Chapter14. Quadrilaterals

Definition1. Quadrilaterals

A polygon with 4 sides is called a *quadrilateral*.

In GMAT, we are asked to master several special quadrilaterals – parallelogram, rectangle, square, and trapezoid.

Theorem1. The sum of all interior angles in a quadrilateral is 360°.

Definition2. Parallelogram

A *parallelogram* is a quadrilateral with two pairs of parallel sides. As shown in Figure 14.1, $AD \parallel BC, AB \parallel CD$, the quadrilateral ABCD is a parallelogram, often denoted as $\square ABCD$.

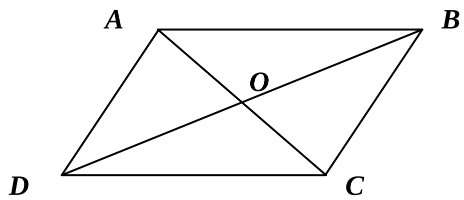

Figure: 14.1

Properties of Parallelograms

As shown in Figure 14.1, If the quadrilateral ABCD is a parallelogram, then:

1. The opposite angles are equal, i.e. $\angle A = \angle C, \angle B = \angle D$.
2. The opposite sides are equal, i.e. *AD = BC, AB = DC.*
3. The two diagonals bisect each other, i.e. *AO = CO, BO = DO.*

Then, how can we judge whether one quadrilateral is a parallelogram? We can use the following theorems:

Theorem2: Determine whether a quadrilateral is a parallelogram

As shown in Figure 14.1, we have a quadrilateral ABCD:

1. If the opposite angles are both equal, i.e. $\angle A = \angle C \text{ and } \angle B = \angle D$, then quadrilateral ABCD is a parallelogram.
2. If the opposite sides are both equal, i.e. *AD = BC, AB = DC,* then quadrilateral ABCD is a parallelogram.
3. If the two diagonals bisect each other, i.e. *AO = CO, BO = DO,* then quadrilateral ABCD is a parallelogram.
4. If one group of opposite sides are parallel and equal to each other, i.e. $AB \parallel CD, AB = CD$, then quadrilateral ABCD is a parallelogram.

EQ1. *In a quadrilateral ABCD, as shown in Figure 14.1, if AD = CB, AB = CD, and angle ADC = 60°, what are the values of the angle ABC and angle BCD?*

Answer: As $AD = CB$, $AB = CD$, thus the quadrilateral ABCD is a parallelogram.

$$\because \angle ADC = 60^\circ \Rightarrow \angle ABC = \angle ADC = 60^\circ$$

$$\because AB \parallel CD \Rightarrow \angle BCD = 180^\circ - \angle ABC = 120^\circ$$

Theorem3. The Area of a Parallelogram

Similar to a triangle, each side of a parallelogram can be taken as a **base.** The **height** then is defined as the perpendicular distance between the base and the opposite side.

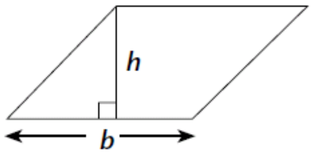

Figure: 14.2

If we know the base *b* and the height *h*, as shown in Figure 14.2, then we can use the following formula to calculate the area of a parallelogram:

Area of a parallelogram = *bh*

EQ2. *As shown in Figure 14.3,* $\angle ADC = 60^\circ$

$AD = 5,\ DC = 7$. *What is the area of* $\square ABCD$ *?*

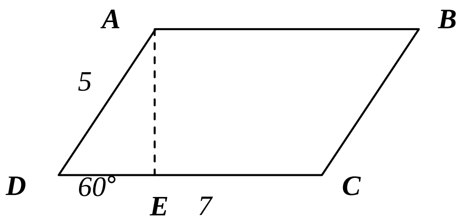

Figure: 14.3

Answer: After paying attention to the purpose, the area of a parallelogram, we think of the only theorem for it:

Area of a parallelogram = *bh*

Thus, if we take DC as the base, we need to draw the height (translation). We draw *AE*, which is perpendicular to *DC* at *E*.

Thus, *AE = h*, *DC = 7 = b*.

Our new purpose is AE. Now, we link it with the known, *AD = 5* and $\angle ADC = 60^\circ$, and think of theorems we learnt – trigonometric functions:

$$\sin \angle ADC = \sin 60^\circ = \frac{AE}{AD} = \frac{\sqrt{3}}{2}$$

$$\therefore AE = \frac{\sqrt{3}}{2} \times 5 = \frac{5\sqrt{3}}{2}$$

$$\therefore S_{\square ABCD} = bh = 7 \times \frac{5\sqrt{3}}{2} = \frac{35\sqrt{3}}{2}$$

Definition3. Rectangles

If one of the interior angles of a parallelogram is 90 degree, then this parallelogram is called a *rectangle*. As shown in Figure 14.4, the quadrilateral ABCD is a parallelogram, and angle D is a right angle. Thus, quadrilateral ABCD is a rectangle.

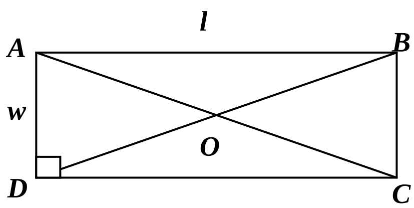

Figure: 14.4

This definition clearly tells that that a rectangle is a special parallelogram and thus all the properties of a parallelogram still apply to a rectangle. Hence, the opposite sides are equal. Then, we call the longer side the **length** and the shorter side the **width**, and denoted them as ***l*** and ***w***.

A rectangle also has the following additional properties:

Properties of Rectangles

As shown in Figure 14.4

1. The four interior angles in a rectangle are right angles, i.e. $\angle A = \angle B = \angle C = \angle D = 90^\circ$.
2. The diagonals are equal, i.e. $AC = BD$.

From the second property, we can get a new theorem for right triangles:

Theorem4.

In a right triangle, the length of the median on the hypotenuse is half the length of the hypotenuse.

As shown in Figure 14.4, triangle ADC is a right triangle. $DO = \frac{1}{2} AC$.

EQ3. *In Figure 14.4, in the rectangle ABCD, AD = 4 cm,* $\angle AOB = 120^\circ$, *what is the length of the diagonal AC?*

Answer: We just link the known with the purpose directly:

$$\because ABCD \text{ is a rectangle and } \angle AOB = 120^\circ$$

$$\Rightarrow OA = OB \Rightarrow \angle OAB = \angle OBA = \frac{180^\circ - 120^\circ}{2} = 30^\circ$$

$$\because AB \parallel DC$$

$$\Rightarrow \angle ACD = \angle BAC = 30^\circ$$

$$\Rightarrow AC = 2AD = 8\ cm$$

Theorem5: Determine whether a quadrilateral is a rectangle

1. If three interior angles in a quadrilateral are right angles, then this quadrilateral is a rectangle.
2. If the diagonals in a parallelogram are equal, then this parallelogram is a rectangle.

Theorem6. The Perimeter and the Area of a Rectangle

For a rectangle, if the length is *l* and the width is *w*, then:

The perimeter = *2 (l + w)*

The area = *lw*

EQ4. As shown in Figure 14.5, i*n parallelogram ABCD, the diagonal AC intersects with another diagonal BD at E. Triangle AED is an equilateral. AB = 6. What is the area of the parallelogram ABCD.*

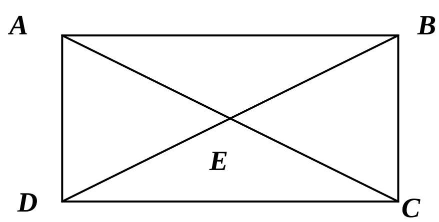

Figure: 14.5

Answer: Firstly, we link the known to see what kind of parallelogram it is. (We start to build the bridge between the known and the purpose from the known.)

$\because ABCD$ is a parallelogram

$\therefore AE = EC = \frac{1}{2}AC, DE = EB = \frac{1}{2}DB$

$\because AE = DE$

$\Rightarrow AC = BD$

$\Rightarrow \square ABCD$ is a rectangle (Theorem5-2).

Now, the purpose, the area, is $AB \cdot AD$. As we know *AB =6*. Thus, our new purpose is AD. Now, we use STRATEGY3 again by linking our new purpose with the known - triangle ADE is an equilateral. Thus,

$\because \triangle ADE$ is an equilateral

$\Rightarrow \angle DAE = 60^\circ$

$\Rightarrow \tan\angle DAE = \tan 60^\circ = \sqrt{3} = \frac{DC}{AD}$

$\because DC = AB = 6$

$\Rightarrow AD = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

$\Rightarrow S_{\square ABCD} = AB \cdot AD = 12\sqrt{3}$

Definition4. Squares

Square is a rectangle with four equal sides. As shown in Figure 14.5, quadrilateral *ABCD* is a rectangle and *AB* = *AD*. Then this rectangle *ABCD* is a square.

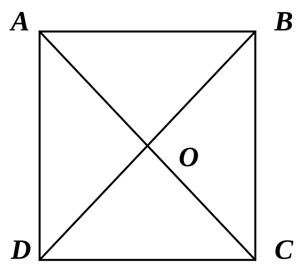

Figure: 14.5

According to the definition, a square is a rectangle and thus a parallelogram. Therefore it shares all properties of a rectangle and a parallelogram. In addition, it has one additional property:

Property of Squares

The diagonals of a square are equal and perpendicular to each other, bisect each other, and bisect the corresponding opposite angles.

In other words, as shown in Figure 14.5, AC and BD are the diagonals of square *ABCD*.

We have: *AC* = *BD*, *AO* = *CO*, *DO* = *BO*, $AO \perp BD$, *AC* is the angle bisector of angle *DAB* and angle *DCB*, *BD* is the angle bisector of angle *ABC* and angle *ADC*.

Theorem7: Determine whether a rectangle is a square

In addition to the definition of square, we have another theorem:

If the diagonals of a rectangle are perpendicular to each other, then this rectangle is a square.

Theorem8. The Perimeter and the Area of a Rectangle

If the length of each side of a square is *a*, then
The perimeter of a square = *4a*

The area of a square = a^2

Here, in Table 14.1, is a summary of the properties of parallelograms, rectangles, and squares:

	Parallelogram	**Rectangle**	**Square**
sides:	Opposite sides are equal.	Opposite sides are equal.	All sides are equal.
	Opposite sides are parallel.	Opposite sides are parallel.	Opposite sides are parallel.
angles:	Opposite angles are equal.	All angles are right angles.	All angles are right angles.
diagonals:	Diagonals bisect each other.	Diagonals bisect each other. Diagonals are equal.	Diagonals bisect each other. Diagonals are equal. Diagonals are perpendicular. Diagonals bisect corresponding angles.
perimeter:	Sum of sides	$2(l+w)$	$4a$
area:	$S=bh$	$S=lw$	$S=a^2$

Table 14.1

Definition5. Trapezoid

Trapezoid is a quadrilateral with one pair of parallel sides and one pair of nonparallel sides. As shown in Figure 14.6, $AB \parallel CD$, *AD* is not parallel with *BC*. The quadrilateral ABCD is a trapezoid. We often call the two parallel sides the **bases**. As shown in Figure 14.6, AB and CD are both bases. The perpendicular distance between them is called the height. As shown in Figure 14.6, AE is perpendicular to CD at E. Thus AE is the height.

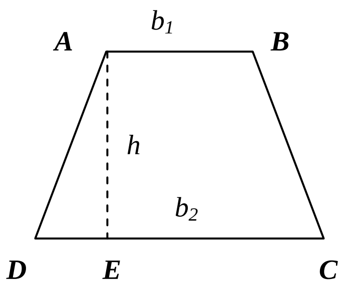

Figure: 14.6

Theorem9. The Area of a Trapezoid

If the length of the two bases of a trapezoid are b_1 and b_2, and the length of its height is *h*, then, the area of the trapezoid = $\frac{(b_1+b_2)}{2}h$.

EQ5. As shown in Figure 14.7, AB = 2, AE = DE = 8, triangle BCE is an equilateral triangle. Angle A and D are both right angles. What is the area of the region ABCDE?

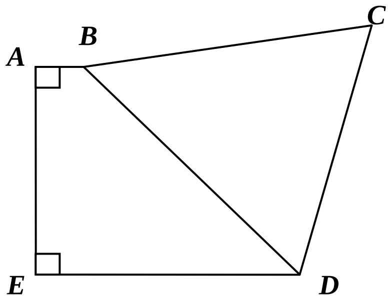

Figure: 14.7

Answer: We start to solve this question with the help of STRATEGY3.
When we pay attention to the purpose, the area of ABCDE, we realize that the area is composed of two parts – the area of the trapezoid ABDE and the area of the equilateral triangle BCD – these are our two new purposes. Firstly, if we pay attention to the area of the trapezoid ABDE, we think of the only theorem we learnt:

$$S_{Trapezoid} = \frac{(b_1 + b_2)}{2} h$$

Here, as the lengths of the two bases and the height are known, thus:

$$S_{TrapezoidABDE} = \frac{(b_1 + b_2)}{2} h = \frac{(2+8)}{2} 8 = 40$$

The only purpose left is the area of the equilateral triangle BCE. We use STRATEGY3 and think of related theorem - $S_{\triangle BCE} = \frac{\sqrt{3}}{4} a^2$, a is the length of each side. Thus, our "new new" purpose the a, i,e, BD. In order to calculate this side, we need to link it with the known, such as DE = AE =8. In fact, we can put the known and our purpose into one triangle to link them. We do so by drawing the segment BF that is perpendicular to DE at F. as shown in Figure 14.8. Thus, in right triangle BFE, we can use Pythagorean theorem:

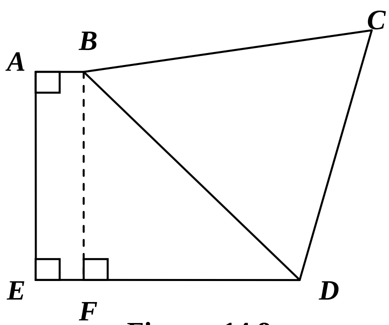

Figure: 14.8

$$BD^2 = BF^2 + FD^2 = 8^2 + (8-2)^2 = 64 + 36 = 100$$

$$\Rightarrow S_{\triangle BCE} = \frac{\sqrt{3}}{4} BD^2 = 25\sqrt{3}$$

$$\Rightarrow S_{\text{RegionABCDE}} = 25\sqrt{3} + 40$$