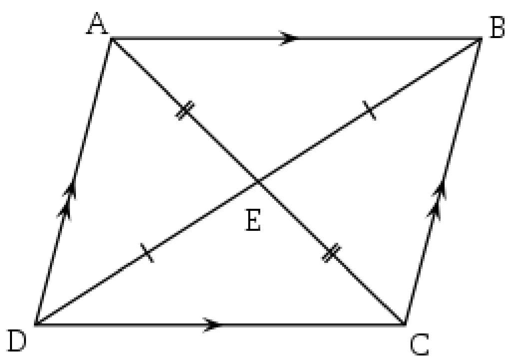
Rectangle Opposite sides equal, all angles 90°. See Definition of a rectangle.

Parallelogram Opposite sides parallel. See Definition of a parallelogram.

Trapezoid Two sides parallel. See Definition of a trapezoid.

Rhombus Opposite sides parallel and equal. See Definition of a rhombus.

Parallelogram *A quadrilateral with two pairs of parallel sides.*



Properties and Tips

- Opposite sides of a parallelogram are equal in length.
- Opposite angles of a parallelogram are equal in measure.
- Opposite sides of a parallelogram will never intersect.
- The diagonals of a parallelogram bisect each other.
- Consecutive angles are supplementary, add to 180°.
- **The area**, A of a parallelogram is $A = bh \; where \; b$ is the base of the parallelogram and h is

its height.
• The area of a parallelogram is twice the area of a triangle created by one of its diagonals.

A parallelogram is a quadrilateral with opposite sides parallel and congruent. It is the "parent" of some other quadrilaterals, which are obtained by adding restrictions of various kinds:
• A rectangle is a parallelogram but with all angles fixed at 90°
• A rhombus is a parallelogram but with all sides equal in length
• A square is a parallelogram but with all sides equal in length and all angles fixed at 90°

Rectangle *A 4-sided polygon where all interior angles are 90°*

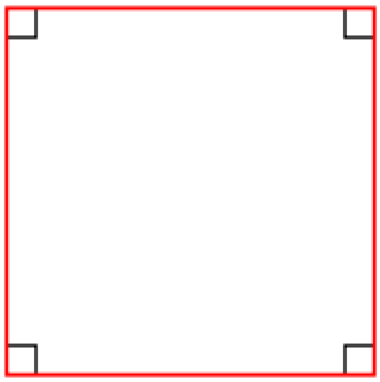
Properties and Tips

• Opposite sides are parallel and congruent
• The diagonals bisect each other
• The diagonals are congruent
• A square is a special case of a rectangle where all four sides are the same length.
• It is also a special case of a parallelogram but with extra limitation that the angles are fixed at 90°.
• The two diagonals are congruent (same length).
• Each diagonal bisects the other. In other words, the point where the diagonals intersect (cross), divides each diagonal into two equal parts.
• Each diagonal divides the rectangle into two congruent right triangles. Because the triangles are congruent, they have the same area, and each triangle has half the area of the rectangle.
• $Diagonal = \sqrt{w^2 + h^2}$where: $w$is the width of the rectangle, **h** is the height of the rectangle.
• The **area** of a rectangle is given by the formula $Width * Height$
A rectangle can be thought about in other ways:
• A square is a special case of a rectangle where all four sides are the same length. Adjust the

rectangle above to create a square.

• It is also a special case of a parallelogram but with extra limitation that the angles are fixed at 90°.

Squares *A 4-sided regular polygon with all sides equal and all internal angles 90°*



Properties and Tips

• If the diagonals of a rhombus are equal, then that rhombus must be a square. The diagonals of a square are (about 1.414) times the length of a side of the square.

• A square can also be defined as a rectangle with all sides equal, or a rhombus with all angles equal, or a parallelogram with equal diagonals that bisect the angles.

• If a figure is both a rectangle (right angles) and a rhombus (equal edge lengths), then it is a square. (Rectangle (four equal angles) + Rhombus (four equal sides) = Square)

• If a circle is circumscribed around a square, the area of the circle is $\frac{\pi}{2}$(about 1.57) times the area of the square.

• If a circle is inscribed in the square, the area of the circle is $\frac{\pi}{4}$ (about 0.79) times the area of the square.

• A square has a larger area than any other quadrilateral with the same perimeter.

• Like most quadrilaterals, the **area** is the length of one side times the perpendicular height. So, in a square this is simply: $area = s^2 s$is the length of one side.

• **The "diagonals" method.** If you know the lengths of the diagonals, the area is half the product of the diagonals. Since both diagonals are congruent (same length), this simplifies to: $area = \frac{d^2}{2} d$is the length of either diagonal

• Each diagonal of a square is the perpendicular bisector of the other. That is, each cuts the other into two equal parts, and they cross and right angles (90°).

• **The length of each diagonal** is $s\sqrt{2}$ where s is the length of any one side.

A square is both a rhombus (equal sides) and a rectangle (equal angles) and therefore has all the properties of both these shapes, namely:

The diagonals of a square bisect each other.

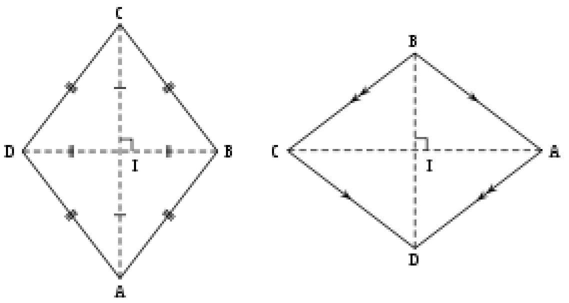
• The diagonals of a square bisect its angles.

• The diagonals of a square are perpendicular.

• Opposite sides of a square are both parallel and equal.
• All four angles of a square are equal. (Each is 360/4 = 90 degrees, so every angle of a square is a right angle.)
• The diagonals of a square are equal.

A square can be thought of as a special case of other quadrilaterals, for example
• a rectangle but with adjacent sides equal
• a parallelogram but with adjacent sides equal and the angles all 90°
• a rhombus but with angles all 90°

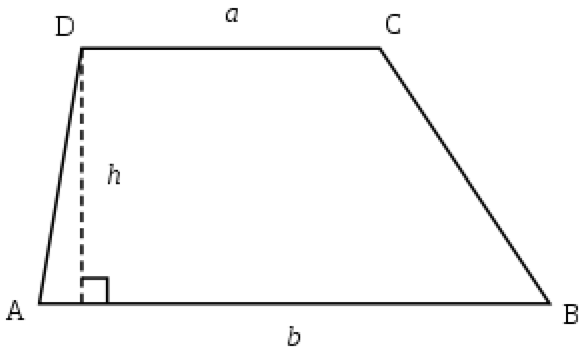
Rhombus *A quadrilateral with all four sides equal in length.*



Properties and Tips

• **A rhombus is actually just a special type of parallelogram.** Recall that in a parallelogram each pair of opposite sides are equal in length. With a rhombus, all four sides are the same length. **It therefore has all the properties of a parallelogram.**
• **The diagonals of a rhombus always bisect each other at 90°.**
• There are several ways to find the **area of a rhombus**. The most common is: $base * altitude$
• **The "diagonals" method.** Another simple formula for the area of a rhombus when you know the lengths of the diagonals. The area is half the product of the diagonals. As a formula: $\frac{d1*d2}{2}$ $d1$is the length of a diagonal $d2$is the length of the other diagonal.

Trapezoid *A quadrilateral which has at least one pair of parallel sides.*



Properties and Tips

• **Base** - One of the parallel sides. Every trapezoid has two bases.
• **Leg** - The non-parallel sides are legs. Every trapezoid has two legs.
• **Altitude** - The altitude of a trapezoid is the perpendicular distance from one base to the other. (One base may need to be extended).
• If both legs are the same length, this is called an isosceles trapezoid, and both base angles are the same.
• **If the legs are parallel, it now has two pairs of parallel sides, and is a parallelogram.**
• **Median** - The median of a trapezoid is a line joining the midpoints of the two legs.
• **The median line is always parallel to the bases.**
• **The length of the median is the average length of the bases, or using the formula:** $\frac{AB+DC}{2}$
• **Area** - The usual way to calculate the area is the average base length times altitude. The area of a trapezoid is given by the formula $h * \frac{a+b}{2}$where
$ab$are the lengths of the two bases $h$is the altitude of the trapezoid
• The **area** of a trapezoid is the altitude$*$median

We could stop with the previous lesson about the quadrilateral figures because they are already polygons: a polygon is any closed figure with just three or more sides. A triangle (special or not) is a polygon as well as a square. However, when in math we refer to a polygon , considering that a square, a triangle, and a rectangle have their own unique peculiarities, to a figure that has five or more sides.

On the GRE we might have the following possible cases

1) Irregular shapes

The probability to encounter such questions that will test your reasoning skills is rather low but it is still a possibility to have polygons with irregular shapes.

2) Polygons with regular shapes

Regular polygons can be used to test your skills thanks to their special properties. Figures such as square, regular pentagon, and regular hexagon.

3) Polygons with hidden features

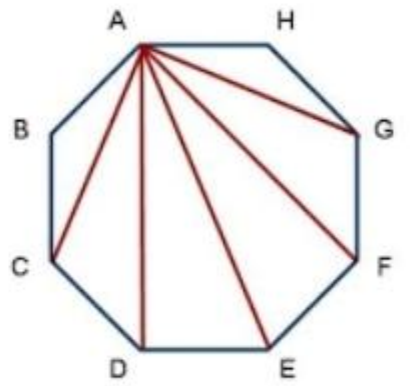
Polygons in which the properties of hidden triangles - and relative angles - can be tested.

4) Interior angles

Regular Figures	**# of sides**	**Sum of the Interior Angles**	**Measure of each interior angle**
Triangle	# 3	180°	180°/3=60°
Quadrilateral	# 4	360°	360°/4=90°
Pentagon	# 5	540°	540°/5=108°
Hexagon	# 6	720°	720°/6=120°
Octagon	# 8	1080°	1080°/8=135°

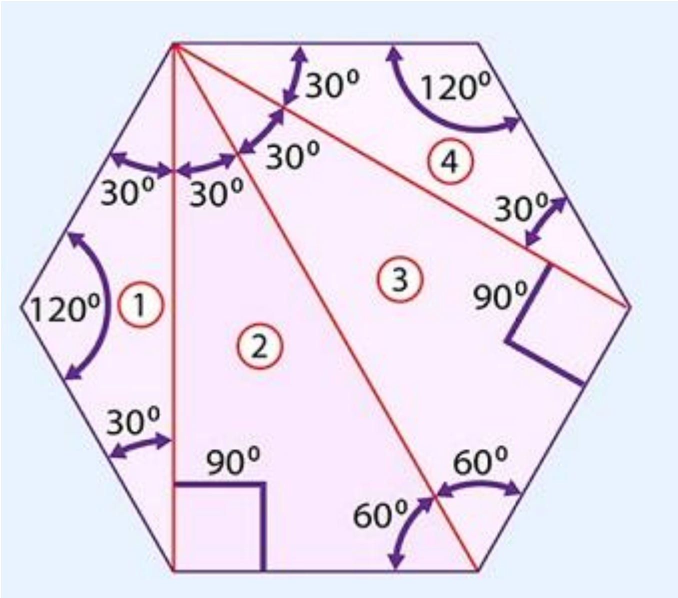
If you are not able to memorize the sums of the interior's angles, divide the regular polygon into triangles with a common vertex. Since the interior angles sum up to 180°, multiply the number of triangles by 180°

Look for instance to the following regular pentagon divided into 6 triangle



Sum of the interior angles = 5 triangles ×180° = 900°

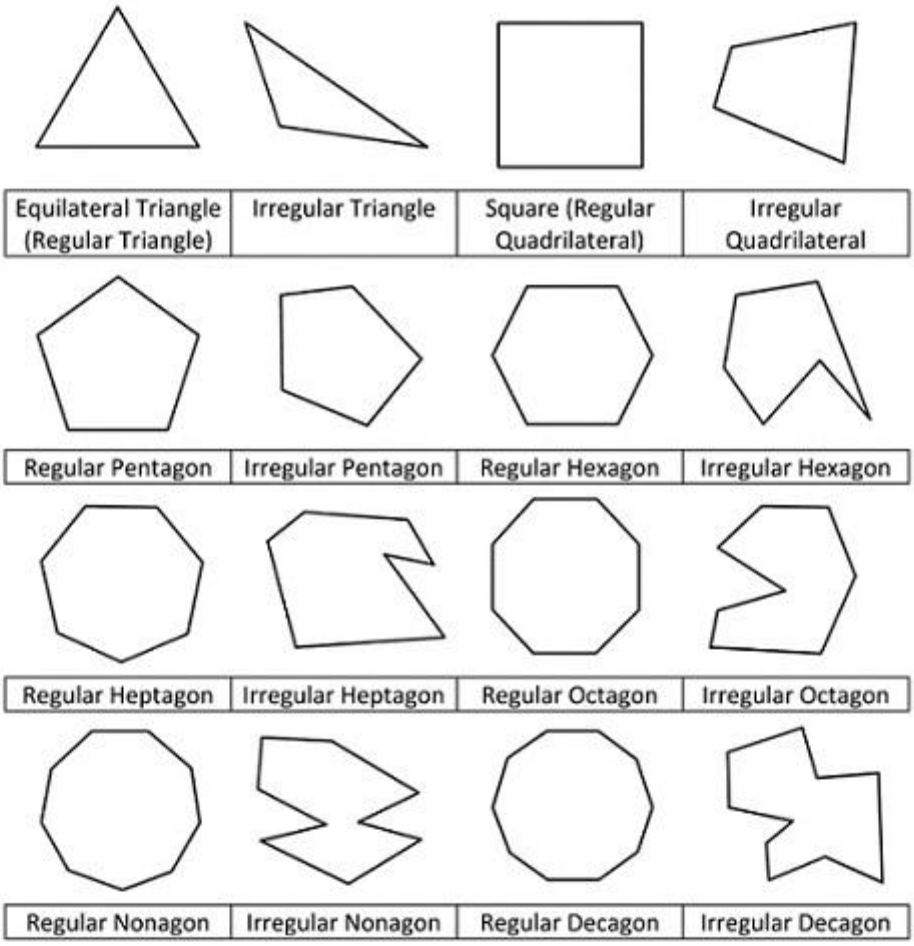
Another important example to show you the overlap between several different kind of properties tested on the GRE at the same time: regular polygon, interior angles, measurements of angles, triangles, lines and angles



5) Same polygons but different aspects

To recap, **regular** polygons and **irregular** polygons are just different ways to see the same geometry shape. Look at the following table, we are still able to figure out a different solution when both we are in from a regular or irregular shape with the same exactly fundamental properties.

REGULAR AND IRREGULAR SHAPES

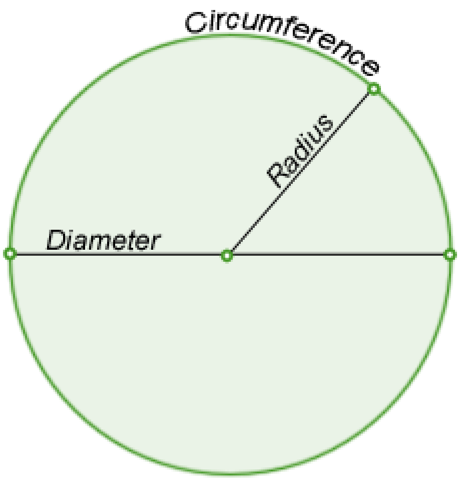


3.5 – CIRCLES

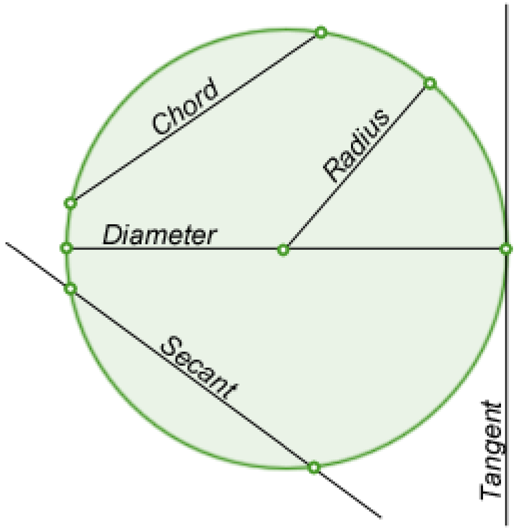
Frequency of the concepts tested: High

Definition

A line forming a closed loop, every point on which is a fixed distance from a center point. Circle could also be defined as the set of all points equidistant from the center.



Center -a point inside the circle. All points on the circle are equidistant (same distance) from the center point.
Radius - the distance from the center to any point on the circle. It is half the diameter.
Diameter -the distances across the circle. The length of any chord passing through the center. It is twice the radius.
Circumference - the distance around the circle.
Area - strictly speaking a circle is a line, and so has no area. What is usually meant is the area of the region enclosed by the circle.
Chord - line segment linking any two points on a circle.
Tangent -a line passing a circle and touching it at just one point.
The tangent line is always at the 90 degree angle (perpendicular) to the radius of a circle.
Secant A line that intersects a circle at two points.
π In any circle, if you divide the circumference (distance around the circle) by it's diameter (distance across the circle), you always get the same number. This number is called Pi and is approximately 3.142.



• A circle is the shape with the largest area for a given length of perimeter (has the highest area to length ratio when compared to other geometric figures such as triangles or rectangles)
• All circles are similar
• To form a unique circle, it needs to have 3 points which are not on the same line.

Circumference, Perimeter of a circle

Given a radius r of a circle, the circumference can be calculated using the formula: $Circumference = 2\pi r$

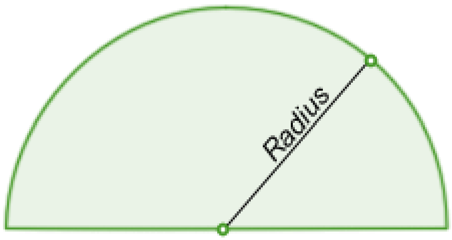
If you know the diameter D of a circle, the circumference can be found using the formula: $Circumference = \pi D$

If you know the area A of a circle, the circumference can be found using the formula: $Circumference = \sqrt{4\pi A}$

Area enclosed by a circle

Given the radius r of a circle, the area can be calculated using the formula: $Area = \pi r^2$
If you know the diameter D of a circle, the area can be found using the formula: $Area = \frac{\pi D^2}{4}$
If you know the circumference C of a circle, the area can be found using the formula: $Area = \frac{C^2}{4\pi}$

Semicircle
Half a circle. A closed shape consisting of half a circle and a diameter of that circle.

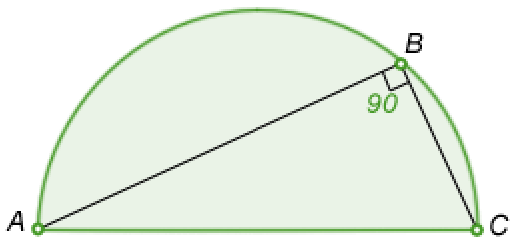


• The area of a semicircle is half the area of the circle from which it is made: $Area = \frac{\pi r^2}{2}$

The perimeter of a semicircle is not half the perimeter of a circle. From the figure above, you can see that the perimeter is the curved part, which is half the circle, plus the diameter line across the bottom. So, the formula for the perimeter of a semicircle is:
$Perimeter = \pi r + 2r = r(\pi + 2)$.

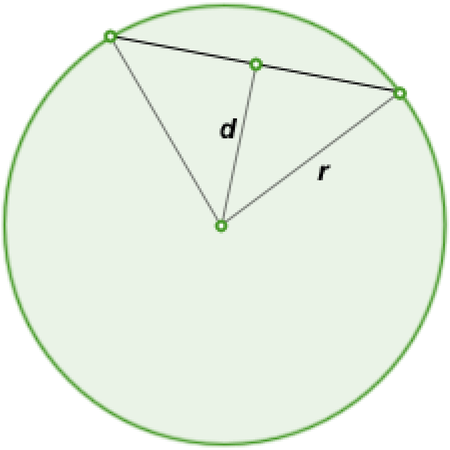
• The angle inscribed in a semicircle is always 90°.

• Any diameter of a circle subtends a right angle to any point on the circle. No matter where the point is, the triangle formed with diameter is always a right triangle.



Chord

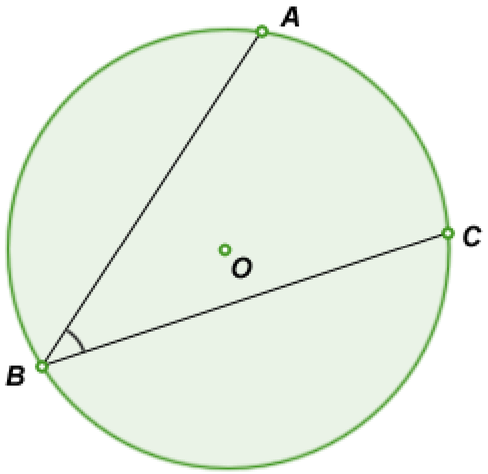
A line that links two points on a circle or curve.



• A diameter is a chord that contains the center of the circle.
• Below is a formula for the length of a chord if you know the radius and the perpendicular distance from the chord to the circle center. This is a simple application of Pythagoras' Theorem.
$Length = 2\sqrt{r^2 - d^2}$ $where$ r is the radius of the circle, $d$is the perpendicular distance from the chord to the circle center.
• In a circle, a radius perpendicular to a chord bisects the chord. *Converse:* In a circle, a radius that bisects a chord is perpendicular to the chord, *or* In a circle, the perpendicular bisector of a chord passes through the center of the circle.

Angles in a circle

An ***inscribed angle*** is an angle ABC formed by points A, B, and C on the circle's

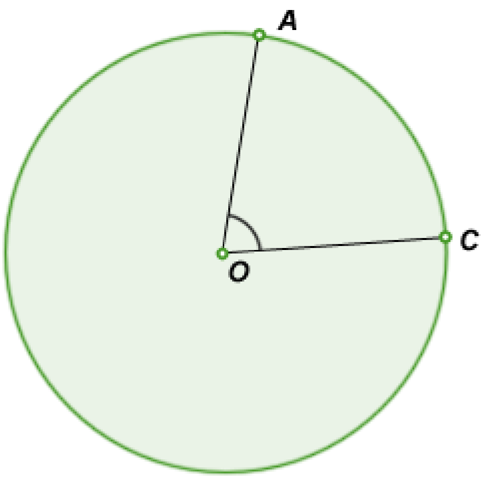


circumferenc

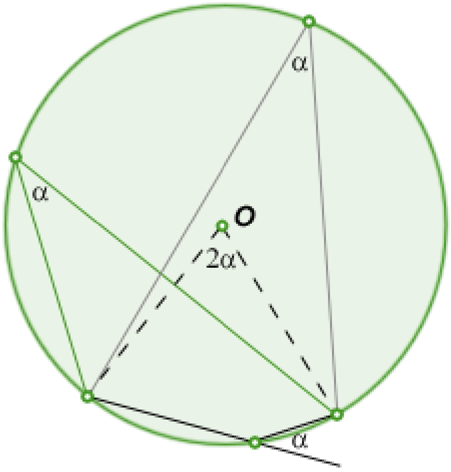
• Given two points A and C, lines from them to a third point B form the inscribed angle ∠ABC. ***Notice*** that the inscribed angle is constant. It only depends on the position of A and C.

• If you know the length L of the minor arc and radius, the inscribed angle is: $Angle = \frac{90L}{\pi r}$

A central angle is an angle AOC with endpoints A and C located on a circle's circumference and vertex O located at the circle's center. A central angle in a circle determines an arc AC.

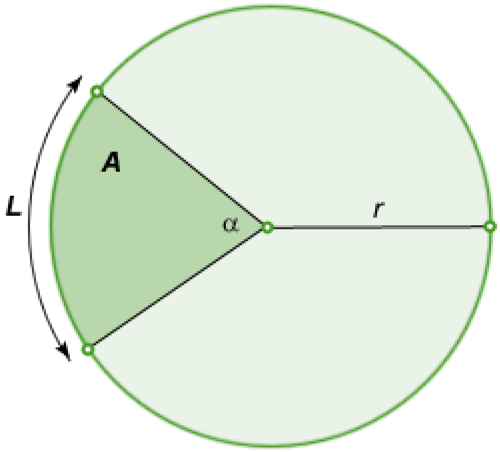


• ***The Central Angle Theorem*** states that the measure of inscribed angle is always half the measure of the central angle.



• An inscribed angle is exactly half the corresponding central angle. Hence, all inscribed angles that subtend the same arc are equal. Angles inscribed on the arc are supplementary. In particular, every inscribed angle that subtends a diameter is a right angle (since the central angle is 180 degrees).

Arcs and Sectors
A portion of the circumference of a circle.



• ***Major and Minor Arcs*** Given two points on a circle, the minor arc is the shortest arc linking them. The major arc is the longest. On the GRE, we usually assume the minor (shortest) arc.
• ***Arc Length*** The formula the arc measure is: $L = 2\pi r \frac{C}{360}$ where C is the central angle of the arc recall that $2\pi r$ is the circumference of the whole circle, so the formula simply reduces this by the ratio of the arc angle to a full angle (360). By transposing the above formula, you solve for the radius, central angle, or arc length if you know any two of them.

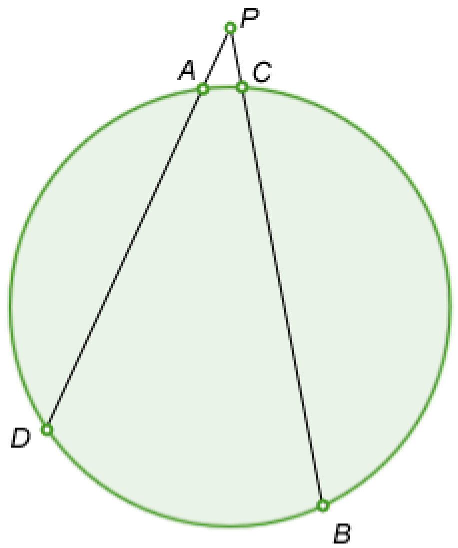
• ***Sector*** is the area enclosed by two radii of a circle and their intercepted arc. A pie-shaped part of a circle.

• ***Area of a sector*** is given by the formula: $Area = \pi r^2 \frac{C}{360}$ where: C is the central angle in degrees. What this formula is doing is taking the area of the whole circle, and then taking a fraction of that depending on the central angle of the sector. So, for example, if the central angle was 90°, then the sector would have an area equal to one quarter of the whole circle.

Power of a Point Theorem

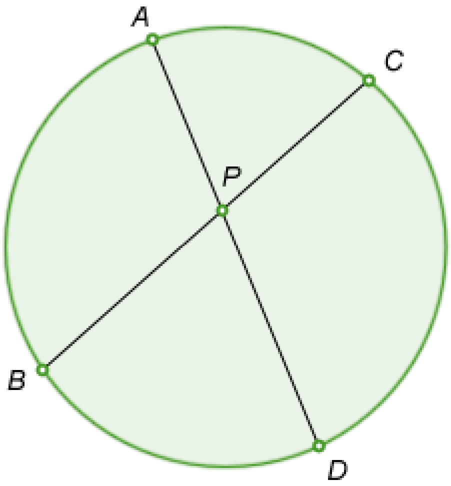
Given circle O, point P *not on the circle*, and a line through P intersecting the circle in two points. The product of the length from P to the first point of intersection and the length from P to the second point of intersection is constant for any choice of a line through P that intersects the circle. This constant is called the ***"power of point P"***.

If P is outside the circle:



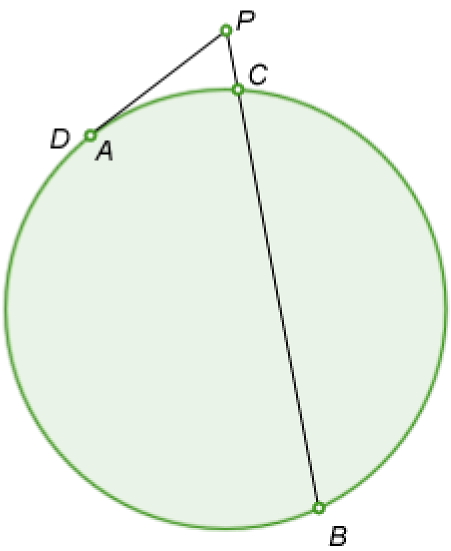
$PA * PD = PC * PB = Constant$- This becomes the theorem we know as the theorem of intersecting secants.

If P is inside the circle:



$PA * PD = PC * PB = Constant$- This becomes the theorem we know as the theorem of intersecting chords.

Tangent-Secant

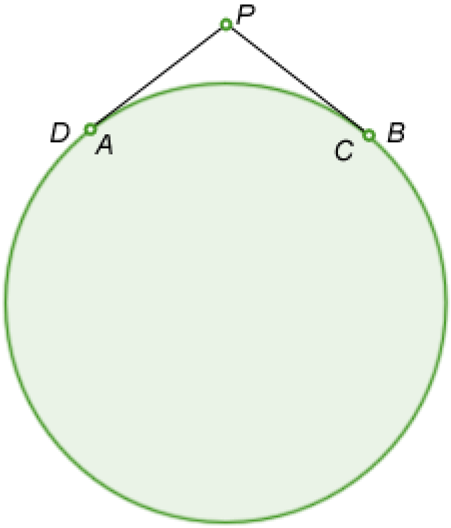


Should one of the lines be tangent to the circle, point A will coincide with point D, and the theorem still applies:

$PA * PD = PC * PB = Constant$

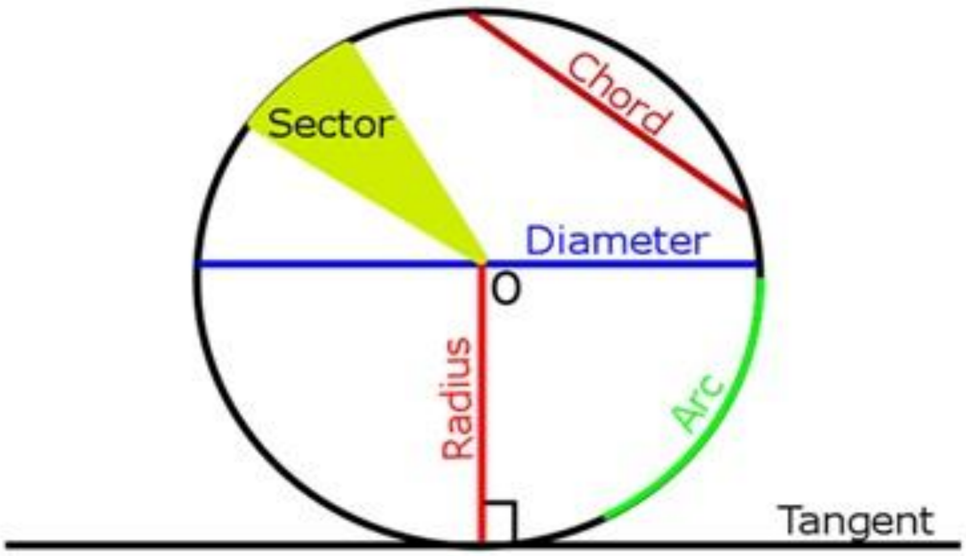
$PA^2 = PC * PB = Constant$- This becomes the theorem we know as the theorem of secant-tangent theorem.

Two tangents



Should both of the lines be tangents to the circle, point A coincides with point D, point C coincides with point B, and the theorem still applies:

$PA * PD = PC * PB = Constant$
$PA^2 = PC^2$
PA=PC

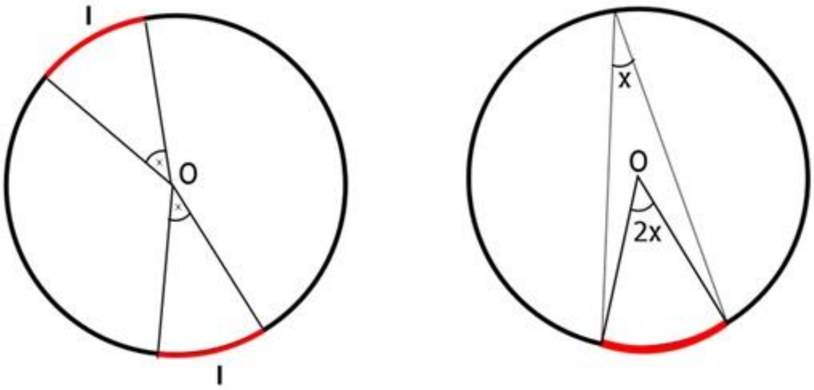


Radius: A line segment drawn joining the center and the boundary line (known as circumference) is called the Radius of the circle.
Diameter: A line segment crossing the circle passing through the center of the circle.
Circumference: The boundary or the perimeter of the circle.
Chord: A line segment joining any two points on the circumference of the circle.
Arc: Part of the circumference of the circle.
Tangent: A line that touches the circumference of the circle at exactly one point.
Sector: An area enclosed by two radii of a circle.

Basic formulae of a circle:
circumference = $2 * pi * r$ is the radius of the circle.
area = $pi * r^2$
diameter = $2 * r$

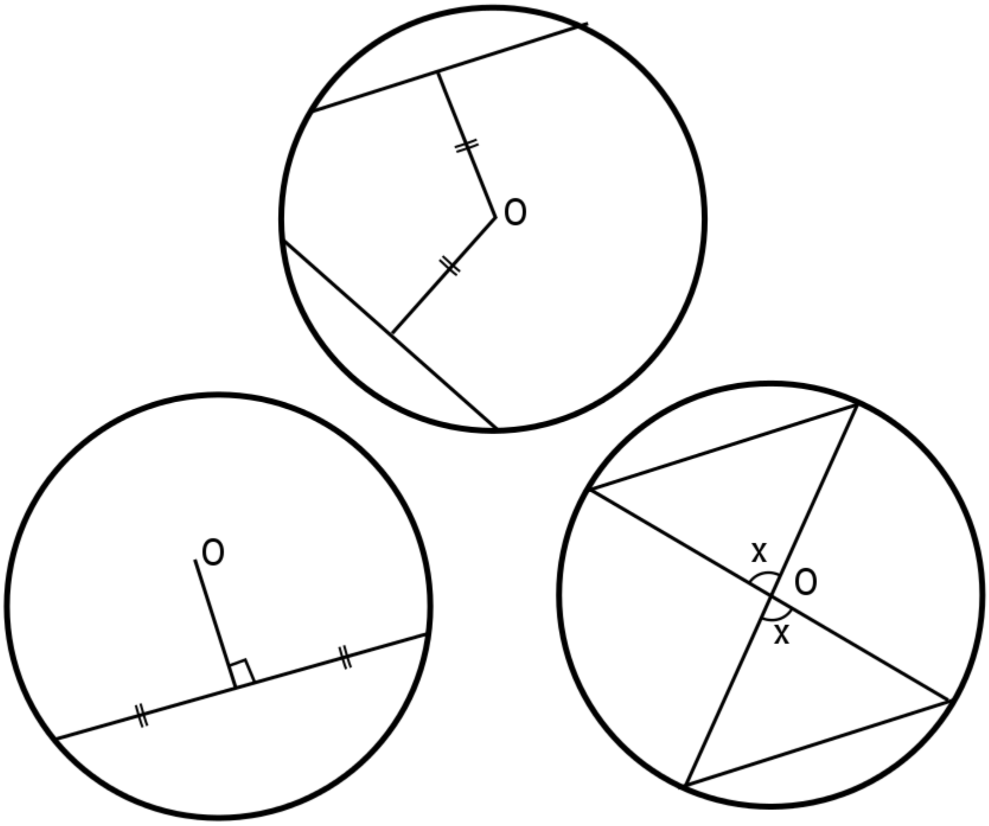
The angle made by the ends of the arc at the center of the circle is called its angle.

- Equal arcs subtend equal angles at the center of the circle and vice versa.
- The angle at the center of the circle is twice the angle at the circumference subtended by the same arc.

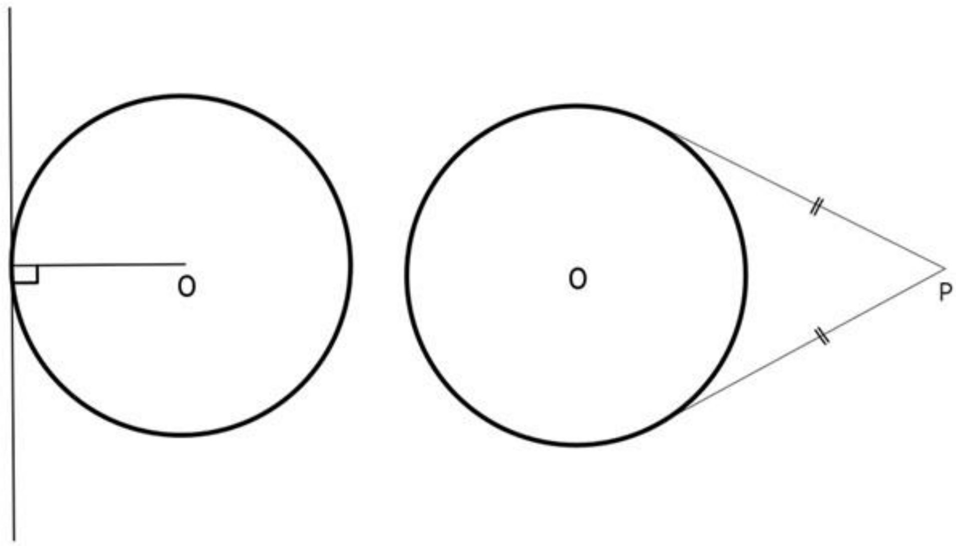


Properties of Chords and tangents:

- Diameter is the longest chord of a circle.
- A perpendicular line from the center to the chord bisects the chord.
- Equal chords are equidistant from the center of the circle.
- Equal chords subtend equal angles at the center of the circle.

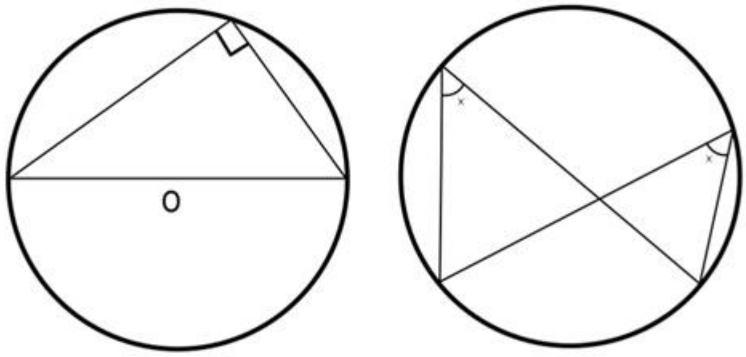


- A tangent to a circle is perpendicular to the radius drawn at the point of contact.
- Tangents to a circle from an exterior point are equal.
- The angle between the tangent and the chord is equal to the inscribed angle on the opposite side of the chord.



Basic properties of a circle

- Angle in a semicircle is the right angle.
- Angle in the same segment of a circle are equal.



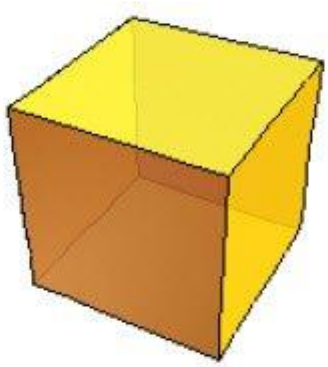
3.6 - THREE-DIMENSIONAL FIGURES

Frequency of the concepts tested: Medium

Scope

The GRE often tests on the knowledge of the geometries of 3-D objects such cylinders, cones, cubes & spheres. The purpose of this document is to summarize some of the important ideas and formulae and act as a useful cheat sheet for such questions.

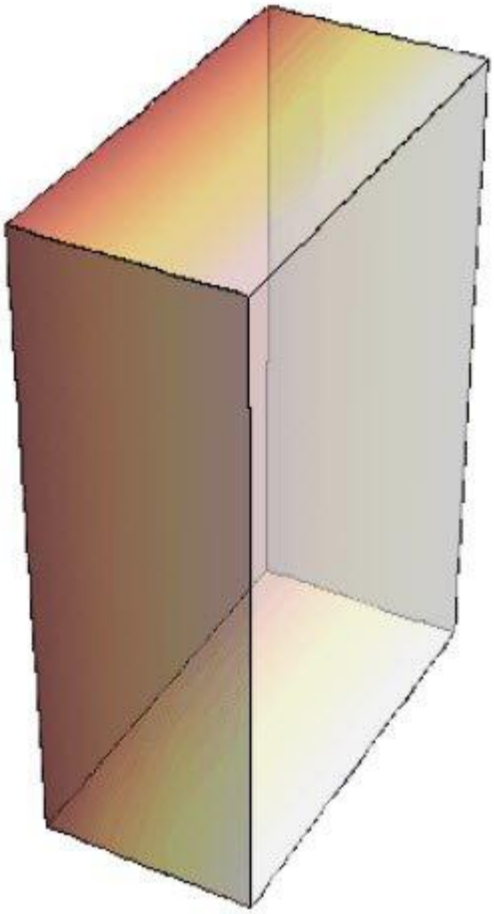
Cube



A cube is the 3-D generalization of a square, and is characterized by the length of the side, a Important results include:

- Volume = a^3
- $6a^2$
- $\sqrt{3}a$

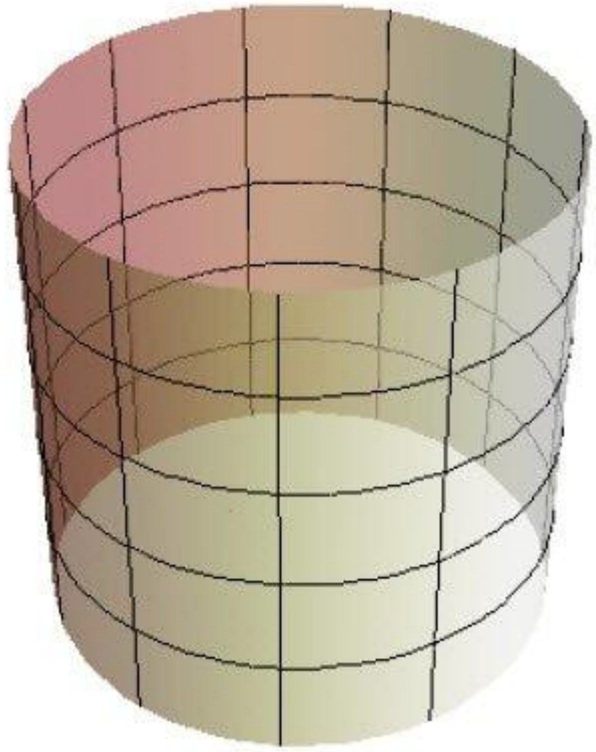
Cuboid



A cube is the 3-D generalization of a rectangle, and is characterized by the length of its sides, a, b, c Important results include:

- Volume = abc
- $2(ab + bc + ca)$
- $\sqrt{a^2 + b^2 + c^2}$

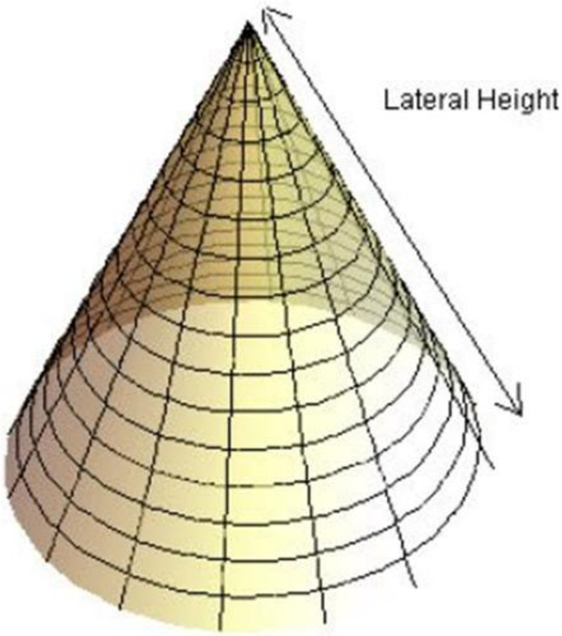
Cylinder



A cylinder is a 3-D object formed by rotating a rectangular sheet along one of its sides. It is characterized by the radius of the base, r, and the height, h. Important results include:

- Volume = $\pi r^2 h$
- $2\pi rh$
- $2\pi r(r+h)$

Cone



A cone is a 3-D object obtained by rotating a right-angled triangle around one of its sides. It is characterized by the radius of its base, r and the height, h. The hypotenuse of the triangle formed by the height and the radius (running along the diagonal side of the cone), is known as its lateral height, $l = \sqrt{r^2 + h^2}$. Important results include:

- Volume = $\frac{1}{3}\pi r^2 h$
- Outer surface area w/o base = $\pi r l = \pi r\sqrt{r^2 + h^2}$
- Outer surface area including base = $\pi r(r + l) = \pi r(r + \sqrt{r^2 + h^2})$

Sphere

A sphere is a 3-D generalization of a circle. It is characterized by its radius, r. Important results include:

- Volume = $\frac{4}{3}\pi r^3$
- $4\pi r^2$

Hemisphere

A hemisphere is a sphere cut in half and is also characterised by its radius r. Important results include:

- Volume = $\frac{2}{3}\pi r^3$
- Surface Area w/o base = $2\pi r^2$
- Surface Area with base = $3\pi r^2$

Some simple configurations

These may appear in various forms on the GRE, and are good practice to derive on one's own:

- Sphere inscribed in cube of side a: Radius of sphere is $\frac{a}{2}$
- Cube inscribed in sphere of radius r: Side of cube is $\frac{2r}{\sqrt{3}}$
- Cylinder inscribed in cube of side a: Radius of cylinder is $\frac{a}{2}, height\ a$
- Cone inscribed in cube of side a: Radius of cone is $\frac{a}{2}, height\ a$
- Cylinder of radius $r$in sphere of radius $R(R > r)$:Height of cylinder is $2\sqrt{R^2 - r^2}$

IMPORTANT LINK!

The Definitive GRE Vocabulary Masterclass

CHAPTER 4. COORDINATE GEOMETRY

4.1 - COORDINATE GEOMETRY

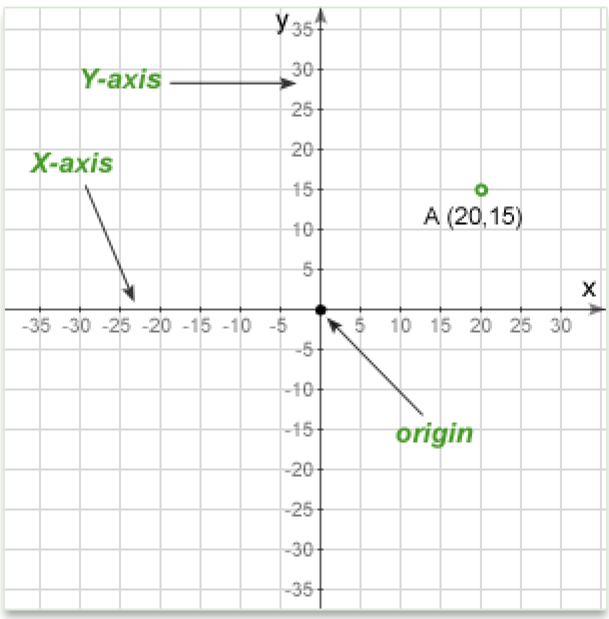
Frequency of the concepts tested: **High**

Definition

Coordinate geometry, or *Cartesian geometry*, is the study of geometry using a coordinate system and the principles of algebra and analysis.

The Coordinate Plane

In coordinate geometry, points are placed on the "coordinate plane" as shown below. The coordinate plane is a two-dimensional surface on which we can plot points, lines and curves. It has two scales, called the x-axis and y-axis, at right angles to each other. The plural of axis is 'axes' (pronounced "AXE-ease").



A point's location on the plane is given by two numbers, one that tells where it is on the x-axis and another which tells where it is on the y-axis. Together, they define a single, unique position on the plane. So, in the diagram above, the point A has an x value of 20 and a y value of 15. These are the coordinates of the point A, sometimes referred to as its "rectangular coordinates".

X axis

The horizontal scale is called the x-axis and is usually drawn with the zero point in the middle. Values to the right are positive and those to the left are negative.

Y axis

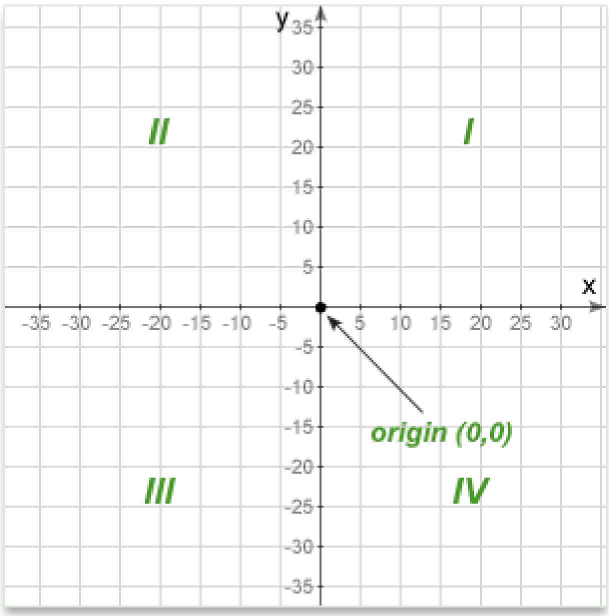
The vertical scale is called the y-axis and is also usually drawn with the zero point in the middle. Values above the origin are positive and those below are negative.

Origin

The point where the two axes cross (at zero on both scales) is called the origin.

Quadrants

When the origin is in the center of the plane, they divide it into four areas called quadrants.



The first quadrant, by convention, is the top right, and then they go around counter-clockwise. In the diagram above they are labeled Quadrant 1, 2 etc. It is conventional to label them with numerals but we talk about them as "first, second, third, and fourth quadrant".

Point (x,y)

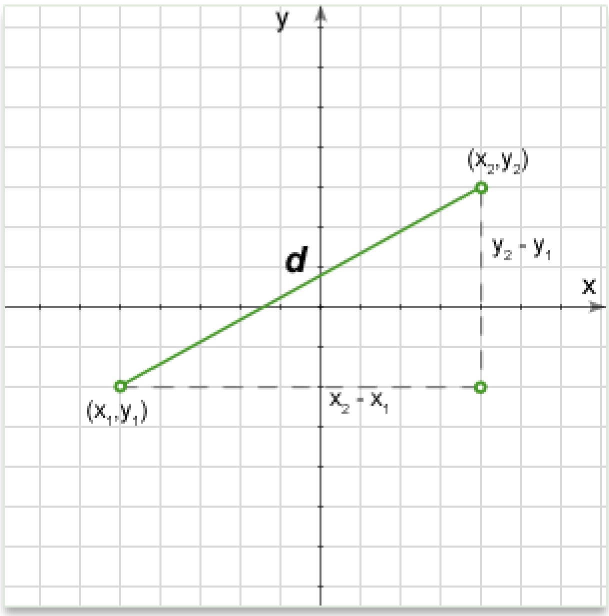
The coordinates are written as an "ordered pair". The letter P is simply the name of the point and is used to distinguish it from others.

The two numbers in parentheses are the x and y coordinate of the point. The first number (x) specifies how far along the x (horizontal) axis the point is. The second is the y coordinate and specifies how far up or down the y axis to go. It is called an ordered pair because the order of the two numbers matters - the first is always the x (horizontal) coordinate.

The sign of the coordinate is important. A positive number means to go to the right (x) or up (y). Negative numbers mean to go left (x) or down (y).

Distance between two points

Given coordinates of two points, distance D between two points is given by:
$D = \sqrt{dx^2 + dy^2}$ (where dx is the difference between the x-coordinates and dy is the difference between the y-coordinates of the points).



As you can see, the distance formula on the plane is derived from the Pythagorean theorem. Above formula can be written in the following way for given two points $(x_1, y_1)$and (x_2, y_2)
$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Vertical and horizontal lines

If the line segment is exactly vertical or horizontal, the formula above will still work fine, but there is an easier way. For a horizontal line, its length is the difference between the x-coordinates. For a vertical line its length is the difference between the y-coordinates.

Distance between the point A (x,y) and the origin

As the one point is origin with coordinate O (0,0) the formula can be simplified to:

$D = \sqrt{x^2 + y^2}$

Example #1

Q: Find the distance between the point A (3, -1) and B (-1, 2)

Solution: Substituting values in the equation we'll get

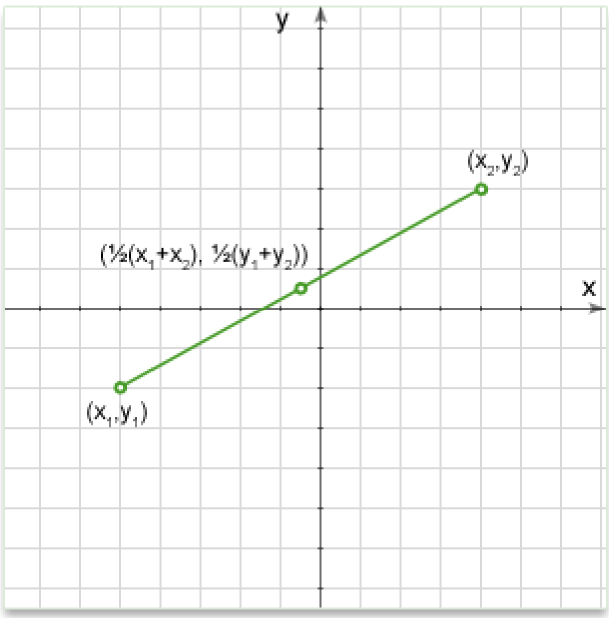
$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$D = \sqrt{(-1-3)^2 + (2-(-1))^2} = \sqrt{16+9} = 5$$

Midpoint of a Line Segment

A line segment on the coordinate plane is defined by two endpoints whose coordinates are known. The midpoint of this line is exactly halfway between these endpoints and it's location can be found using the Midpoint Theorem, which states:

• The x-coordinate of the midpoint is the average of the x-coordinates of the two endpoints.

• Likewise, the y-coordinate is the average of the y-coordinates of the endpoints.



Coordinates of the midpoint $M\ (x_m, y_m)$ of the line segment AB, $A\ (x_1, y_1)$and $B\ (x_2, y_2)$ $x_m = \frac{x_1+x_2}{2}$and ym=$\frac{y_1+y_2}{2}$

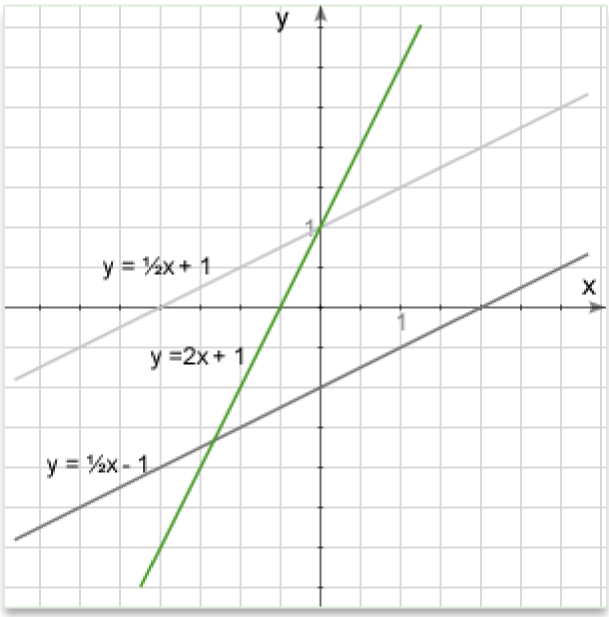
4.2 - COORDINATE GEOMETRY LINES

Frequency of the concepts tested: **Medium**

Lines in Coordinate Geometry

In Euclidean geometry, a line is a straight curve. In coordinate geometry, lines in a Cartesian plane can be described algebraically by linear equations and linear functions.

Every straight line in the plane can represented by a first-degree equation with two variables.



There are several approaches commonly used in coordinate geometry. It does not matter whether we are talking about a line, ray or line segment. **In all cases any of the below methods will provide enough information to define the line exactly.**

1. General form.

The general form of the equation of a straight line is

$ax + by + c = 0$

a, b and c are arbitrary constants. This form includes all other forms as special cases. For an equation in this form the slope is $-\frac{a}{b}$and the y intercept is $-\frac{c}{b}$

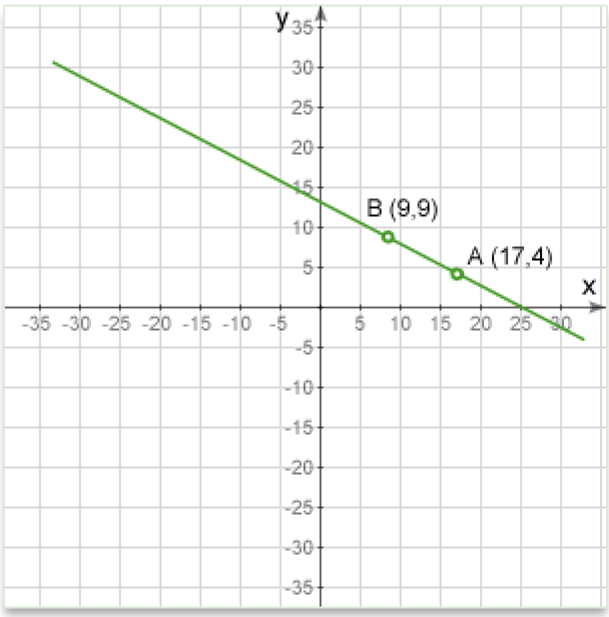
2. Point-intercept form.

$y = mx + b$

m is the slope of the line; b is the y-intercept of the line; x is the independent variable of the function y

3. Using two points

In figure below, a line is defined by the two points A and B. By providing the coordinates of the two points, we can draw a line. No other line could pass through both these points and so the line they define is unique.



The equation of a straight line passing through points $P_1(x_1, y_1)$and $P_2(x_2, y_2)$ is:

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

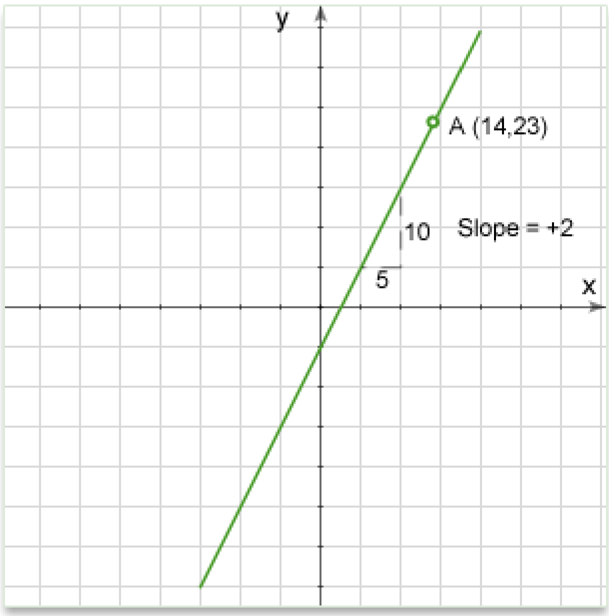
Q: Find the equation of a line passing through the points **A (17,4)** and **B (9,9)**.

Solution: Substituting the values in equation $\frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2}$ we'll get: $\frac{y-4}{x-17} = \frac{4-9}{17-9}$ $\frac{y-4}{x-17} = \frac{-5}{8}$ --> $8y - 32 = -5x + 85$--> $8y + 5x - 117 = 0$ OR if we want to write the equation in the slope-intercept form: $y = -\frac{5}{8}x + \frac{117}{8}$

4. Using one point and the slope

Sometimes on the GRE you will be given a point on the line and its slope and from this information you will need to find the equation or check if this line goes through another point. You can think of the slope as the direction of the line. So, once you know that a line goes through a certain point, and which direction it is pointing, you have defined one unique line.

In figure below, we see a line passing through the point A at (14,23). We also see that its slope is +2 (which means it goes 2 up for every one across). With these two facts we can establish a unique line.



The equation of a straight line that passes through a point $P_1(x_1, y_1)$ with a slope m is:

$y - y_1 = m(x - x_1)$

Q: Find the equation of a line passing through the point A (14,23) and the slope 2.

Solution: Substituting the values in equation $y - y_1 = m(x - x_1)$ we'll get $y - 23 = 2(x - 14)$ --> $y = 2x - 5$

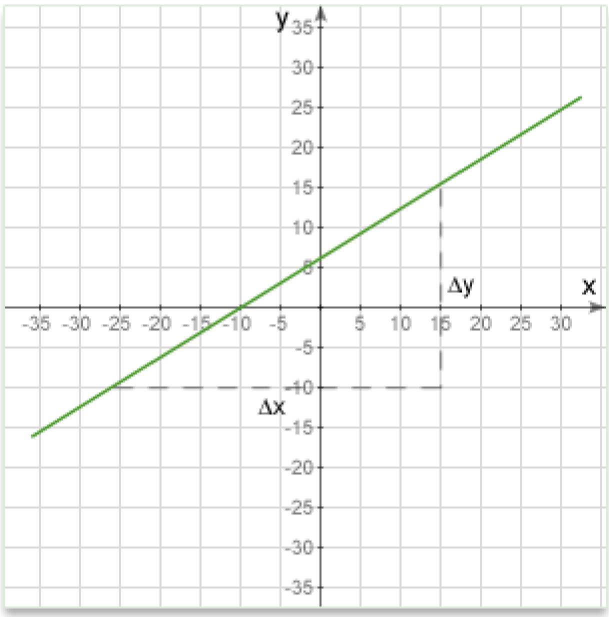
4. Intercept form.
The equation of a straight line whose x and y intercepts are a and b, respectively, is:
$\frac{x}{a} + \frac{y}{b} = 1$

Q: Find the equation of a line whose x intercept is 5 and y intercept is 2.

Solution: Substituting the values in equation $\frac{x}{a} + \frac{y}{b} = 1$ we'll get $\frac{x}{5} + \frac{y}{2} = 1$--> $5y + 2x - 10 = 0$ OR if we want to write the equation in the slope-intercept form: $y = -\frac{2}{5}x + 2$

Slope of a Line
The slope or gradient of a line describes its steepness, incline, or grade. **A higher slope value indicates a steeper incline.**

The slope is defined as the ratio of the "rise" divided by the "run" between two points on a line, or in other words, the ratio of the altitude change to the horizontal distance between any two points on the line.



Given two points $(x_1, y_1)$and (x_2, y_2)on a line, the slope m of the line is:

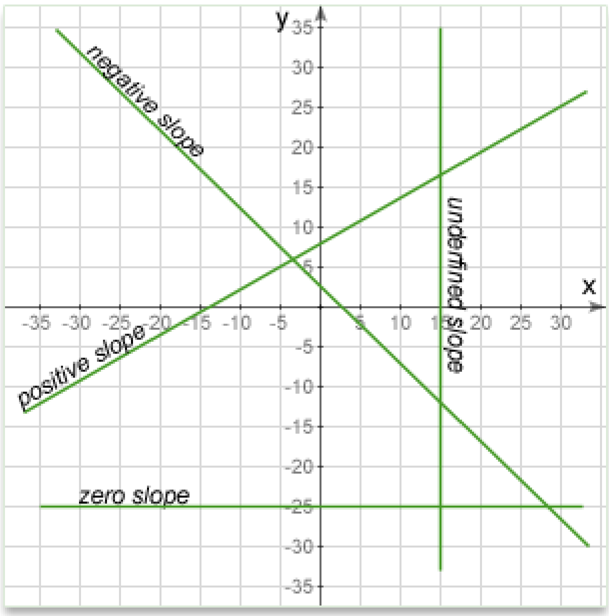
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If the equation of the line is given in the **Point-intercept form**: $y = mx + b\ then\ m$ is the slope. This form of a line's equation is called the slope-intercept form, because $b$can be interpreted as the y-intercept of the line, the y-coordinate where the line intersects the y-axis.

If the equation of the line is given in the **General form:** $ax + by + c = 0, then\ the\ slip = -\frac{a}{b}$ and the y intercept is $-\frac{c}{b}$

SLOPE DIRECTION

The slope of a line can be positive, negative, zero or undefined.



Positive slope
Here, y increases as x increases, so the line slopes upwards to the right. The slope will be a positive number. The line below has a slope of about +0.3, it goes up about 0.3 for every step of 1 along the x-axis.

Negative slope
Here, y decreases as x increases, so the line slopes downwards to the right. The slope will be a negative number. The line below has a slope of about -0.3, it goes down about 0.3 for every step of 1 along the x-axis.

Zero slope
Here, y does not change as x increases, so the line in exactly horizontal. The slope of any horizontal line is always zero. The line below goes neither up nor down as x increases, so its slope is zero.

Undefined slope
When the line is exactly vertical, it does not have a defined slope. The two x coordinates are the same, so the difference is zero. The slope calculation is then something like $slope = \frac{15}{0}$ When you divide anything by zero the result has no meaning. The line above is exactly vertical, so it has no defined slope.

SLOPE AND QUADRANTS:

1. **If the slope of a line is negative**, the line WILL intersect quadrants II and IV. X and Y intercepts of the line with negative slope have the same sign. Therefore, if X and Y intersect are positive, the line intersects quadrant I; if negative, quadrant III.

2. **If the slope of line is positive**, line WILL intersect quadrants I and III. Y and X intercepts of the line with positive slope have opposite signs. Therefore, if X intersect is negative, line intersects the quadrant II too, if positive quadrant IV.

3. **Every line (but the one crosses origin OR parallel to X or Y axis OR X and Y axis themselves) crosses three quadrants**. Only the line which crosses origin (0,0) OR is parallel to either of axis crosses only two quadrants.

4. **If a line is horizontal,** it has a slope of 0, is parallel to X-axis and crosses quadrant I and II if the Y intersect is positive OR quadrants III and IV, if the Y intersect is negative. Equation of such line is y=b, where b is y intersect.

5. **If a line is vertical**, the slope is not defined, line is parallel to Y-axis and crosses quadrant I and IV, if the X intersect is positive and quadrant II and III, if the X intersect is negative. Equation of such line is $x = a$ is x-intercept.

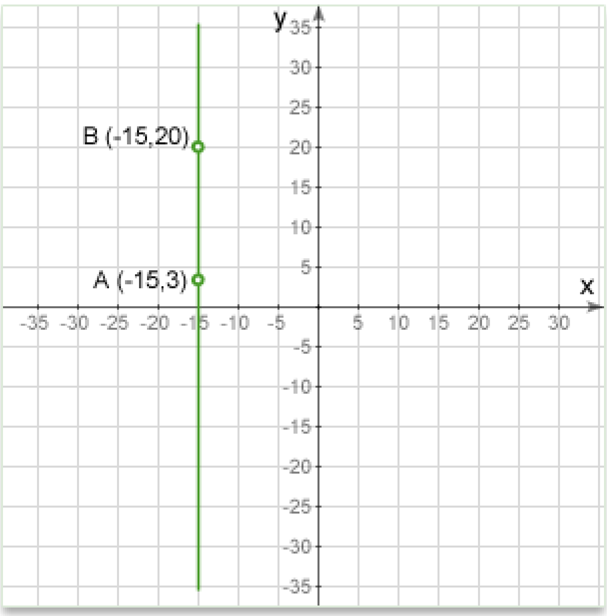
6. **For a line that crosses two points** (x_1, y_1) **and** (x_2, y_2), slope $m = \frac{y_2 - y_1}{x_2 - x_1}$

7. If the slope is 1 the angle formed by the line is 45 degrees.

8. **Given a point and slope, equation of a line can be found**. The equation of a straight line that passes through a point (x_1, y_1) with a slope m is: $y - y_1 = m(x - x_1)$

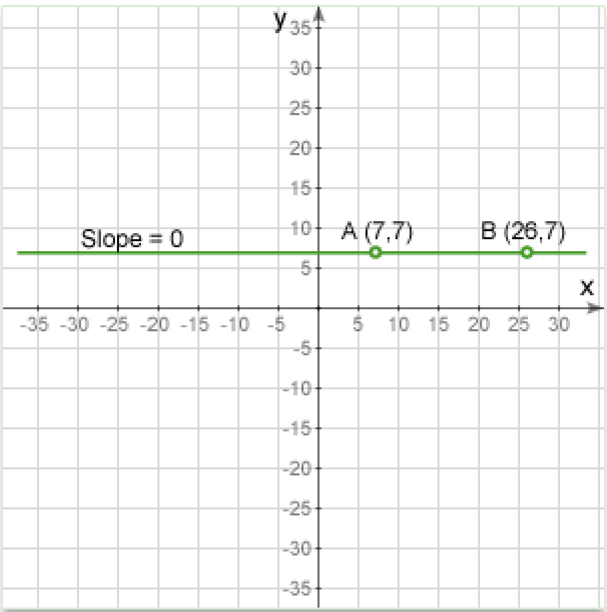
Vertical and horizontal lines

A vertical line is parallel to the y-axis of the coordinate plane. All points on the line will have the same x-coordinate.



A vertical line has no slope. Or put another way, for a vertical line the slope is undefined.
The equation of a vertical line is: $x = a$

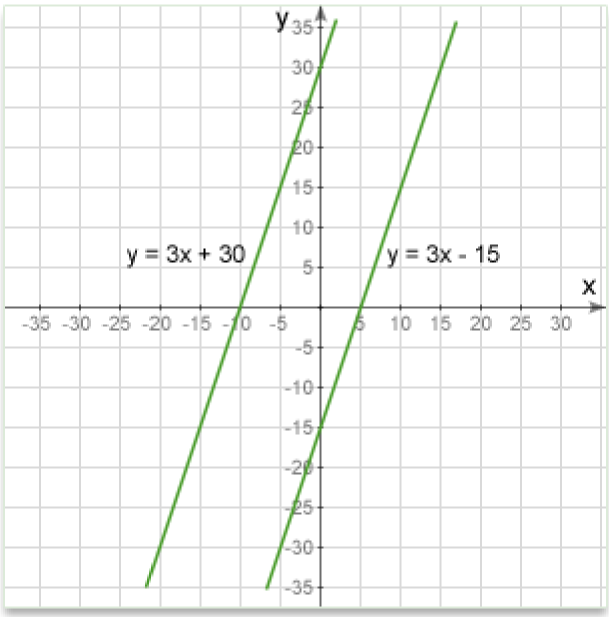
Where: x is the coordinate of any point on the line; a is where the line crosses the x-axis (x intercept). Notice that the equation is independent of y. Any point on the vertical line satisfies the equation.
A horizontal line is parallel to the x-axis of the coordinate plane. All points on the line will have the same y-coordinate.



A horizontal line has a slope of zero.
The equation of a horizontal line is: $y = b$

Where: y is the coordinate of any point on the line; b is where the line crosses the y-axis (y intercept). Notice that the equation is independent of x. Any point on the horizontal line satisfies the equation.

Parallel lines
Parallel lines have the same slope.



The slope can be found using any method that is convenient to you:

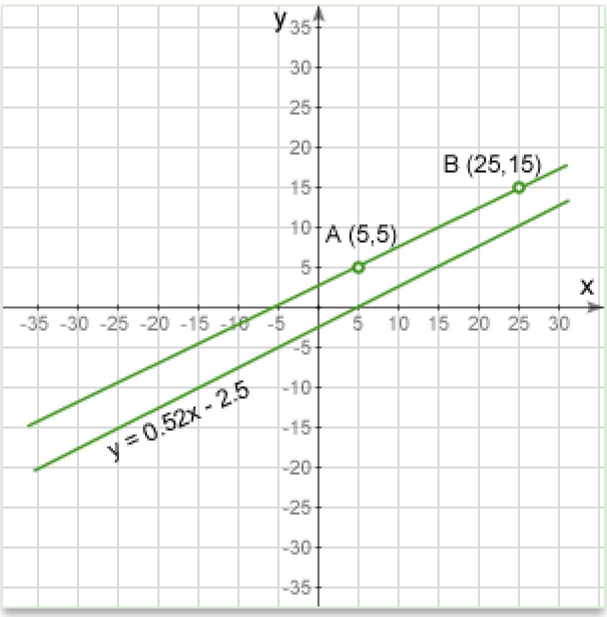
From two given points on the line.
From the equation of the line in slope-intercept form
From the equation of the line in point-slope form

The equation of a line through the point $P_1(x_1, y_1)$and parallel to line $ax + by + c = 0$ is: $a(x - x_1) + b(y - y_1) = 0$

Distance between two parallel lines $y = mx + b$ and $y = mx + c$ can be found by the formula:

$$D = \frac{|b - c|}{\sqrt{m^2 + 1}}$$

Example #1
Q: There are two lines. One line is defined by two points at (5,5) and (25,15). The other is defined by an equation in slope-intercept form y = 0.52x - 2.5. Are two lines parallel?

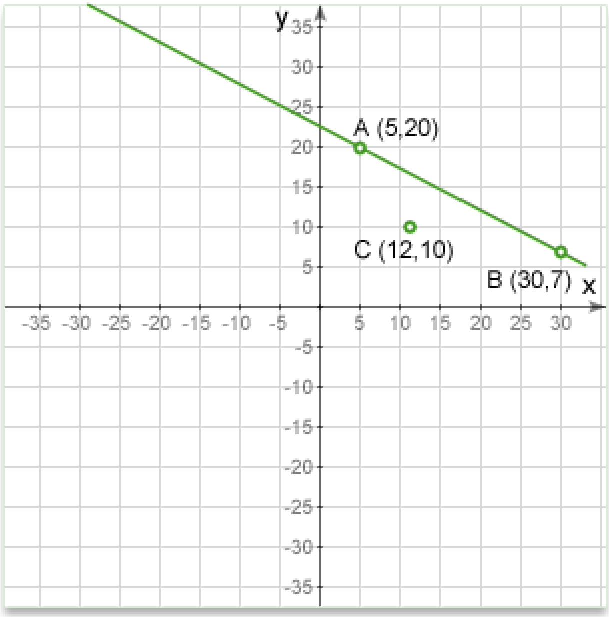


Solution:
For the top line, the slope is found using the coordinates of the two points that define the line. $Slope = \frac{15-5}{25-5} = 0.5$

For the lower line, the slope is taken directly from the formula. Recall that the slope intercept formula is y = mx + b, where m is the slope. So looking at the formula we see that the slope is 0.52.

So, the top one has a slope of 0.5, the lower slope is 0.52, which are not equal. Therefore, the lines are not parallel.

Q: Define a line through a point C parallel to a line passes through the points A and B.



Solution: We first find the slope of the line AB using the same method as in the example above.

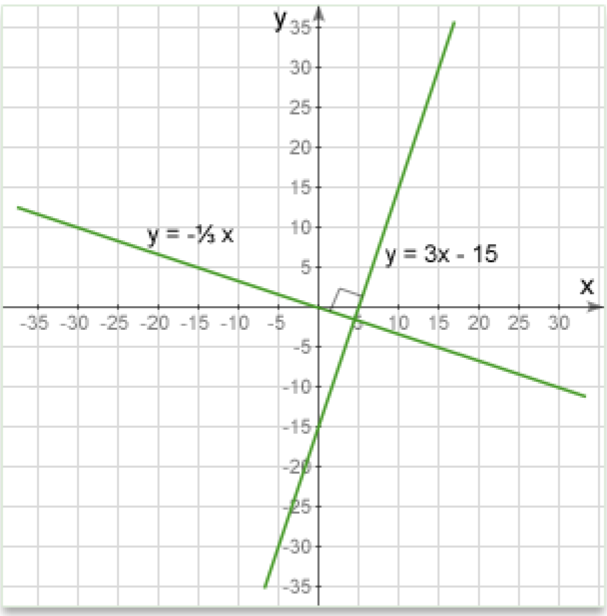
$$Slope\ AB = \frac{20-7}{5-30} = -0.52$$

For the line to be parallel to AB it will have the same slope, and will pass through a given point, C(12,10). We therefore have enough information to define the line by its equation in point-slope form:

$y = -0.52(x-12) + 10$--> $y = -0.52x + 16.24$

Perpendicular lines

For one line to be perpendicular to another, the relationship between their slopes has to be **negative reciprocal** $-\frac{1}{m} - 1$. In other words, the two lines are perpendicular if and only if the product of their slopes is -1

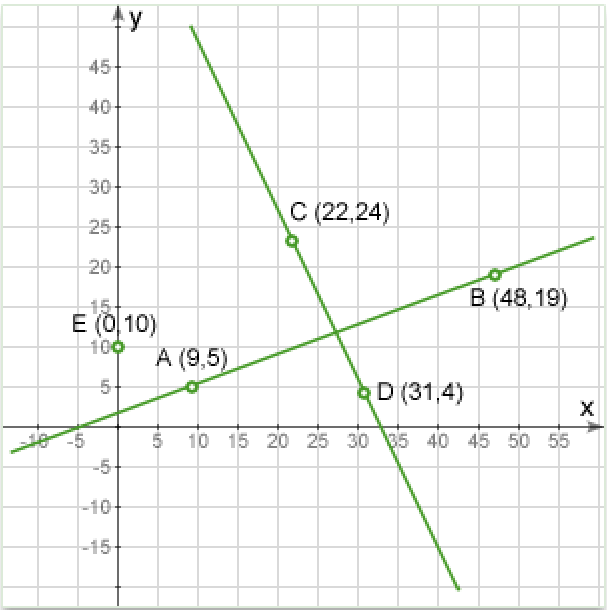


$=a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular if $a_1a_2 + b_1b_2 = 0$

The equation of a line passing through the point (x_1, y_1) and perpendicular to line $ax + by + c = 0$ is $b(x - x_1) - a(y - y_1) = 0$

Example #1:

Q: Are the two lines below perpendicular?
Solution:



To answer, we must find the slope of each line and then check to see if one slope is the negative reciprocal of the other or if their product equals to -1.

$$Slope\ AB = \frac{5-19}{9-48} = \frac{-14}{-39} = 0.358$$

$$Slope\ CD = \frac{24-4}{22-31} = \frac{20}{-9} = -2.22$$

If the lines are perpendicular, each will be the negative reciprocal of the other. It doesn't matter which line we start with, so we will pick AB:

Negative reciprocal of 0.358 is

$$-\frac{1}{0.358} = -2.79$$

So, the slope of CD is -2.22, and the negative reciprocal of the slope of AB is -2.79. These are not the same, so the lines are not perpendicular, even though they may look as though they are. However, if you looked carefully at the diagram, you might have noticed that point C is a little too far to the left for the lines to be perpendicular.

Example # 2.

Q: Define a line passing through the point E and perpendicular to a line passing through the points C and D on the graph above.

Solution: The point E is on the y-axis and so is the y-intercept of the desired line. Once we know the slope of the line, we can express it using its equation in slope-intercept form y=mx+b, where m is the slope and b is the y-intercept.

First find the slope of line CD:

$$Slope\ CD = \frac{24-4}{22-31} = \frac{20}{-9} = -2.22$$

The line we seek will have a slope which is the negative reciprocal of:

$$-\frac{1}{-2.22} = 0.45$$

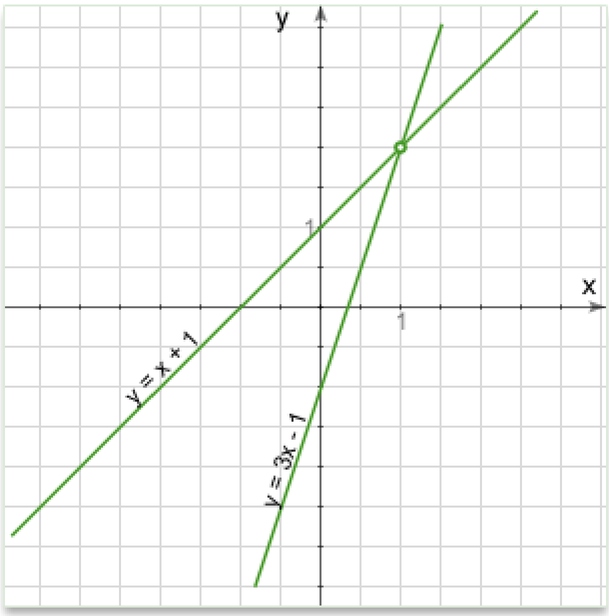
Since E is on the Y-axis, we know that the intercept is 10. Plugging these values into the line equation, the line we need is described by the equation.

$$y = 0.45x + 10$$

This is one of the ways a line can be defined and so we have solved the problem. If we wanted to plot the line, we would find another point on the line using the equation and then draw the line through that point and the intercept.

Intersection of two straight lines

The point of intersection of two non-parallel lines can be found from the equations of the two lines.



To find the intersection of two straight lines:

1. First, we need their equations
2. Then, since at the point of intersection, the two equations will share a point and thus have the same values of x and y, we set the two equations equal to each other. This gives an equation that we can solve for x
3. We substitute the x value in one of the line equations (it doesn't matter which) and solve it for y. This gives us the x and y coordinates of the intersection.

Example #1
Q: Find the point of intersection of two lines that have the following equations (in slope-intercept form):
$y = 3x - 3$
$y = 2.3x + 4$

Solution: At the point of intersection they will both have the same y-coordinate value, so we set the equations equal to each other:
$3x - 3 = 2.3x + 4$

This gives us an equation in one unknown (x) which we can solve:
$x = 10$

To find y, simply set x equal to 10 in the equation of either line and solve for y:
Equation for a line $y = 3x - 3$ (Either line will do)
Set x equal to 10: $y = 30 - 3$

$y = 27$
We now have both x and y, so the intersection point is (10, 27)

Example #2
Q: Find the point of intersection of two lines that have the following equations (in slope-intercept form): $y = 3x - 3$and $x = 12$(A vertical line)

Solution: When one of the lines is vertical, it has no defined slope. We find the intersection slightly differently.

On the vertical line, all points on it have an x-coordinate of 12 (the definition of a vertical line), so we simply set x equal to 12 in the first equation and solve it for y.
Equation for a line $y = 3x - 3$
Set x equal to $y = 36 - 3$
$y = 33$
So, the intersection point is at (12,33).

Note: If both lines are vertical or horizontal, they are parallel and have no intersection

Distance from a point to a line

The distance from a point to a line is the **shortest** distance between them - the length of a perpendicular line segment from the line to the point.

The distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is given by the formula:

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

When the line is horizontal the formula transforms to: $D|P_y - L_y|$

Where P_y is the y-coordinate of the given point P; L_y is the y-coordinate of **any** point on the given vertical line L. | | the vertical bars mean "absolute value" - make it positive even if it calculates to a negative.

When the line is vertical the formula transforms to: $D = |P_x - L_x|$
Where P_x is the x-coordinate of the given point P; L_x is the x-coordinate of **any** point on the given vertical line L. | | the vertical bars mean "absolute value" - make it positive even if it calculates to a negative.

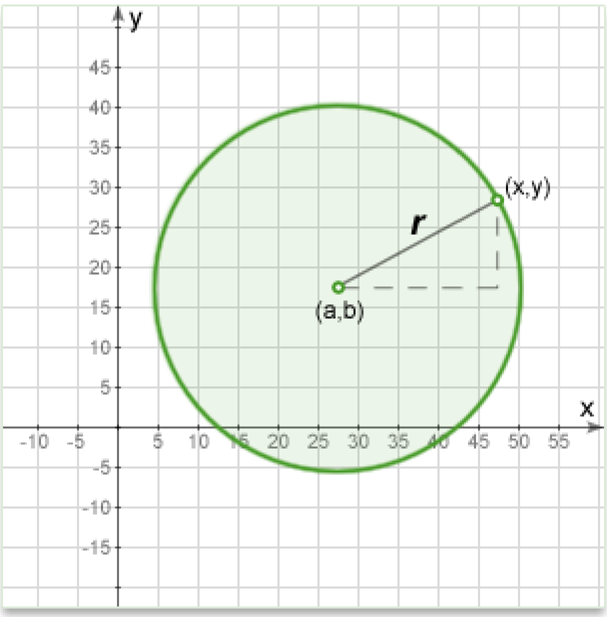
When the given point is origin, then the distance between origin and line ax+by+c=0 is given by the formula: $D = \frac{|c|}{\sqrt{a^2 + b^2}}$

4.3 - COORDINATE GEOMETRY FIGURES

Frequency of the concepts tested: **Very Low**

Circle on a plane

In an x-y Cartesian coordinate system, the circle with center (a, b) and radius r is the set of all points (x, y) such that: $(x-a)^2+(y-b)^2=r^2$

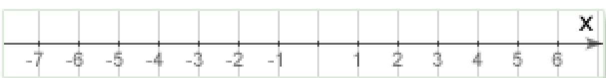


This equation of the circle follows from the Pythagorean theorem applied to any point on the circle: as shown in the diagram above, the radius is the hypotenuse of a right-angled triangle whose other sides are of length x-a and y-b.

If the circle is centered at the origin (0, 0), then the equation simplifies to: $\boldsymbol{x^2+y^2=r^2}$

Number line

A number line is a picture of a straight line on which every point corresponds to a real number and every real number to a point.



On the GRE we can often see such statement: k is halfway between m and $n$on the number line. Remember this statement can ALWAYS be expressed as:

$$\frac{m+n}{2}=k$$

Also, on the GRE we can often see another statement: The distance between p and m on the number line is the same as the distance between p and n. Remember this statement can ALWAYS be expressed as:

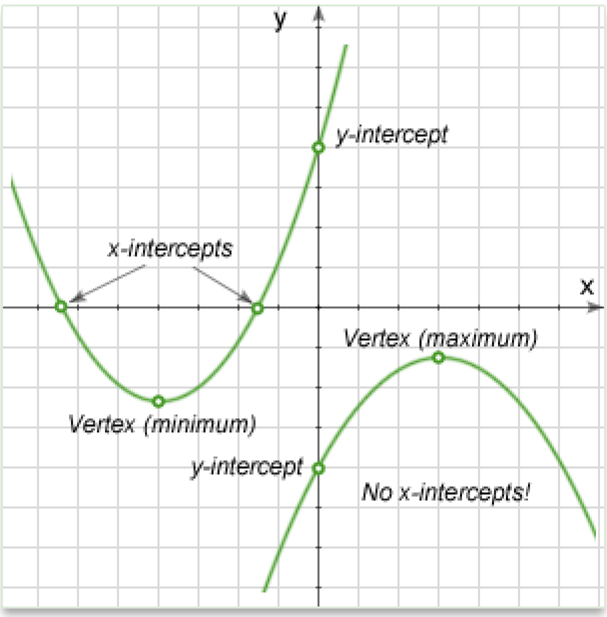
$$|p-m|=|p-n|$$

4.4 - COORDINATE GEOMETRY PARABOLAS

Frequency of the concepts tested: **Low**

Parabola

A parabola is the graph associated with a quadratic function, i.e. a function of the form $y = ax^2 + bx + c$



The general or standard form of a quadratic function is $y = ax^2 + bx + c$, or in function form, $f(x) = ax^2 + bx + cx$ is the independent variable, $y$is the dependent variable, and $abc$are constants.

- The larger the absolute value of a the steeper (or thinner) the parabola is, since the value of y is increased more quickly.
- If a is positive, the parabola opens upward, if negative, the parabola opens downward.

x-intercepts: The x-intercepts, **if any**, are also called the roots of the function. The x-intercepts are the solutions to the equation $0 = ax^2 + bx + c$ and can be calculated by the formula:

$x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

Expression $b^2 - 4ac$ is called *discriminant*:

- If discriminant is positive parabola has two intercepts with x-axis;
- If discriminant is negative parabola has no intercepts with x-axis;

- If discriminant is zero parabola has one intercept with x-axis (tangent point).

y-intercept: Given $y = ax^2 + bx + c$ the y intercept is c, as y intercept means the value of y when x=0.

Vertex: The vertex represents the maximum (or minimum) value of the function, and is very important in calculus.
The vertex of the parabola is located at point $(-\frac{b}{2a}, c - \frac{b^2}{4a})$

Note: Typically, just $-\frac{b}{2a}$, is calculated and plugged in for x to find y.

IMPORTANT LINK!

GRE® Score Calculator

CHAPTER 5. DATA ANALYSIS

5.1 - AVERAGE, MEDIAN, AND MODE

Frequency of the concepts tested: **High**

Mean /Average / Arithmetic Mean

• Mean is the average of the all the numbers in the set.

• Mean = $\frac{SumOfAllTheNumbersInTheSet}{TotalNumberOfNumbersInTheSet}$

$$\frac{(1+2+3+4+5)}{5} = \frac{15}{5} = 3$$

Properties of Mean

1. If all the numbers in the set are increased/decreased by the same number(k) then the mean also gets increased/decreased by the same number(k)

1. Suppose the set is {a,b,c,d,e}
then the Mean = $\frac{(a+b+c+d+e)}{5}$

Now, let's increase all the numbers by k. So, the new set is {a+k,b+k,c+k,d+k,e+k)

New Mean = $\frac{(a+k+b+k+c+k+d+k+e+k)}{5}$

$\frac{(a+b+c+d+e+5k)}{5} = \frac{(a+b+c+d+e)}{5}$ + k = Old Mean + k

2. If all the numbers in the set are multiplied/divided by the same number(k) then the mean also gets multiplied/divided by the same number(k). Proof same as above. In this case if we multiple all the numbers by k then

New Mean = k* (Old Mean)

SUGGESTION: Don't try remembering the points 1 and 2 above. It does not take much time to calculate them!

Median

• Median is the middle value of the set.

• In case of even number of numbers in the set: Median is the mean of the two middle numbers (after the numbers are arranged in the increasing / decreasing order)

Example: If the set is {5,1,4,6,3,2} then we will arrange the set as {1,2,3,4,5,6} and median will be mean of middle two terms. Middle two terms in this case are 3 and 4 so Median = (3+4)/2 = 3.5

• In case of odd number of numbers in the set: Median is the middle number (after the numbers are arranged in increasing/ decreasing order)

Example: If the set is {4,5,3,1,2} then we will arrange the set as {1,2,3,4,5} and the median will be the middle number which is 3

<u>Properties of Median</u>

1. If all the numbers in the set are increased/decreased by the same number(k) then the median also gets increased/decreased by the same number(k)

2. If all the numbers in the set are multiplied/divided by the same number(k) then the median also gets multiplied/divided by the same number(k)

3. In Case of evenly spaced set
Mean = Median = Middle term (if the number of terms is odd)
= Mean of middle terms (if the number of terms is even)

4. In case of consecutive integers: IF the number of integers is even then then the Mean = Median ≠ Integer
Suppose the set is {1,2,3,4,5,6}
then Mean = Median = 3.5

SUGGESTION: Don't try remembering the points 1 and 2 above. It does not take much time to calculate them!

<u>Mode</u>
• Mode is the number which has occurred the maximum number of times in the set.
Suppose the set is {1,1,2,2,3,3,3,3,4,5}
then the mode is 3, as 3 has occurred the maximum number of times in the set.

5.2 - RANGE, WEIGHTED MEAN, VARIANCE, SD

Frequency of the concepts tested: **Medium**

Range

Range of a set is the difference between the highest and lowest value of the set.
Example: Suppose the set is {-1,2,3,6,8} then the range will be 8 -(-1) = 9

Properties of Range

1. If all the numbers in the set are increased/decreased by the same number(k) then the range DOES NOT CHANGE!
Suppose the set is {a,b,c} (in increasing order)
Range = c-a
Now, let's increase all the numbers by k then the set will become {a+k, b+k, c+k}
New range = c+k -(a+k) = c-a = Old range

2. If all the numbers in the set are multiplied/divided by the same number(k) then the range also gets multiplied/divided by the same number(k)

Weighted Average

• Weighted Average = $\frac{((Weight1*Value1)+(Weight2*Value2)...+(WeightN*ValueN))}{(Weight1+Weight2+...WeightN)}$

Variance

• Variance, V = Mean of (Square of difference of each number from the mean)

V = $\frac{Sumof(SquaresOfDifferenceOfEachNumberFromMean)}{TotalNumberOfNumbers}$

Q1 Find the Variance of the set { 1, 2, 3, 4, 5 }

Sol: Mean of this set is 3.

$$\frac{((3-1)^2+(3-2)^2+(3-3)^2+(3-4)^2+(3-5)^2)}{5}$$

$\frac{(4+1+0+1+4)}{5} = 2$

Properties of Variance

1. If all the numbers in a set are increased/ decreased by the same number(k) then the variance DOES NOT change.
2. If all the numbers in a set are multiplied/ divided by the same number(k) then the variance gets multiplied/divided by the square of the number (k2)

Standard Deviation (SD)

- SD is an indication of how spread the numbers are as compared to the Mean
- SD is equal to the Root Mean Square(RMS) of the distance of the values from the mean
- Standard Deviation =$\sqrt{Variance}\sqrt{V}$

Q1 Find the SD of the set { 1, 2, 3, 4, 5 }
Sol: V = 2 (calculated above)

$SD = \sqrt{V} = \sqrt{2}$

Properties of SD

1. If all the numbers in the set are increased/decreased by the same number(k) then the Standard Deviation DOES NOT CHANGE!
(This happens because the mean also gets increased/decreased by the same number and the Variance or Standard Deviation are calculated by subtracting all the numbers by the mean and taking square of them and taking their average.)

2. If all the numbers in the set are multiplied/divided by the same number(k) then the Standard Deviation also gets multiplied by the same number.

Recap of Properties

Statistics	If all the numbers in the set are + or – or * or / by the same number k			
	+	-	*	/
Mean	**Mean + k**	**Mean - k**	**Mean * k**	**Mean / k**
Median	**Median + k**	**Median - k**	**Median * k**	**Median / k**
Range	**Range**	**Range**	**Range * k**	**Range / k**
Variance	**Variance**	**Variance**	**Variance * k^2**	**Variance / k^2**
SD	**SD**	**SD**	**SD * k**	**SD / k**

5.3 - COUNTING METHODS

Frequency of the concepts tested: **Medium/High**

Definition

Combinatorics is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties.

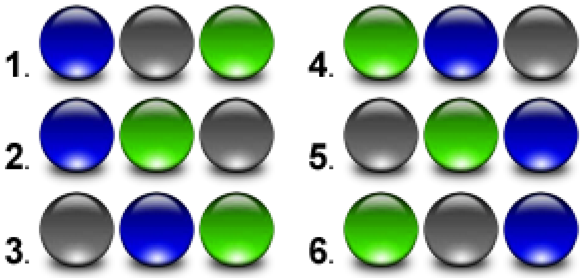
Enumeration

Enumeration is a method of counting all possible ways to arrange elements. Although it is the simplest method, it is often the fastest method to solve hard GRE problems and is a pivotal principle for any other combinatorial method. In fact, combination and permutation is shortcuts for enumeration. The main idea of enumeration is writing down all possible ways and then count them. Let's consider a few examples:

Example #1

Q:. There are three marbles: 1 blue, 1 gray and 1 green. In how many ways is it possible to arrange
marbles in a row?

Solution: Let's write out all possible ways:



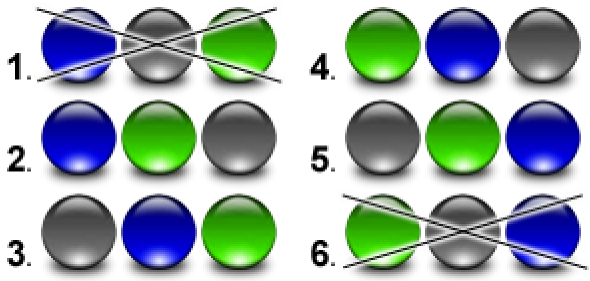
Answer is 6.

In general, the number of ways to arrange n different objects in a row

Example #2

Q:. There are three marbles: 1 blue, 1 gray and 1 green. In how many ways is it possible to arrange marbles in a row if blue and green marbles have to be next to each other?

Solution: Solution: Let's list all possible ways to arrange the marbles in a row and then identify the arrangements that satisfy the question's condition.

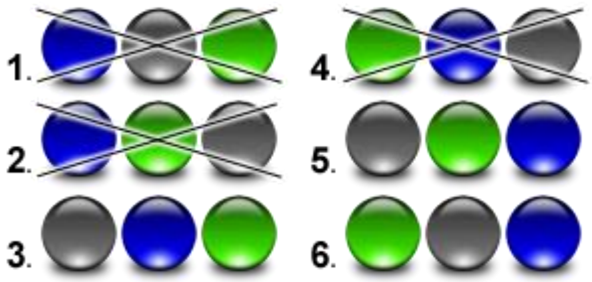


Answer is 4.

Example #3

Q:. There are three marbles: 1 blue, 1 gray and 1 green. In how many ways is it possible to arrange marbles in a row if gray marble have to be left to blue marble?

Solution: Solution: Let's list all possible ways to arrange the marbles in a row and then identify the arrangements that satisfy the question’s condition.



Answer is 3.

Arrangements of n different objects

Enumeration is a great way to count a small number of arrangements. But when the total number of arrangements is large, enumeration can't be very useful, especially taking into account GRE time restriction. Fortunately, there are some methods that can speed up counting of all arrangements.

The number of arrangements of n different objects in a row is a typical problem that can be solve this way:

1. How many objects we can put at 1st place? n.
2. How many objects we can put at 2nd place? n - 1.

We can't put the object that already placed at 1st place. n.

How many objects we can put at n-th place?

1. Only one object remains.

Therefore, the total number of arrangements of n different objects in a row is

$N = n * (n-1) * (n-2) \ldots 2 * 1 = n!$

Combination

A combination is an unordered collection of k objects taken from a set of n distinct objects. The number of ways how we can choose k objects out of n distinct objects is denoted as:

C_k^n

knowing how to find the number of arrangements of n distinct objects we can easily find formula for combination:

1. The total number of arrangements of n distinct objects is n!
2. Now we have to exclude all arrangements of k objects (k!) and remaining (n-k) objects ((n-k)!) as the order of chosen k objects and remained (n-k) objects doesn't matter.

$$C_k^n = \frac{n!}{k!\,(n-k)!}$$

Permutation

A permutation is an ordered collection of k objects taken from a set of n distinct objects. The number of ways how we can choose k objects out of n distinct objects is denoted as:

P_k^n

Knowing how to find the number of arrangements of n distinct objects we can easily find formula for combination:

1. The total number of arrangements of n distinct objects is n!
2. Now we have to exclude all arrangements of remaining (n-k) objects ((n-k)!) as the order of remained (n-k) objects doesn't matter.

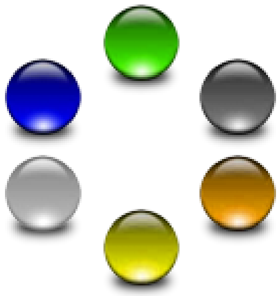
$$P_k^n = \frac{n!}{(n-k)!}$$

If we exclude order of chosen objects from permutation formula, we will get combination formula:

$$\frac{P_k^n}{k!} = C_k^n$$

Circular arrangements

Let's say we have 6 distinct objects, how many relatively different arrangements do we have if those objects should be placed in a circle.



The difference between placement in a row and that in a circle is following: if we shift all object by one position, we will get different arrangement in a row but the same relative arrangement in a circle. So, for the number of circular arrangements of n objects we have:

$R = \frac{n!}{n} = (n-1)!$

Tips and Tricks

Any problem in Combinatorics is a counting problem. Therefore, a key to solution is a way how to count the number of arrangements. It sounds obvious but a lot of people begin approaching to a problem with thoughts like "Should I apply C- or P formula here?

Don't fall in this trap: define how you are going to count arrangements first, realize that your way is right and you don't miss something important, and only then use C- or P- formula if you need them.

5.4 – PROBABILITY

Frequency of the concepts tested: **Medium**

Definition

A number expressing the probability (p) that a specific event will occur, expressed as the ratio of the number of actual occurrences (n) to the number of possible occurrences (N).

$$p = \frac{n}{N}$$

A number expressing the probability (q) that a specific event will not occur:

$$q = \frac{(N-n)}{N} = 1 - p$$

Coin

There are two equally possible outcomes when we toss a coin: a head (H) or tail (T). Therefore, the probability of getting head is 50% or $\frac{1}{2}$ and the probability of getting tail is 50% or $\frac{1}{2}$

All possibilities: {H,T}

Dice

There are 6 equally possible outcomes when we roll a die. The probability of getting any number out of 1-6 is $\frac{1}{6}$

Marbles, Balls, Cards...

Let's assume we have a jar with 10 green and 90 white marbles. If we randomly choose a marble, what is the probability of getting a green marble?
The number of all marbles: N = 10 + 90 =100
The number of green marbles: n = 10
Probability of getting a green marble: $p = \frac{n}{N} = \frac{10}{100} = \frac{1}{10}$

There is one important concept in problems with marbles/cards/balls. When the first marble is removed from a jar and not replaced, the probability for the second marble differs ($\frac{9}{99}$ vs. $\frac{10}{100}$) Whereas in case of a coin or dice the probabilities are always the same ($\frac{1}{6}$ and $\frac{1}{2}$) Usually, a problem explicitly states: it is a problem with replacement or without replacement.

Independent events

Two events are independent if occurrence of one event does not influence occurrence of other events. For n independent events the probability is the product of all probabilities of independent events:

p = p1 * p2 * ... * pn-1 * pn
or
P(A and B) = P(A) * P(B) - A and B denote independent events

Example #1
Q: There is a coin and a die. After one flip and one toss, what is the probability of getting heads and a "4"?
Solution: Tossing a coin and rolling a die are independent events. The probability of getting heads is $\frac{1}{2}$and probability of getting a "4" is $\frac{1}{6}$

$$P = \frac{1}{2} * \frac{1}{6} = \frac{1}{12}$$

Example #2
Q: If there is a 20% chance of rain, what is the probability that it will rain on the first day but not on the second?
Solution: The probability of rain is 0.2; therefore, probability of sunshine is q = 1 - 0.2 = 0.8.

This yields that the probability of rain on the first day and sunshine on the second day is:
P = 0.2 * 0.8 = 0.16

Example #3
Q: There are two sets of integers: {1,3,6,7,8} and {3,5,2}. If Robert chooses randomly one integer from the first set and one integer from the second set, what is the probability of getting two odd integers?
Solution: There is a total of 5 integers in the first set and 3 of them are odd: {1, 3, 7}. Therefore, the probability of getting odd integer out of first set is $\frac{3}{5}$. There are 3 integers in the second set and 2 of them are odd: {3, 5}. Therefore, the probability of getting an odd integer out of second set is $\frac{2}{3}$ Finally, the probability of getting two odd integers is:

$$P = \frac{3}{5} * \frac{2}{3} = \frac{2}{5}$$

Mutually exclusive events

Shakespeare's phrase "To be, or not to be: that is the question" is an example of two mutually exclusive events.
Two events are mutually exclusive if they cannot occur at the same time. For n mutually exclusive events the probability is the sum of all probabilities of events:

$$\mathrm{p} = \mathrm{p1} + \mathrm{p2} + \ldots + \mathrm{pn} - 1 + \mathrm{pn}$$

or
P(A or B) = P(A) + P(B) - A and B denotes mutually exclusive events.

Example #1
Q: If Jessica rolls a die, what is the probability of getting at least a "3"?

Solution: There are 4 outcomes that satisfy our condition (at least 3): {3, 4, 5, 6}. The probability of each outcome is 1/6. The probability of getting at least a "3" is:

$$P = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

Combination of independent and mutually exclusive events

Many probability problems contain combination of both independent and mutually exclusive events.

To solve those problems, it is important to identify all events and their types. One of the typical problems can be presented in a following general form:

Q: If the probability of a certain event is p, what is the probability of it occurring k times in n-time sequence? (Or in English, what is the probability of getting 3 heads while tossing a coin 8 times?)

Solution: All events are independent. So, we can say that:

$P' = p^k * (1-p)^{n-k}(1)$

But it isn't the right answer. It would be right if we specified exactly each position for events in the sequence. So, we need to take into account that there is more than one outcome. Let's consider our example with a coin where "H" stands for Heads and "T" stands for Tails:

HHHTTTTT and HHTTTTTH are different mutually exclusive outcomes but they both have 3 heads and 5 tails. Therefore, we need to include all combinations of heads and tails. In our general question, probability of occurring event k times in n-time sequence could be expressed as:

$P = C_k^n * p^k * (1-p)^{n-k}(2)$

In the example with a coin, right answer is $P = C_3^8 * 0.5^3 * 0.5^5 = C_3^8 * 0.5^8$

Example #1
Q.: If the probability of raining on any given day in Atlanta is 40 percent, what is the probability of raining on exactly 2 days in a 7-day period?

Solution: We are not interested in the exact sequence of event and thus apply formula #2:

$P = C_2^7 * 0.4^2 * 0.6^5$

A few ways to approach a probability problem

There are a few typical ways that you can use for solving probability questions. Let's consider example, how it is possible to apply different approaches:

Example #1
Q: There are 8 employees including Bob and Rachel. If 2 employees are to be randomly chosen to form a committee, what is the probability that the committee includes both Bob and Rachel?

Solution:
1) combinatorial approach: The total number of possible committees is $N = C_2^8 n.$ The number of possible committees that includes both Bob and Rachel is $n = 1$

$$P = \frac{n}{N} = \frac{1}{C_2^8} = \frac{1}{28}$$

2) reversal combinatorial approach: Instead of counting probability of occurrence of certain event, sometimes it is better to calculate the probability of the opposite and then use formula p = 1 - q. The total number of possible committees is $N = C_2^8$. The $m = C_2^6 + 2 * C_1^6$ where, C_2^6- the number of committees formed from 6 other people.

$2 * C_1^6$- the number of committees formed from Rob or Rachel and one out of 6 other people.

$$P = 1 - \frac{m}{N} = 1 - \frac{C_2^6 + 2 * C_1^6}{C_2^8}$$

$$P = 1 - \frac{15 + 2 * 6}{28} = 1 - \frac{27}{28} = \frac{1}{28}$$

3) probability approach: The probability of choosing Bob or Rachel as a first person in committee is 2/8. The probability of choosing Rachel or Bob as a second person when first person is already chosen is 1/7. The probability that the committee includes both Bob and Rachel is.

$$P = \frac{2}{8} * \frac{1}{7} = \frac{2}{56} = \frac{1}{28}$$

4) reversal probability approach: We can choose any first person. Then, if we have Rachel or Bob as first choice, we can choose any other person out of 6 people. If we have neither Rachel nor Bob as first choice, we can choose any person out of remaining 7 people. The probability that the committee includes both Bob and Rachel is.

$$P = 1 - (\frac{2}{8} * \frac{6}{7} + \frac{6}{8} * 1) = \frac{2}{56} = \frac{1}{28}$$

Example #2

Q: Given that there are 5 married couples. If we select only 3 people out of the 10, what is the probability that none of them are married to each other?

Solution:

1) combinatorial approach:

C_3^5- we choose 3 couples out of 5 couples.

C_1^2- we chose one person out of a couple.

$(C_1^2)^3$- we have 3 couple and we choose one person out of each couple.

C_3^{10}- The total number of ways to choose 3 people from 10 people..

$$p = \frac{C_3^5 * (C_1^2)^3}{C_3^{10}} = \frac{10 * 8}{10 * 3 * 4} = \frac{2}{3}$$

2) reversal combinatorial approach: In this example reversal approach is a bit shorter and faster.

C_1^5- we choose 1 couple out of 5 couples.

C_1^8- we chose one person out of remaining 8 people.

C_3^{10}- The total number of ways to choose 3 people from 10 people..

$$p = 1 - \frac{C_1^5 * C_1^8}{C_3^{10}} = 1 - \frac{5 * 8}{10 * 3 * 4} = \frac{2}{3}$$

3) probability approach:

1st person: $\frac{10}{10} = 1$- we choose any person out of 10.

2nd person: $\frac{8}{9}$- we choose any person out of 8=10-2(one couple from previous choice)

3rd person: $\frac{6}{8}$- we choose any person out of 6=10-4(two couples from previous choices).

$$p = 1 * \frac{8}{9} * \frac{6}{8} = \frac{2}{3}$$

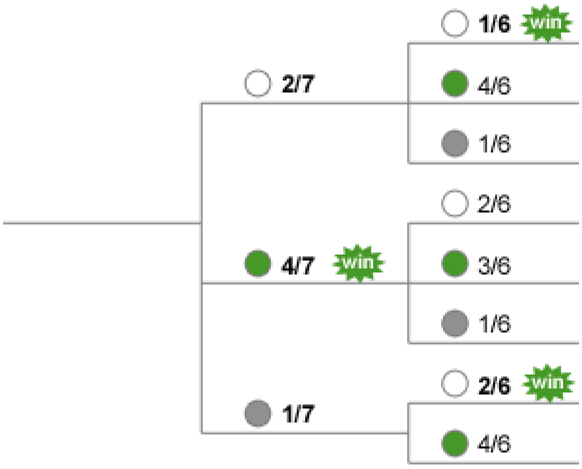
Probability tree

Sometimes, at hard question you may see complex probability problems that include conditions or restrictions. For such problems it could be helpful to draw a probability tree that include all possible outcomes and their probabilities

Example #1

Q: Julia and Brian play a game in which Julia takes a ball and if it is green, she wins. If the first ball is not green, she takes the second ball (without replacing first) and she wins if the two balls are white or if the first ball is gray and the second ball is white. What is the probability of Julia winning if the jar contains 1 gray, 2 white and 4 green balls?

Solution: Let's draw all possible outcomes and calculate all probabilities.



Now, it is pretty obvious that the probability of Julia's win is:

$$P = \frac{4}{7} + \frac{2}{7} * \frac{1}{6} + \frac{1}{7} * \frac{2}{6} = \frac{2}{3}$$

Tips and Tricks: Symmetry

Symmetry sometime lets you solve seemingly complex probability problem in a few seconds. Let's consider an example.

Example #1

Q: There are 5 chairs. Bob and Rachel want to sit such that Bob is always left to Rachel. How many ways it can be done?

Solution: Because of symmetry, the number of ways that Bob is left to Rachel is exactly 1/2 of all possible ways:

$$N = \frac{1}{2} * P_2^5 = 10$$

5.5 - SEQUENCES & PROGRESSIONS

Frequency of the concepts tested: **Medium**

Definition

Sequence: It is an ordered list of objects. It can be finite or infinite. The elements may repeat themselves more than once in the sequence, and their ordering is important unlike a set.

Arithmetic Progressions

Definition

It is a special type of sequence in which the difference between successive terms is constant.

General Term

$a_n = a_{n-1} + d = a_1 + (n-1)d$

a_i is the ith term

d is the common difference

a_1 is the first term

Defining Properties

Each of the following is necessary & sufficient for a sequence to be an AP :

- $a_i - a_{i-1}$ =Constant
- If you pick any 3 consecutive terms, the middle one is the mean of the other two
- For all i,j > k >= 1 : $\frac{a_i - a_k}{i-k} = \frac{a_j - a_k}{j-k}$

Summation

The sum of an infinite AP can never be finite except if $a_1 = 0$ & $d = 0$

The general sum of a n term AP with common difference d is given by $\frac{n}{2}(2a + (n-1)d)$

The sum formula may be re-written as $n * Avg(a_1, a_n) = \frac{n}{2} * (FirstTerm + LastTerm)$

Examples

- All odd positive integers: {1,3,5, 7...} $a_1 = 1, d = 2$
- All positive multiples of 23: {23,46,69,92,...} $a_1 = 23, d = 23$
- All negative reals with decimal part 0.1 : {-0.1,-1.1,-2.1,-3.1,...} $a_1 = -0.1, d = -1$

Geometric Progressions

Definition

It is a special type of sequence in which the ratio of consecutive terms is constant

General Term

$b_n = b_{n-1} * r = a_1 * r^{n-1}$

$b_i$is the ith term

$r$is the common ratio

$b_1$is the first term

Defining Properties

Each of the following is necessary & sufficient for a sequence to be an GP:

- $\frac{b_i}{b_{i-1}}$ =Constant
- If you pick any 3 consecutive terms, the middle one is the geometric mean of the other two
- For all i,j > k >= 1: $(\frac{b_i}{b_k})^{j-k} = (\frac{b_j}{b_k})^{i-k}$

Summation

The sum of an infinite GP will be finite if absolute value of r < 1

The general sum of a n term GP with common ratio r is given by $b_1 * \frac{r^n-1}{r-1}$

If an infinite GP is summable (|r|<1) then the sum is $\frac{b_1}{1-r}$

Examples

- All positive powers of 2: {1,2,4, 8...} $b_1 = 1, r = 2$
- All negative powers of 4: {1/4,1/16,1/64,1/256,...} $b_1 = \frac{1}{4}, r = \frac{1}{4}, sum = \frac{1/4}{(1-1/4)} = (1/3)$

Harmonic Progressions

Definition

It is a special type of sequence in which if you take the inverse of every term, this new sequence forms an HP.

<u>Important Properties</u>

Of any three consecutive terms of a HP, the middle one is always the harmonic mean of the other two, where the harmonic mean (HM) is defined as:

$\frac{1}{2} * \left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{HM(a,b)}$ or in other words

$$HM(a, b) = \frac{2ab}{a + b}$$

APs, GPs, HPs: Linkage

Each progression provides us a definition of "mean" :

Arithmetic Mean: $\frac{a+b}{2}$ OR $\frac{a1+..+an}{n}$

Geometric Mean: $\sqrt{ab}$ OR $(a1 *..* an)^{\frac{1}{n}}$

Harmonic Mean: $\frac{2ab}{a+b}$ OR $\frac{n}{\frac{1}{a1}+..+\frac{1}{an}}$

For all non-negative real numbers: AM >= GM >= HM

In particular for 2 numbers: AM * HM = GM * GM

Example:

Let a=50 and b=2,

then the AM = (50+2)*0.5 = 26;

the GM = sqrt(50*2) = 10;

the HM = (2*50*2)/(52) = 3.85

AM > GM > HM

AM*HM = 100 = GM^2

<u>How to Solve: Arithmetic and Geometric Progression</u>

What is Arithmetic Progression (AP)?

A sequence of numbers such that the difference between the consecutive terms is constant. It is also known as Arithmetic Sequence or Arithmetic Series

Example: 2 , 5 , 8 , 11…. (Consecutive terms have the same common difference of 3)

AP Formulas

N^{th} **Term of an Arithmetic Series**

Arithmetic Series is given by a, a+d, a+2d...

T_1= a = a + (1-1)d
T_2= a + d = a + (2-1)d
T_3= a + 2d = a + (3-1)d
.
.
.
T_n= a + (n-1)d

N^{th} of an Arithmetic Series, T_n**= a + (n-1)d**
where,
a is the first term of the sequence
d is the difference between consecutive terms in the sequence (common difference)
n is the number of terms
T_n is the nth term in the sequence

Sum of n terms of a AP is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_n = n * \frac{(a + (a + (n-1)d))}{2}$$

S_n**= Number of terms * Mean of First term and Last term**

Number of terms in an AP is given by

n = $\frac{(T_n - T_1)}{d} + 1$

For Arithmetic Series
Mean = Median = Avg. of 1st and Last term = Avg. of 2nd term from the starting and second term from the end = Avg. of 3rd term from the starting and third term from the end and so on....

AP Problems

Q. Find the sum of first "n" positive integers (i.e. 1 + 2 + 3 +… + n)

Sol: Series is given by 1, 2, 3, 4, …, n

Sum of the series
= Number of terms * Mean of First and Last term
= n * $\frac{((1+n))}{2}$

Sum of First n Positive integers

Sum of first n integers $= \frac{(n(n+1))}{2}$

only when
Terms are starting from 1 and
Series comprises of consecutive integers

What is Geometric Progression (GP)?
Geometric Series is a series in which consecutive terms have the same ratio.
It is also known as Geometric Sequence or Geometric Series
Example: 2, 6, 18, 54…. (Consecutive terms have the same ratio of 3:1)

GP Formulas

N^{th}**Term of a Geometric Series**

Geometric Series is given by

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$

T_1= a = ar^{1-1}

T_2= ar = ar^{2-1}

T_3= ar^2= ar^{3-1}

T_n= ar^{n-1}

Sum of n terms of a GP is given by

$$S_n = a * \frac{((r^n - 1)}{(r-1))}$$

Miscellaneous Problems

Following is not an Arithmetic or a Geometric series. Find the n^{th}term of this series $T_n$of the series:

1. 1, 4, 9, 16,…
2. 1, 8, 27, 64, …
3. 2, 5, 10, 17,…
4. 2, 9, 28, 65,…
5. 2, 6, 12, 20,…
6. 2, 4, 8, 16, …
7. 1, 2, 3, 4, 5, 8, 7, 16,…
8. 1, 2, 3, 5, 8, 13,…
9. **1, 2, 2, 4, 8, 32, …**

Answers

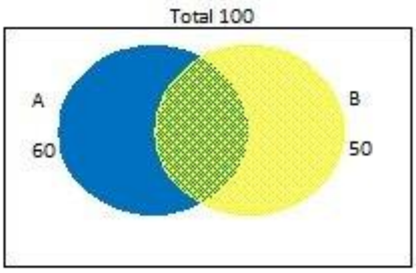
1. T_n= n^2
2. T_n= n^3
3. T_n= n^2+ 1
4. T_n= n^3+ 1
5. T_n= n^2+ n
6. T_n= 2n
7. T_{Odd}= n T_{Even}= $2^{n/2}$
8. T_n= T_{n-1}+ $T_{n-2}$for n>= 3
9. T_n= $T_{n-1}T_{n-2}$for n>= 3

5.6 - OVERLAPPING SETS

Frequency of the concepts tested: **Medium**

A rare question involves an overlapping set for group members, These questions a fairly easy but they could result intimidating to those students who have never seen them before.

Say, there are a total of 100 people in a housing society. There are two clubs close to the society – A and B. You are given that of the 100 people of the housing society, 60 people are members of club A and 50 people are members of club B.



Question 1: How many people are members of both the clubs?

We are looking for the number of people in the green region. The answer here is not 10. It is 'cannot be determined' i.e. you cannot say how many people are members of both the clubs. The reason is that you do not know how many people belong to neither club.

Say, for all future questions (unless mentioned otherwise), you are given that 20 people belong to neither club. What can you say about the number of people who belong to both the clubs? Now, out of the pool of 100, 20 are out. Only 80 people are club members. Since 60 are members of club A and 50 people are members of club B which gives us a total of 110, there must be an overlap of 30 people i.e. 30 people must belong to both the clubs (80 = 60 + 50 – Both)

Question 2: How many people belong to only one club?

We found above that 30 people belong to both the clubs. So out of the 60 people of club A, 30 belong to only club A. Out of the 50 people of club B, 20 belong to only club B. So a total of 30+20 = 50 people belong to only one club, either A or B but not both. (60 – Both + 50 – Both = 30 + 20 = 50)

Question 3: Say, you don't know the number of people who belong to neither club. What is the minimum number of people who must belong to both the clubs?

We know that there is a total of 100 people. 60 belong to club A and 50 belong to club B which adds up to 110. Therefore, AT LEAST 10 people must have membership of both the clubs. Now if you increase the number of people who do not belong to either club, the number of people who belong to both will increase by the same number. Think in terms of the Venn diagram. If the 'Neither' number increases, the number of people who are members decreases. Hence, the overlap increases to keep A = 60 and B = 50.

Let's look at the promised question which will make this concept clear.

Question: Of the 400 members at a health club, 260 use the weight room and 300 use the pool. If at least 60 of the members do not use either, then the number of members using both the weight room and the pool must be between:

(A) 40 to 100
(B) 80 to 140
(C) 160 to 260
(D) 220 to 260
(E) 220 to 300

Solution: When we minimize "number of members who do not use either", we are minimizing the "number of members who use both" as well.

Look at the equation:
Total = A + B – Both + Neither

Eventually, ***this is the only formula you may need to memorize to tackle the overlapping sets on the GRE***

Since the total sum 400 is constant, if we increase the 'Neither' i.e. 60, we will have to increase the 'Both' terms too to maintain the sum of 400 (Assuming A and B are constant which they are since they are given to us).

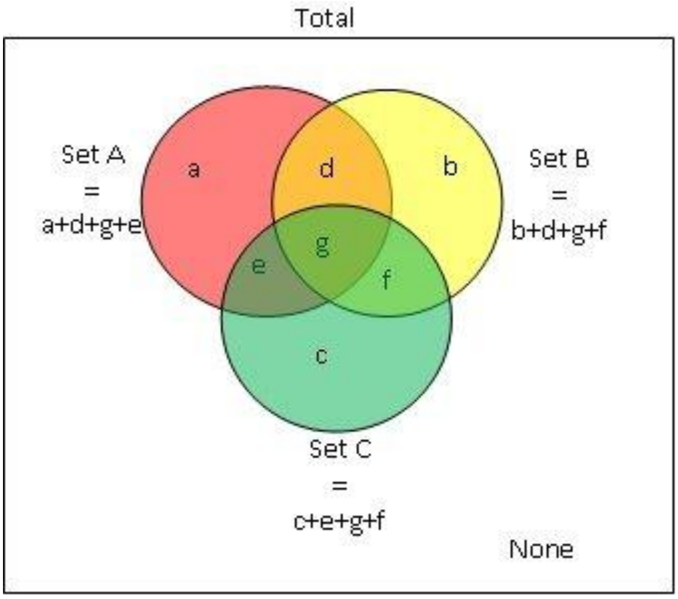
Least value of 'number of members who use neither' is 60. We will get the least value of 'number of members who use both' when we put 'Neither' = 60. 400 = 260 + 300 – 'Minimum value of both' + 60 Minimum value of both = 220

On the same lines, if we maximize "number of members who use neither", we are maximizing the "number of members who use both" as well.
What is the maximum number of people who use neither? Out of a total of 400 people, 300 people use the pool. Hence at least 300 people use at least one of the two facilities. This means that there can be AT MOST 100 people (total 400 – 300 who use pool) who use neither facility.

400 = 260 + 300 – 'Maximum value of both' + 100 Maximum value of both = 260

Three Overlapping Sets

How the three overlapping sets diagram looks like.



Notice that the total comprises of the elements that do not fall in any of the three sets and the elements that are a part of at least one of the three sets.

The elements falling in the red, yellow or green region (region a, b or c) fall in only one set. The elements falling in region d, e or f fall in exactly two sets and the elements falling in region g fall in all three sets.

Now some quick questions to get a clear picture:

Question 1: Which regions represent the elements that belong to at least 2 sets?
Answer 1: d + e + f + g

Question 2: Which regions represent the elements that belong to at least 1 set?
Answer 2: a + b + c + d + e + f + g = Total – None

Question 3: Which regions represent the elements that belong to at most 2 sets?
Answer 3: None + a + b + c + d + e + f = Total – g

Hope there are no doubts up till now. Let's look at a question to see how to apply these concepts.

Question: Three table runners have a combined area of 200 square inches. By overlapping the runners to cover 80% of a table of area 175 square inches, the area that is covered by exactly two layers of runner is 24 square inches. What is the area of the table that is covered with three layers of runner?

(A) 18 square inches
(B) 20 square inches
(C) 24 square inches
(D) 28 square inches
(E) 30 square inches

Solution: Let's first try to understand what exactly is given to us. The area of all the runners is equal to 200 square inches.

Runner 1 + Runner 2 + Runner 3 = 200. In our diagram, this area is represented by $(a + d + g + e) + (b + d + g + f) + (c + e + g + f) = 200$

(We need to find the value of g i.e. the area of the table that is covered with three layers of runner.)

Area of table covered is only 80% of 175 i.e. only 140 square inches. This means that if each section is counted only once, the total area covered is 140 square inches. $a + b + c + d + e + f + g = 140$

So, the overlapping regions are obtained by subtracting second equation from the first. We get $d + e + f + 2g = 60$

But $d + e + f$ (area with exactly two layers of runner) = 24 So $2g = 60 – 24 = 36$ $g = 18$ square inches.

Note that you don't need to make all these equations and can directly jump to $d + e + f + 2g = 60$. We wrote these equations down only for clarity. It is a matter of thinking vs solving. If we think more, we have to solve less. Let's see how.

Combined area of runners is 200 square inches while area of table they cover is only 140 square inches. So, what does the extra 60 square inches of runner do? It covers another runner!

Wherever there are two runners overlapping, one runner is not covering the table but just another runner. Wherever there are three runners overlapping, two runners are not covering the table but just the third runner at the bottom.

So, can we say that $(d + e + f)$ represents the area where one runner is covering another runner and g is the area where two runners are covering another runner?

Put another way, can we say $d + e + f + 2g = 60$?

We know that $d + e + f = 24$ giving us $g = 18$ square inches

This entire ‘thinking process’ takes ten seconds once you are comfortable with it and your answer would be out in about 30 sec!

In the end, **it's not that difficult of a concept to memorize if you simply extrapolate the formula for overlapping sets as follows**:

Total = Group1 + Group2 + Group3 - Those in Exactly Two Groups - 2(Those in All Three Groups) + None

How to Use a Double Matrix on the GRE?

Using a Venn diagram when dealing with most Sets questions is correct and effective, there are some questions in which a double-matrix is necessary (and much more powerful than a wimpy Venn). This little guy will make even the scariest-looking Sets question into a simple set of rows and columns, and its ability to help us determine whether a statement is sufficient in DS is unmatched!

To make a double-matrix, simply create a chart with rows and columns. The rows are assigned to one variable, and the columns to another. At the bottom of each column and at the far-right of each row, place a box for the column-total and row-total. Let’s check out a sample GRE question to see what this might look like!

Question: 33 out of the 47 students in an advanced degree program have a higher-than-average GPA. How many students in the program are receiving some form of academic scholarship?

(1) More students do not have a scholarship than have a scholarship.
(2) The same number of students have a higher-than-average GPA and are receiving some form of academic scholarship as have neither a higher-than-average GPA nor an academic scholarship.

Solution: From the question-stem, we know 33 students have a high GPA, while 14 do not. We need more information about which of these students have scholarships to be able to answer this value question. Statement (1) is insufficient because it does not give us information to find the exact numerical value of the students receiving some form of scholarship.

Statement (2) tells us that the number of students who fit “both” is equal to the number of students who fit “neither.” Let’s set up a chart to visualize the four possible categories for the students. Since “both” = “neither,” let’s fill in “x” for those boxes.

	Scholarship	No Scholarship	TOTAL
Higher GPA	x	33 - x	33
Not a Higher GPA	14 - x	x	14
TOTAL	?	33	47

Since each column and row must total, if there are "x" students receiving no scholarship and not a higher GPA, and the total students who don't have a higher GPA is 14, then 14-x students must not have a higher GPA and have a scholarship. The total we are looking for is represented by the red "?," and we can set up an equation to solve: x + (14 – x) = 14. Sufficient.

The correct response is (B).

To wrap up, keep in mind that it's possible (although highly unlikely) that you might see a Sets question on the GRE involving three variables instead of two. In that case, you'd have 5 columns instead of 4, and 5 rows instead of 4, but the same rules of totaling apply!

How to draw a Venn Diagram?

At a certain school, each of the 150 students takes between 1 and 3 classes. The 3 classes available are Math, Chemistry and English. 53 students study math, 88 study chemistry and 58 study English. If 6 students take all 3 classes, how many take exactly 2 classes?

A. 37
B. 43
C. 45
D. 60

Step 1: Deconstruction

This is where you extract the given information from a problem. Use a symbol for each section given to you. I usually use the first letter in caps. Please remember at this point that any number specified without explicitly mentioning that it's "exactly" or "only" for a certain subject is not to be taken so.

For instance, in this case, it says 53 math students. This doesn't mean students who are taking only 1 subject (Math). This could include students who are taking Math and Chemistry or Math and English or all three too. So now break down the numbers given to us.

Total = 150
M (Total) = 53

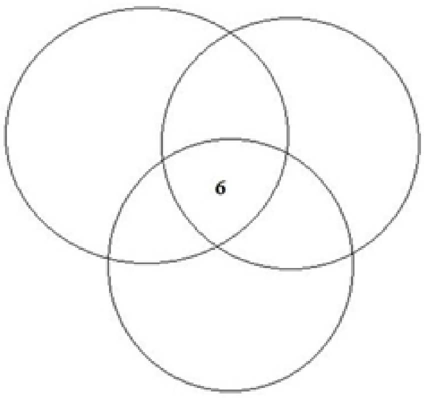
C (Total) = 88 [Why the hell are people studying more Chemistry than Math?]
E (Total) = 58
MCE = 6

We are asked to find MC + CE + EM.

Step 2: Drawing the diagram

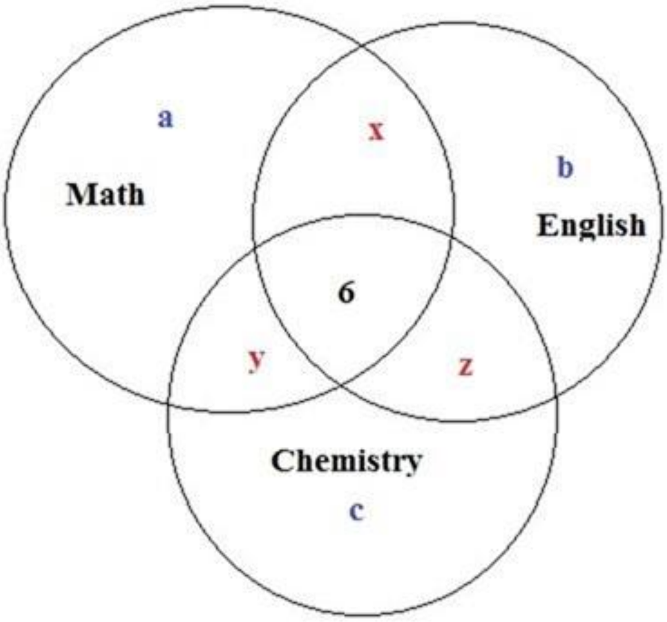
ALWAYS start from the center of the Venn Diagram wherever information is available. This will make life infinitely easier. In this case, the center is the intersection of all three circles, i.e MCE = 6

So, fill that in to the diagram you've drawn



.

Now that you've gotten that, let's start filling in a variable for each section not known to us. Here, considers each letter to represent only that specific section and not the entire circle or a larger portion.



So now that you have the parts filled in, what you need to do is write down what you have in the diagram in terms of numbers. So, we are given the totals for each subject.

Look at math first. There are four types of people taking math (each group of these people mutually exclusive, and not in common with any other group)
1. Only math: a
2. Math and Chemistry: y
3. Math and English: x
4. All three: 6

So now represent this as a sum and you get

$a + x + y + 6 = 53$ and hence $a + x + y = 47$

Similarly for the other subjects you get:

$b + x + z + 6 = 58$ and hence $b + x + z = 52$

$c + y + z + 6$= 88 and hence $c + y + z = 82$

And then you have the total:

$x + y + z + a + b + c + 6 = 150$ and hence $x + y + z + a + b + c = 144$

Step 3: Solution

This is perhaps the most intuitive part, but in my experience the first part of this step is the same in all overlapping sets problem. It's only what's asked for that's different.

Add all the individual equations together to get a combined equation with all the variables and a number.

So, you get:

$2(x + y + z) + a + b + c = 47 + 52 + 82 = 181$

Rearranging this to get $a + b + c$ $we\ get$ $a + b + c = 181 - 2(x + y + z)$

Substitute this into the total equation we derived earlier saying $x + y + z + a + b + c = 144$ so, you get:

$x + y + z + 181 - 2(x + y + z) = 144$

Upon rearranging this you get: $x + y + z = 181 - 144 = 37$ which is option A the right answer.

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